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COMPOSITION OF CONFORMAL AND PROJECTIVE MAPPINGS OF GENERALIZED RIEMANNIAN SPACES IN EISENHART'S SENSE PRESERVING CERTAIN TENSORS

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Abstract. The composition of conformal and projective mappings between Riemannian spaces that were at the same time harmonic had been studied by S. E. Stepanov, I. G. Shandra in 2003 and further developed in I. Hinterleitner's Ph.D. thesis in 2009. Conformal and projective mappings of Riemannian spaces preserving certain tensors were studied by O. Chepurna in the 2012 Ph.D. thesis. We consider conformal and projective mappings of generalized Riemannian spaces in Eisenhart's sense and find necessary and sufficient conditions for these mappings to preserve curvature, Ricci and traceless Ricci tensors and some of their linear combinations. Particularly, as an additional contribution to related results collected in the Ph.D. thesis by O. Chepurna, we find that the following result holds in the case of Riemannian spaces: if a conformal mapping $f_1 : M \to \hat{M}$ is preserving the traceless Ricci tensor and a projective mapping $f_2 : \hat{M} \to \overline{M}$ is preserving the traceless Ricci tensor then the Yano tensor of concircular curvature is invariant with respect to the composition $f_3 = f_1 \circ f_2 : M \to \overline{M}$.
Keywords: conformal mapping, geodesic mapping, generalized Riemannian space, Riemannian curvature tensor, traceless Ricci tensor, Weyl's tensor of projective curvature, Weyl's conformal curvature tensor, Yano's tensor of concircular curvature.

1. Introduction and preliminaries

A generalized Riemannian space in Eisenhart's sense [3] $(M, \mathcal{G} = g + \omega)$ is a differentiable manifold M endowed with a bilinear form $\mathcal{G} = g + \omega$, or in local components $\mathcal{G}_{ij} = g_{ij} + \omega_{ij}$, where g is a non-degenerate (i.e., $\det(g) \neq 0$),

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symmetric bilinear form and ω is a skew-symmetric bilinear form. The curvature tensors $\underset{\theta}{R}{}^h_{ijk}$, $\theta = 1, \dots, 5$ that correspond to generalized Christoffel symbols $\Gamma^h_{ij} = g^{hp}\Gamma_{pij} = \frac{1}{2}g^{hp}\left(\partial_j \mathcal{G}_{ip} - \partial_p \mathcal{G}_{ij} + \partial_i \mathcal{G}_{pj}\right)$, are related with the Riemannian curvature tensor

$$\overset{g}{R}{}^h_{ijk} = \partial_k \overset{g}{\Gamma}{}^h_{ij} - \partial_j \overset{g}{\Gamma}{}^h_{ik} + \overset{g}{\Gamma}{}^p_{ij}\overset{g}{\Gamma}{}^h_{pk} - \overset{g}{\Gamma}{}^p_{ik}\overset{g}{\Gamma}{}^h_{pj}, \tag{1.1}$$

where $\overset{g}{\Gamma}{}^h_{ij}$ are components of the Levi-Civita connection $\overset{g}{\nabla}$ of the Riemannian metric g, by (see [6], pp. 37–38)

$$\begin{aligned} \underset{\theta}{R}{}^h_{ijk}[\mu,\nu,\alpha,\beta,\gamma] = \overset{g}{R}{}^h_{ijk} + \mu \overset{g}{\nabla}_k T^h_{ij} + \nu \overset{g}{\nabla}_j T^h_{ik} \\ + \alpha T^h_{pk} T^p_{ij} + \beta T^h_{pj} T^p_{ik} + \gamma T^h_{pi} T^p_{jk}, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} (\theta,\mu,\nu,\alpha,\beta,\gamma) \in \Big\{ & \Big(1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, 0\Big), \Big(2, -\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, 0\Big), \\ & \Big(3, \frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{2}\Big), \Big(4, \frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\Big), \\ & \Big(5, 0, 0, \frac{1}{4}, \frac{1}{4}, 0\Big) \Big\}. \end{aligned} \tag{1.3}$$

Obviously, by contracting the relation between the curvature tensors $\underset{\theta}{R}{}^h_{ijk}[\mu,\nu,\alpha,\beta,\gamma]$ and $\overset{g}{R}{}^h_{ijk}$, given by (1.2), with respect to the indices h and k, we get the relation between the Ricci tensors $\underset{\theta}{Ric}_{ij}[\mu,\beta,\gamma] = \underset{\theta}{R}{}^p_{ijp}[\mu,\beta,\gamma]$ and $\overset{g}{Ric}_{ij} = \overset{g}{R}{}^p_{ijp}$

$$\underset{\theta}{Ric}_{ij}[\mu,\beta,\gamma] = \overset{g}{Ric}_{ij} + \mu \overset{g}{\nabla}_p T^p_{ij} + \beta T^q_{pj} T^p_{iq} + \gamma T^q_{pi} T^p_{jq}. \tag{1.4}$$

Furthermore, after contracting the relation (1.4) with g^{ij} we find the relation between the scalar curvatures $\underset{\theta}{S}[\mu,\beta,\gamma] = g^{pq}\underset{\theta}{Ric}_{pq}[\mu,\beta,\gamma]$ and $\overset{g}{S} = g^{pq}\overset{g}{R}_{pq}$

$$\underset{\theta}{S}[\mu,\beta,\gamma] = \overset{g}{S} + \mu g^{pq} \overset{g}{\nabla}_r T^r_{pq} + \beta g^{pq} T^s_{rq} T^r_{ps} + \gamma g^{pq} T^s_{rp} T^r_{qs}. \tag{1.5}$$

From (1.2) it directly follows that the $(0,4)$ curvature tensors $\underset{\theta}{R}_{hijk}[\mu,\nu,\alpha,\beta,\gamma] = g_{hp}\underset{\theta}{R}{}^p_{ijk}, \theta = 1, \dots, 5$ and $\overset{g}{R}_{hijk} = g_{hp}\overset{g}{R}{}^p_{ijk}$ are related by

$$\begin{aligned} \underset{\theta}{R}_{hijk}[\mu,\nu,\alpha,\beta,\gamma] = \overset{g}{R}_{hijk} + \mu \overset{g}{\nabla}_k g_{hp} T^p_{ij} + \nu \overset{g}{\nabla}_j g_{hp} T^p_{ik} \\ + \alpha g_{hp} T^p_{qk} T^q_{ij} + \beta g_{hp} T^p_{qj} T^q_{ik} + \gamma g_{hp} T^p_{qi} T^q_{jk}, \end{aligned} \tag{1.6}$$

where $(\theta,\mu,\alpha,\beta,\gamma)$ are given by (1.3).

2. Conformal mappings between generalized Riemannian spaces in Eisenhart's sense

For preliminaries about conformal mappings between generalized Riemannian spaces in Eisenhart's sense see Chapter 6 of the monograph [6].

2.1. Weyl conformal curvature tensor

In what follows, we shall give a more clear overview of the basic idea used in the paper [9] and give some remarks and comments. On page 84 of the monograph [7] the Riemannian curvature tensor is defined by $R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^p_{ik}\Gamma^h_{pj} - \Gamma^p_{ij}\Gamma^h_{pk}$, which is different than the Riemannian curvature tensor that we use $\overset{g}{R}{}^h_{ijk}$ by a sign.

The relation between the Riemannian curvature tensors R^h_{ijk} and $\overline{R}{}^h_{ijk}$ of two Riemannian spaces (M, g) and $(\overline{M}, \overline{g})$, respectively, with respect to the conformal mapping between these spaces $f : M \to \overline{M}$ is well-known, which, by using the notation from page 239 of the monograph [7], reads

$$\overline{R}{}^h_{ijk} = R^h_{ijk} + \delta^h_k\,\sigma_{ij} - \delta^h_j\,\sigma_{ik} + \sigma^h_k g_{ij} - \sigma^h_j g_{ik} + \left(\delta^h_k\, g_{ij} - \delta^h_j\, g_{ik}\right)\Delta_1\sigma, \tag{2.1}$$

where

$$\sigma_{ij} = \overset{g}{\nabla}_j \sigma_i - \sigma_i\sigma_j, \quad \sigma^h_k = g^{hp}\sigma_{pk}, \quad \Delta_1\sigma = g^{pq}\sigma_p\sigma_q. \tag{2.2}$$

By multiplying the relation (2.1) with -1 we get the relation between the Riemannian curvature tensors $\overset{g}{R}{}^h_{ijk}$ and $\overset{\overline{g}}{R}{}^h_{ijk}$

$$\overset{\overline{g}}{R}{}^h_{ijk} = \overset{g}{R}{}^h_{ijk} - \delta^h_k\,\sigma_{ij} + \delta^h_j\,\sigma_{ik} - \sigma^h_k g_{ij} + \sigma^h_j g_{ik} - \left(\delta^h_k\, g_{ij} + \delta^h_j\, g_{ik}\right)\Delta_1\sigma, \tag{2.3}$$

where σ_{ij}, σ^h_k and $\Delta_1\sigma$ are determined by (2.2), the Riemannian curvature tensor $\overset{g}{R}{}^h_{ijk}$ is determined by (1.1) and the Riemannian curvature tensor $\overset{\overline{g}}{R}{}^h_{ijk}$ is determined in the same manner in the Riemannian space $(\overline{M}, \overline{g})$.

In [9] it was observed that starting from the relation (2.3) and following the procedure for deriving the Weyl conformal curvature tensor, described, for instance, on page 239 of the monograph [7], we get Theorem 1 from [9]. Theorem 1 from [9] states that for arbitrary $(\theta, \mu, \nu, \alpha, \beta, \gamma)$, given by (1.3), the tensor $\underset{\theta}{C}{}^h_{ijk}$, given by

$$\underset{\theta}{C}{}^h_{ijk} = \underset{\theta}{R}{}^h_{ijk} + \delta^h_j\,\underset{\theta}{L}_{ik} - \delta^h_k\,\underset{\theta}{L}_{ij} + g_{ik}\underset{\theta}{L}{}^h_j - g_{ij}\underset{\theta}{L}{}^h_k, \tag{2.4}$$

where we used the notation analogous as in [7], p. 239

$$\underset{\theta}{L}_{ij} = \frac{1}{n-2}\left(\underset{\theta}{Ric}_{ij} - \frac{\underset{\theta}{S}}{2(n-1)} g_{ij}\right) \tag{2.5}$$

and

$$L_{\theta}{}^{h}_{j} = g^h_p \underset{\theta}{L}_{pj} = \frac{1}{n-2}\left(\underset{\theta}{Ric}{}^h_j - \frac{\underset{\theta}{S}}{2(n-1)}\delta^h_j\right), \tag{2.6}$$

is invariant with respect to the conformal mapping $f : M \to \overline{M}$ between the generalized Riemannian spaces $(M, \mathcal{G} = g + \omega)$ and $(\overline{M}, \overline{\mathcal{G}} = \overline{g} + \overline{\omega})$. However, the assumption that must be satisfied for the result in Theorem 1 from [9] to hold is given by [10]

$$\begin{aligned}
&\mu \overset{\overline{g}}{\nabla}_k \overline{T}^h_{ij} + \nu \overset{\overline{g}}{\nabla}_j \overline{T}^h_{ik} + \alpha \overline{T}^h_{pk}\overline{T}^p_{ij} + \beta \overline{T}^h_{pj}\overline{T}^p_{ik} + \gamma \overline{T}^h_{pi}\overline{T}^p_{jk} \\
&= \mu \overset{g}{\nabla}_k T^h_{ij} + \nu \overset{g}{\nabla}_j T^h_{ik} + \alpha T^h_{pk} T^p_{ij} + \beta T^h_{pj} T^p_{ik} + \gamma T^h_{pi} T^p_{jk},
\end{aligned} \tag{2.7}$$

where $\overset{\overline{g}}{\nabla}$ and $\overset{g}{\nabla}$ are the Levi-Civita connections that correspond to the Riemannian metrics $\overline{g}$ and g, respectively. Here $\overline{T}^h_{ij}$ and T^h_{ij} are the torsion tensors that correspond to the generalized Christoffel symbols with respect to the generalized Riemannian metrics $\overline{\mathcal{G}}$ and $\mathcal{G}$, respectively.

The assumption (2.7) does not affect the result given in Theorem 1 from [9] and this assumption is quite obvious and natural, because the left and right sides of this assumption are expressions which we can add to the left and right sides of the relation (2.3), respectively, in order to obtain the same relation where instead of the Riemannian curvature tensors $\overset{g}{R}{}^h_{ijk}$ and $\overset{\overline{g}}{R}{}^h_{ijk}$ we will have the curvature tensors $\underset{\theta}{R}{}^h_{ijk}[\mu, \nu, \alpha, \beta, \gamma]$ and $\underset{\theta}{\overline{R}}{}^h_{ijk}[\mu, \nu, \alpha, \beta, \gamma]$, according to the relation (1.2).

Obviously, when we lower the upper index in (2.4) we get the relation between the $(0, 4)$ tensors $\underset{\theta}{C}_{hijk} = g_{hp}\underset{\theta}{C}{}^p_{ijk}$ and $\underset{\theta}{R}_{hijk} = g_{hp}\underset{\theta}{R}{}^p_{ijk}$

$$\underset{\theta}{C}_{hijk} = \underset{\theta}{R}_{hijk} + g_{hj}\underset{\theta}{L}_{ik} - g_{hk}\underset{\theta}{L}_{ij} + g_{ik}\underset{\theta}{L}_{hj} - g_{ij}\underset{\theta}{L}_{hk}, \tag{2.8}$$

where the tensors $\underset{\theta}{L}_{ij}$ are given by (2.5).

Proposition 2.1. *Let $f : M \to \overline{M}$ be a conformal mapping between generalized Riemannian spaces $(M, \mathcal{G} = g + \omega)$ and $(\overline{M}, \overline{\mathcal{G}} = \overline{g} + \overline{\omega})$ of dimension $n > 2$. For an arbitrary $(\theta, \mu, \nu, \alpha, \beta, \gamma)$ given by (1.3), the tensor $\underset{\theta}{C}{}^h_{ijk}$ given by (2.4) is related with the Weyl conformal curvature tensor $\overset{g}{C}{}^h_{ijk}$ by*

$$\begin{aligned}
\underset{\theta}{C}{}^h_{ijk}[\mu, \nu, \alpha, \beta, \gamma] = \overset{g}{C}{}^h_{ijk} &+ \mu \overset{g}{\nabla}_k T^h_{ij} + \nu \overset{g}{\nabla}_j T^h_{ik} \\
&+ \alpha T^h_{pk} T^p_{ij} + \beta T^h_{pj} T^p_{ik} + \gamma T^h_{pi} T^p_{jk} \\
&+ \delta^h_j\, Q_{ik}[\mu, \beta, \gamma] - \delta^h_k\, Q_{ij}[\mu, \beta, \gamma] \\
&+ g_{ik} Q^h_j[\mu, \beta, \gamma] - g_{ij} Q^h_k[\mu, \beta, \gamma],
\end{aligned} \tag{2.9}$$

where the tensors

$$Q_{ij}[\mu,\beta,\gamma] = \frac{1}{n-2}S_{ij}[\mu,\beta,\gamma] - \frac{1}{2(n-2)}g^{pq}S_{pq}[\mu,\beta,\gamma]g_{ij} \tag{2.10}$$

and

$$Q^h_k = g^{hp}Q_{pk}$$

depend on the tensor

$$S_{ij}[\mu,\beta,\gamma] = \mu\overset{g}{\nabla}_p T^p_{ij} + \beta T^q_{pj}T^p_{iq} + \gamma T^q_{pi}T^p_{jq}. \tag{2.11}$$

Proof. By using the relations (1.2), (1.4) and (1.5) into (2.4), (2.5) and (2.6) we obtain that

$$\underset{\theta}{L}_{ij}[\mu,\beta,\gamma] = L_{ij} + Q_{ij}[\mu,\beta,\gamma], \tag{2.12}$$

where the tensor $Q_{ij}[\mu,\beta,\gamma]$ is defined by (2.10). Also,

$$\underset{\theta}{L}{}^h_j[\mu,\beta,\gamma] = g^{hp}\left(L_{pj} + Q_{pj}[\mu,\beta,\gamma]\right) = L^h_j + Q^h_j[\mu,\beta,\gamma]. \tag{2.13}$$

By using (2.12) and (2.13) into (2.4) we obtain (2.9), which completes the proof. □

Corollary 2.1. *Let $f : M \to \overline{M}$ be a conformal mapping between generalized Riemannian spaces $(M, \mathcal{G} = g + \omega)$ and $(\overline{M}, \overline{\mathcal{G}} = \overline{g} + \overline{\omega})$ of dimension $n > 2$. For an arbitrary $(\theta,\mu,\nu,\alpha,\beta,\gamma)$ given by (1.3), the $(0,4)$ tensor $\underset{\theta}{C}_{hijk}$ given by (2.8) is related with the $(0,4)$ Weyl conformal curvature tensor $\overset{g}{C}_{hijk}$ by*

$$\begin{aligned}\underset{\theta}{C}_{hijk}[\mu,\nu,\alpha,\beta,\gamma] = \overset{g}{C}_{hijk} &+ \mu\overset{g}{\nabla}_k T_{hij} + \nu\overset{g}{\nabla}_j T_{hik}\\ &+ \alpha T_{hpk}T^p_{ij} + \beta T_{hpj}T^p_{ik} + \gamma T_{hpi}T^p_{jk}\\ &+ g_{hj}Q_{ik}[\mu,\beta,\gamma] - g_{hk}Q_{ij}[\mu,\beta,\gamma]\\ &+ g_{ik}Q_{hj}[\mu,\beta,\gamma] - g_{ij}Q_{hk}[\mu,\beta,\gamma],\end{aligned}$$

where the tensor $Q_{ij}[\mu,\beta,\gamma]$ is defined by (2.10), or in a more compact way as

$$\underset{\theta}{C}_{hijk}[\mu,\nu,\alpha,\beta,\gamma] = \overset{g}{C}_{hijk} + P_{hijk}[\mu,\nu,\alpha,\beta,\gamma] + Q_{hijk}[\mu,\nu,\alpha,\beta,\gamma],$$

where $P_{hijk}[\mu,\nu,\alpha,\beta,\gamma]$ and $Q_{hijk}[\mu,\nu,\alpha,\beta,\gamma]$ are defined by

$$\begin{aligned}P_{hijk}[\mu,\nu,\alpha,\beta,\gamma] = &\mu g_{hp}\overset{g}{\nabla}_k T^p_{ij} + \nu g_{hp}\overset{g}{\nabla}_j T^p_{ik}\\ &+ \alpha g_{hp}T^p_{qk}T^q_{ij} + \beta g_{hp}T^p_{qj}T^q_{ik} + \gamma g_{hp}T^p_{qi}T^q_{jk}\end{aligned}$$

and

$$\begin{aligned}Q_{hijk}[\mu,\nu,\alpha,\beta,\gamma] = &g_{hj}Q_{ik}[\mu,\beta,\gamma] - g_{hk}Q_{ij}[\mu,\beta,\gamma]\\ &+ g_{ik}Q_{hj}[\mu,\beta,\gamma] - g_{ij}Q_{hk}[\mu,\beta,\gamma].\end{aligned}$$

2.2. Conformal mappings between generalized Riemannian spaces in Eisenhart's sense preserving curvature, Ricci and traceless Ricci tensors

V. E. Berezovski, S. Bácsó and J. Mikeš [1] studied diffeomorphisms between affine connected spaces preserving certain tensors, including the Riemannian curvature tensor and the Ricci tensor. Here we will use such approach to consider diffeomorphisms between the generalized Riemannian spaces in Eisenhart's sense.

Theorem 2.1. *Let $(M, \mathcal{G} = g+\omega)$ and $(\overline{M}, \overline{\mathcal{G}} = \overline{g}+\overline{\omega})$ be generalized Riemannian spaces in Eisenhart's sense of dimension $n > 2$. For an arbitrary $(\theta, \mu, \nu, \alpha, \beta, \gamma)$ given by* (1.3) *a conformal mapping $f : M \to \overline{M}$ is preserving the Riemannian curvature tensor $\overset{g}{R}{}^h_{ijk}$ if and only if*

$$
\begin{aligned}
&\delta^h_j \overline{L}_{ik} - \delta^h_k \overline{L}_{ij} + \overline{L}^h_j \overline{g}_{ik} - L^h_k \overline{g}_{ij} \\
&+ \mu \overset{\overline{g}}{\nabla}_k \overline{T}^h_{ij} + \nu \overset{\overline{g}}{\nabla}_j \overline{T}^h_{ik} + \alpha \overline{T}^h_{pk} \overline{T}^p_{ij} + \beta \overline{T}^h_{pj} \overline{T}^p_{ik} + \gamma \overline{T}^h_{pi} \overline{T}^p_{jk} \\
&+ \delta^h_j \overline{Q}_{ik}[\mu, \beta, \gamma] - \delta^h_k \overline{Q}_{ij}[\mu, \beta, \gamma] \\
&+ \overline{g}_{ik} \overline{Q}^h_j[\mu, \beta, \gamma] - \overline{g}_{ij} \overline{Q}^h_k[\mu, \beta, \gamma] \\
=&\delta^h_j L_{ik} - \delta^h_k L_{ij} + L^h_j g_{ik} - L^h_k g_{ij} \\
&+ \mu \overset{g}{\nabla}_k T^h_{ij} + \nu \overset{g}{\nabla}_j T^h_{ik} + \alpha T^h_{pk} T^p_{ij} + \beta T^h_{pj} T^p_{ik} + \gamma T^h_{pi} T^p_{jk} \\
&+ \delta^h_j Q_{ik}[\mu, \beta, \gamma] - \delta^h_k Q_{ij}[\mu, \beta, \gamma] \\
&+ g_{ik} Q^h_j[\mu, \beta, \gamma] - g_{ij} Q^h_k[\mu, \beta, \gamma],
\end{aligned}
\tag{2.14}
$$

where $\overline{Q}^h_j = \overline{g}^{hp} \overline{Q}_{pj}$ and $Q^h_j = \overline{g}^{hp} Q_{pj}$. The tensors L_{ij} and $Q_{ij}[\mu, \beta, \gamma]$ are determined in the space $(M, \mathcal{G} = g + \omega)$ by $L_{ij} = \frac{1}{n-2}\left(\overset{g}{Ric}_{ij} - \frac{\overset{g}{S}}{2(n-1)} g_{ij}\right)$ and (2.10), *respectively, while the tensors $\overline{L}_{ij}$ and $\overline{Q}_{ij}[\mu, \beta, \gamma]$ are determined in the same manner in the space $(\overline{M}, \overline{\mathcal{G}} = \overline{g} + \overline{\omega})$.*

Corollary 2.2. *Let $(M, \mathcal{G} = g+\omega)$ and $(\overline{M}, \overline{\mathcal{G}} = \overline{g}+\overline{\omega})$ be generalized Riemannian spaces in Eisenhart's sense of dimension $n > 2$. For an arbitrary $(\theta, \mu, \nu, \alpha, \beta, \gamma)$ given by* (1.3) *a conformal mapping $f : M \to \overline{M}$ is preserving the Ricci curvature tensor $\overset{g}{Ric}_{ij} = \overset{g}{R}{}^p_{ijp}$ if and only if*

$$
\begin{aligned}
&\overline{L}_{ij} - (n-1)\overline{L}_{ij} - \overline{L}^p_p \overline{g}_{ij} + \mu \overset{\overline{g}}{\nabla}_p \overline{T}^p_{ij} + \beta \overline{T}^p_{qj} \overline{T}^q_{ip} + \gamma \overline{T}^p_{qi} \overline{T}^q_{jp} \\
&+ 2\overline{Q}_{ij}[\mu, \beta, \gamma] - n\overline{Q}_{ij}[\mu, \beta, \gamma] - \overline{g}_{ij} \overline{Q}^p_p[\mu, \beta, \gamma] \\
=&L_{ij} - (n-1)L_{ij} - L^p_p g_{ij} + \mu g \nabla_p T^p_{ij} + \beta T^p_{qj} T^q_{ip} + \gamma T^p_{qi} T^q_{jp} \\
&+ 2Q_{ij}[\mu, \beta, \gamma] - nQ_{ij}[\mu, \beta, \gamma] - g_{ij} Q^p_p[\mu, \beta, \gamma],
\end{aligned}
\tag{2.15}
$$

where $\overline{Q}^h_j = \overline{g}^{hp}\overline{Q}_{pj}$ *and* $Q^h_j = \overline{g}^{hp}Q_{pj}$. *The tensors* L_{ij} *and* $Q_{ij}[\mu,\beta,\gamma]$ *are determined in the space* $(M, \mathcal{G} = g+\omega)$ *by* $L_{ij} = \frac{1}{n-2}\left(\overset{g}{Ric}_{ij} - \frac{\overset{g}{S}}{2(n-1)}g_{ij}\right)$ *and* (2.10), *respectively, while* L_{ij} *and* $\overline{Q}_{ij}[\mu,\beta,\gamma]$ *are determined in the same manner in the space* $(\overline{M}, \overline{\mathcal{G}} = \overline{g}+\overline{\omega})$.

Proof. The relation (2.15) is obtained directly from the relation (2.14). □

Theorem 2.2. *Let* $(M, \mathcal{G} = g+\omega)$ *and* $(\overline{M}, \overline{\mathcal{G}} = \overline{g}+\overline{\omega})$ *be generalized Riemannian spaces in Eisenhart's sense of dimension* $n > 2$. *For an arbitrary* $(\theta,\mu,\nu,\alpha,\beta,\gamma)$ *given by* (1.3) *the curvature tensor* $\underset{\theta}{R}{}^h_{ijk}$ *is preserved with respect to the conformal mapping* $f: M \to \overline{M}$ *if and only if*

$$\begin{aligned}
&\mu\overset{\overline{g}}{\nabla}_k\overline{T}^h_{ij} + \nu\overset{\overline{g}}{\nabla}_j\overline{T}^h_{ik} + \alpha\overline{T}^h_{pk}\overline{T}^p_{ij} + \beta\overline{T}^h_{pj}\overline{T}^p_{ik} + \gamma\overline{T}^h_{pi}\overline{T}^p_{jk}\\
&+ \delta^h_j\underset{\theta}{\overline{L}}_{ik} - \delta^h_k\overline{L}_{ij} + \underset{\theta}{\overline{L}}{}^h_j\overline{g}_{ik} - \underset{\theta}{\overline{L}}{}^h_k\overline{g}_{ij}\\
&= \mu\overset{g}{\nabla}_kT^h_{ij} + \nu\overset{g}{\nabla}_jT^h_{ik} + \alpha T^h_{pk}T^p_{ij} + \beta T^h_{pj}T^p_{ik} + \gamma T^h_{pi}T^p_{jk}\\
&+ \delta^h_j\underset{\theta}{L}_{ik} - \delta^h_k L_{ij} + \underset{\theta}{L}{}^h_j g_{ik} - \underset{\theta}{L}{}^h_k g_{ij},
\end{aligned}$$

where the tensors L_{ij} *given by* $L_{ij} = \frac{1}{n-2}\left(\overset{g}{Ric}_{ij} - \frac{\overset{g}{S}}{2(n-1)}g_{ij}\right)$ *and* $L^h_i = g^{hp}L_{pi}$ *are determined in the generalized Riemannian space in Eisenhart's sense* $(M, \mathcal{G} = g+\omega)$, *while the tensors* $\overline{L}_{ij}$ *and* $\overline{L}^h_i = g^{hp}\overline{L}_{pi}$ *are determined in the same manner in the generalized Riemannian space in Eisenhart's sense* $(\overline{M}, \overline{\mathcal{G}} = \overline{g}+\overline{\omega})$.

Corollary 2.3. *Let* $(M, \mathcal{G} = g+\omega)$ *and* $(\overline{M}, \overline{\mathcal{G}} = \overline{g}+\overline{\omega})$ *be generalized Riemannian spaces in Eisenhart's sense of dimension* $n > 2$. *For an arbitrary* $(\theta,\mu,\nu,\alpha,\beta,\gamma)$ *given by* (1.3) *the curvature tensor* $\underset{\theta}{R}{}^h_{ijk}$ *is preserved with respect to the conformal mapping* $f: M \to \overline{M}$ *if and only if*

$$\begin{aligned}
&\mu\overset{\overline{g}}{\nabla}_p\overline{T}^p_{ij} + \beta\overline{T}^p_{qj}\overline{T}^q_{ip} + \gamma\overline{T}^p_{qi}\overline{T}^q_{jp} + \underset{\theta}{\overline{L}}_{ij} - n\overline{L}_{ij} + \underset{\theta}{\overline{L}}_{ij} - \underset{\theta}{\overline{L}}{}^p_p\overline{g}_{ij}\\
&= \mu\overset{g}{\nabla}_pT^p_{ij} + \beta T^p_{qj}T^q_{ip} + \gamma T^p_{qi}T^q_{jp} + \underset{\theta}{L}_{ij} - nL_{ij} + \underset{\theta}{L}_{ij} - \underset{\theta}{L}{}^p_p g_{ij},
\end{aligned}$$

where the tensors L_{ij} *given by* $L_{ij} = \frac{1}{n-2}\left(\overset{g}{Ric}_{ij} - \frac{\overset{g}{S}}{2(n-1)}g_{ij}\right)$ *and* $L^h_i = g^{hp}L_{pi}$ *are determined in the generalized Riemannian space in Eisenhart's sense* $(M, \mathcal{G} = g+\omega)$, *while the tensors* $\overline{L}_{ij}$ *and* $\overline{L}^h_i = g^{hp}\overline{L}_{pi}$ *are determined in the same manner in the generalized Riemannian space in Eisenhart's sense* $(\overline{M}, \overline{\mathcal{G}} = \overline{g}+\overline{\omega})$.

In the Riemannian space (M, g) the traceless Ricci tensor is given by

$$\overset{g}{Ric}_{ij} - \frac{1}{n}\overset{g}{S}g_{ij} \tag{2.16}$$

and in the generalized Riemannian space $(M, \mathcal{G} = g + \omega)$ the traceless Ricci tensors are given by [9]

$$\underset{\theta}{Ric}_{ij} - \frac{1}{n}\underset{\theta}{S}g_{ij}, \qquad \theta \in \{1, \dots, 5\}, \tag{2.17}$$

where $\underset{\theta}{S}$ and $\underset{\theta}{Ric}_{ij}$ are scalar curvatures and Ricci tensors, respectively.

Theorem 2.3. *Let $(M, \mathcal{G} = g + \omega)$ be a generalized Riemannian space in Eisenhart's sense of dimension $n > 2$. For an arbitrary $(\theta, \mu, \nu, \alpha, \beta, \gamma)$ given by* (1.3)*, the tensor given by* (2.17)*, is related with the tensor given by* (2.16)*, by*

$$\begin{aligned}(\underset{\theta}{Ric}_{ij} - \frac{1}{n}\underset{\theta}{S}g_{ij})[\mu, \beta, \gamma] = \overset{g}{Ric}_{ij} - \frac{1}{n}\overset{g}{S}g_{ij} + \mu\overset{g}{\nabla}_p T^p_{ij} + \beta T^q_{pj}T^p_{iq} + \gamma T^q_{pi}T^p_{jq} \\ - \frac{1}{n}\left(\mu g^{ij}\overset{g}{\nabla}_p T^p_{ij} + \beta g^{ij}T^q_{pj}T^p_{iq} + \gamma g^{ij}T^q_{pi}T^p_{jq}\right) g_{ij}.\end{aligned}$$

Proof. Obvious. □

2.3. Yano tensor of concircular curvature

Theorem 4 from [9] claims that the tensors $\underset{\theta}{Y}^h_{ijk}$ and $\underset{\theta}{Ric}_{ij} - \frac{1}{n}\underset{\theta}{S}g_{ij}$, $\theta = 1, \dots, 5$ which are analogous to the Yano tensor of concircular curvature and the traceless Ricci tensor, respectively, given below, are invariant with respect to the concircular mapping $f : M \to \overline{M}$ between generalized Riemannian spaces in Eisenhart's sense (M, g) and $(\overline{M}, \overline{g})$.

The tensor $\underset{\theta}{Y}^h_{ijk}$ is given by

$$\underset{\theta}{Y}^h_{ijk} = \underset{\theta}{R}^h_{ijk} - \frac{\underset{\theta}{S}}{n(n-1)}\left(g_{ki}\delta^h_j - g_{ji}\delta^h_k\right). \tag{2.18}$$

When we lower the upper index in (2.18) we get

$$\underset{\theta}{Y}_{hijk} = \underset{\theta}{R}_{hijk} - \frac{\underset{\theta}{S}}{n(n-1)}\left(g_{ki}g_{hj} - g_{ji}g_{hk}\right), \tag{2.19}$$

where we denoted $\underset{\theta}{Y}_{hijk} = g_{hp}\underset{\theta}{Y}^p_{ijk}$ and $\underset{\theta}{R}_{hijk} = g_{hp}\underset{\theta}{R}^p_{ijk}$.

Theorem 2.4. *For an arbitrary $(\theta, \mu, \nu, \alpha, \beta, \gamma)$ given by* (1.3)*, the tensor $\underset{\theta}{Y}^h_{ijk}$ determined by* (2.18)*, is related with the Yano tensor of concircular curvature $\overset{g}{Y}^h_{ijk}$*

by

$$
\begin{aligned}
\underset{\theta}{Y}{}^h_{ijk}[\mu,\nu,\alpha,\beta,\gamma] = {}& \overset{g}{Y}{}^h_{ijk} + \mu\overset{g}{\nabla}_k T^h_{ij} + \nu\overset{g}{\nabla}_j T^h_{ik} \\
& + \alpha T^h_{pk}T^p_{ij} + \beta T^h_{pj}T^p_{ik} + \gamma T^h_{pi}T^p_{jk} \\
& - \frac{1}{n(n-1)}(\mu g^{pq}\overset{g}{\nabla}_r T^r_{pq} + \beta g^{pq}T^s_{rq}T^r_{ps} \\
& + \gamma g^{pq}T^s_{rp}T^r_{qs})(g_{ki}\delta^h_j - g_{ji}\delta^h_k).
\end{aligned}
\tag{2.20}
$$

Proof. Obvious. □

Corollary 2.4. *For an arbitrary* $(\theta,\mu,\nu,\alpha,\beta,\gamma)$ *given by* (1.3)*, the tensor* $\underset{\theta}{Y}{}_{hijk}$ *determined by* (2.19)*, is related with the Yano tensor of concircular curvature* $\overset{g}{Y}{}_{hijk}$ *by*

$$
\begin{aligned}
\underset{\theta}{Y}{}_{hijk}[\mu,\nu,\alpha,\beta,\gamma] = {}& \overset{g}{Y}{}_{hijk} + \mu g_{hp}\overset{g}{\nabla}_k T^p_{ij} + \nu g_{hp}\overset{g}{\nabla}_j T^p_{ik} \\
& + \alpha g_{hp}T^p_{qk}T^q_{ij} + \beta g_{hp}T^p_{qj}T^q_{ik} + \gamma g_{hp}T^p_{qi}T^q_{jk} \\
& - \frac{\mu g^{pq}\overset{g}{\nabla}_r T^r_{pq} + \beta g^{pq}T^s_{rq}T^r_{ps} + \gamma g^{pq}T^s_{rp}T^r_{qs}}{n(n-1)}\left(g_{ki}g_{hj} - g_{ji}g_{hk}\right).
\end{aligned}
$$

Theorem 2.5. (See [2], p. 26) *A Riemannian space admits a traceless Ricci tensor preserving conformal mapping onto a Riemannian space if and only if the mapping under consideration preserves the Yano tensor of concircular curvature.*

The result given in Theorem 2.5 can easily be extended to generalized Riemannian spaces in Eisenhart's sense as stated in Proposition 2.2.

Proposition 2.2. *If a mapping* $f : M \to \overline{M}$ *between generalized Riemannian spaces in Eisenhart's sense* $(M, \mathcal{G} = g + \omega)$ *and* $(\overline{M}, \overline{\mathcal{G}} = \overline{g} + \overline{\omega})$ *is conformal then for arbitrary* $\theta \in \{1,\dots,5\}$ *the mapping* f *preserves the tensor* $\underset{\theta}{Y}{}^h_{ijk}$ *if and only if it preserves the tensor* $\underset{\theta}{Ric}_{ij} - \frac{1}{n}\underset{\theta}{S}g_{ij}$.

3. Geodesic mappings between generalized Riemannian spaces in Eisenhart's sense

Geodesic mappings between generalized Riemannian spaces were previously studied, see Chapter 7 of the monograph [6] and the papers that follow that direction.

3.1. Weyl tensor of projective curvature

In Theorem 2.1 from [10] we considered a geodesic mapping between manifolds with non-symmetric linear connection. A generalized Riemannian space in Eisenhart's sense is a particular manifold with a non-symmetric linear connection. Here we consider a geodesic mapping between generalized Riemannian spaces in Eisenhart'se sense, hence Theorem 2.1 from [10] can be applied, which gives Proposition 3.1.

Proposition 3.1. *The tensors*

$$\underset{\theta}{W}{}^h_{ijk} = \underset{\theta}{R}{}^h_{ijk} - \frac{1}{(n-1)}\Big(\underset{\theta}{Ric}{}_{ij}\delta^h_k - \underset{\theta}{Ric}{}_{ik}\delta^h_j\Big), \tag{3.1}$$

are invariant with respect to the geodesic mapping $f : M \to \overline{M}$ *between generalized Riemannian spaces* $(M, \mathcal{G} = g + \omega)$ *and* $(\overline{M}, \overline{\mathcal{G}} = \overline{g} + \overline{\omega})$ *if and only if the condition* (2.7) *holds, where for given* $\theta \in \{1, 2, \dots, 5\}$ *the parameters* $(\mu, \nu, \alpha, \beta, \gamma)$ *are chosen from* (1.3).

Obviously, by lowering the upper index h in (3.1) one can get

$$g_{hp}\underset{\theta}{W}{}^p_{ijk} = g_{hp}\underset{\theta}{R}{}^p_{ijk} - \frac{1}{(n-1)}\left(\underset{\theta}{Ric}{}_{ij}g_{hp}\delta^p_k - \underset{\theta}{Ric}{}_{ik}g_{hp}\delta^p_j\right),$$

i.e.,

$$\underset{\theta}{W}{}_{hijk} = \underset{\theta}{R}{}_{hijk} - \frac{1}{(n-1)}\left(\underset{\theta}{Ric}{}_{ij}g_{hk} - \underset{\theta}{Ric}{}_{ik}g_{hj}\right), \tag{3.2}$$

where we denoted $\underset{\theta}{W}{}_{hijk} = g_{hp}\underset{\theta}{W}{}^p_{ijk}$ and $\underset{\theta}{R}{}_{hijk} = g_{hp}\underset{\theta}{R}{}^p_{ijk}$.

Theorem 3.1. *Let* $(M, \mathcal{G} = g + \omega)$ *be a generalized Riemannian space in Eisenhart's sense of dimension* $n > 2$. *For an arbitrary* $(\theta, \mu, \nu, \alpha, \beta, \gamma)$ *given by* (1.3), *the tensor* $\underset{\theta}{W}{}^h_{ijk}$ *given by* (3.1) *and the Weyl projective curvature tensor* $\overset{g}{W}{}^h_{ijk} = \overset{g}{R}{}^h_{ijk} - \frac{1}{(n-1)}\left(\overset{g}{Ric}_{ij}\delta^h_k - \overset{g}{Ric}_{ik}\delta^h_j\right)$, *are related by*

$$\begin{aligned}\underset{\theta}{W}{}^h_{ijk}[\mu,\nu,\alpha,\beta,\gamma] = \overset{g}{W}{}^h_{ijk} &+ \mu\overset{g}{\nabla}_k T^h_{ij} + \nu\overset{g}{\nabla}_j T^h_{ik}\\ &+ \alpha T^h_{pk}T^p_{ij} + \beta T^h_{pj}T^p_{ik} + \gamma T^h_{pi}T^p_{jk}\\ &- \frac{1}{(n-1)}\left(\mu\overset{g}{\nabla}_p T^p_{ij} + \beta T^q_{pj}T^p_{iq} + \gamma T^q_{pi}T^p_{jq}\right)\delta^h_k\\ &+ \frac{1}{(n-1)}\left(\mu\overset{g}{\nabla}_p T^p_{ik} + \beta T^q_{pk}T^p_{iq} + \gamma T^q_{pi}T^p_{kq}\right)\delta^h_j\,.\end{aligned}$$

Proof. By using (1.2) and (1.4) into (3.1) we obtain that

$$\begin{aligned}\underset{\theta}{W}{}^{h}_{ijk}[\mu,\nu,\alpha,\beta,\gamma] = {}& \overset{g}{R}{}^{h}_{ijk} + \mu\overset{g}{\nabla}_k T^h_{ij} + \nu\overset{g}{\nabla}_j T^h_{ik} \\ & + \alpha T^h_{pk}T^p_{ij} + \beta T^h_{pj}T^p_{ik} + \gamma T^h_{pi}T^p_{jk} \\ & - \frac{1}{(n-1)}\left(\overset{g}{Ric}_{ij} + \mu\overset{g}{\nabla}_p T^p_{ij} + \beta T^q_{pj}T^p_{iq} + \gamma T^q_{pi}T^p_{jq}\right)\delta^h_k \\ & + \frac{1}{(n-1)}\left(\overset{g}{Ric}_{ik} + \mu\overset{g}{\nabla}_p T^p_{ik} + \beta T^q_{pk}T^p_{iq} + \gamma T^q_{pi}T^p_{kq}\right)\delta^h_j\,,\end{aligned}$$

which completes the proof. □

Corollary 3.1. *Let $(M, \mathcal{G} = g + \omega)$ be a generalized Riemannian space in Eisenhart's sense of dimension $n > 2$. For an arbitrary $(\theta, \mu, \nu, \alpha, \beta, \gamma)$ given by (1.3), the tensor $\underset{\theta}{W}_{hijk}$ given by (3.2) and the Weyl projective curvature tensor*

$$\overset{g}{W}_{hijk} = \overset{g}{R}_{hijk} - \frac{1}{(n-1)}\left(\overset{g}{Ric}_{ij}g_{hk} - \overset{g}{Ric}_{ik}g_{hj}\right),$$

are related by

$$\begin{aligned}\underset{\theta}{W}_{hijk}[\mu,\nu,\alpha,\beta,\gamma] = {}& \overset{g}{W}_{hijk} + \mu\overset{g}{\nabla}_k T_{hij} + \nu\overset{g}{\nabla}_j T_{hik} \\ & + \alpha T^h_{pk}T^p_{ij} + \beta T^h_{pj}T^p_{ik} + \gamma T^h_{pi}T^p_{jk} \\ & - \frac{1}{(n-1)}\left(\mu\overset{g}{\nabla}_p T^p_{ij} + \beta T^q_{pj}T^p_{iq} + \gamma T^q_{pi}T^p_{jq}\right)g_{hk} \\ & + \frac{1}{(n-1)}\left(\mu\overset{g}{\nabla}_p T^p_{ik} + \beta T^q_{pk}T^p_{iq} + \gamma T^q_{pi}T^p_{kq}\right)g_{hj}.\end{aligned}$$

3.2. Geodesic mappings between generalized Riemannian spaces in Eisenhart's sense preserving curvature and Ricci tensors

We will follow the idea from [1] where geodesic mappings between affine connected spaces preserving the Riemannian and Ricci tensors were studied to consider the geodesic mappings between the Riemannian spaces preserving the curvature and Ricci tensors.

The relation $\underset{\theta}{\overline{W}}{}^h_{ijk} = \underset{\theta}{W}{}^h_{ijk}$ reads

(3.3)
$$\underset{\theta}{\overline{R}}{}^h_{ijk} - \frac{1}{(n-1)}\left(\underset{\theta}{\overline{Ric}}_{ij}\delta^h_k - \underset{\theta}{\overline{Ric}}_{ik}\delta^h_j\right) = \underset{\theta}{R}{}^h_{ijk} - \frac{1}{(n-1)}\left(\underset{\theta}{Ric}_{ij}\delta^h_k - \underset{\theta}{Ric}_{ik}\delta^h_j\right),$$

where $\underset{\theta}{Ric}_{ij} = \underset{\theta}{R}{}^p_{ijp}$ and $\underset{\theta}{\overline{Ric}}_{ij} = \underset{\theta}{\overline{R}}{}^p_{ijp}$, $\theta = 1, \ldots, 5$ are Ricci tensors of generalized Riemannian spaces in Eisenhart's sense $(M, \mathcal{G} = g + \omega)$ and $(\overline{M}, \overline{\mathcal{G}} = \overline{g} + \overline{\omega})$, respectively.

From relation (3.3) it follows that a geodesic mapping $f : M \to \overline{M}$ is preserving the curvature tensor $\underset{\theta}{R}{}^h_{ijk}$ if and only if

$$\Big(\underset{\theta}{\overline{R}}ic_{\,ij} - \underset{\theta}{R}ic_{\,ij}\Big)\delta^h_k - \Big(\underset{\theta}{\overline{R}}ic_{\,ik} - \underset{\theta}{R}ic_{\,ik}\Big)\,\delta^h_j = 0.$$

Particularly, from (3.3) it follows that if a geodesic mapping $f : M \to \overline{M}$ is preserving the Ricci tensor $\underset{\theta}{R}ic_{\,ij}$ then it preserves the curvature tensor $\underset{\theta}{R}{}^h_{ijk}$ as well, which is in accordance with the observation obtained in [1] for an arbitrary diffeomorphism between affine connected spaces without torsion.

4. Composition of conformal and projective mappings between generalized Riemannian spaces in Eisenhart's sense

If we consider a composition of conformal and projective mappings preserving the tensor $\underset{\theta}{R}ic_{ij} - \frac{1}{n}\underset{\theta}{S}g_{ij}$, for an arbitrary $\theta \in \{1, \dots, 5\}$ then we obtain that the tensor $\underset{\theta}{Y}{}^h_{ijk}$ is invariant with respect to this composition.

Theorem 4.1. *Let $(M, \mathcal{G} = g + \omega)$, $(\widehat{M}, \widehat{\mathcal{G}} = \widehat{g} + \widehat{\omega})$ and $(\overline{M}, \overline{\mathcal{G}} = \overline{g} + \overline{\omega})$ be generalized Riemannian spaces of dimension $n > 2$. If there exists a conformal mapping $f_1 : M \to \widehat{M}$ that preserves the tensor $\underset{\theta}{R}ic_{ij} - \frac{1}{n}\underset{\theta}{S}g_{ij}$, for chosen $\theta \in \{1, \dots, 5\}$, and a projective mapping $f_2 : \widehat{M} \to \overline{M}$ that preserves the tensor $\underset{\theta}{\widehat{R}}ic_{ij} - \frac{1}{n}\underset{\theta}{\widehat{S}}\widehat{g}_{ij}$ then the tensor $\underset{\theta}{Y}{}^h_{ijk}$ is invariant with respect to the mapping $f_3 = f_1 \circ f_2 : M \to \overline{M}$.*

The result given in Theorem 4.1 particularly holds when instead of the tensor $\underset{\theta}{R}ic_{ij} - \frac{1}{n}\underset{\theta}{S}g_{ij}$ we consider the traceless Ricci tensor $\overset{g}{R}ic_{ij} - \frac{1}{n}\overset{g}{S}g_{ij}$ and instead of $\underset{\theta}{Y}{}^h_{ijk}$ we consider the Yano tensor of concircular curvature $\overset{g}{Y}{}^h_{ijk}$. From Theorem 4.1 we get Corollary 4.1.

Corollary 4.1. *Let (M, g), $(\widehat{M}, \widehat{g})$ and $(\overline{M}, \overline{g})$ be Riemannian spaces of dimension $n > 2$. Let us assume that there exists a conformal mapping $f_1 : M \to \widehat{M}$ that preserves the traceless Ricci tensor $\overset{g}{R}ic_{ij} - \frac{1}{n}\overset{g}{S}g_{ij}$ and a projective mapping $f_2 : \widehat{M} \to \overline{M}$ that preserves the traceless Ricci tensor $\overset{\widehat{g}}{R}ic_{ij} - \frac{1}{n}\overset{\widehat{g}}{S}\widehat{g}_{ij}$, then the Yano tensor of concircular curvature is invariant with respect to the mapping $f_3 = f_1 \circ f_2 : M \to \overline{M}$.*

4.1. Conformally-projective harmonic mappings of generalized Riemannian spaces in Eisenhart's sense

Let $(M, \mathcal{G} = g + \omega)$ and $(\overline{M}, \overline{\mathcal{G}} = \overline{g} + \overline{\omega})$ be generalized Riemannian spaces of dimension $n > 2$. A diffeomorphism $f : M \to \overline{M}$ is said to be *harmonic* if and only if (see for instance [4], p. 46, Eq. (4.12))

$$\left(\overset{\overline{g}}{\Gamma}{}^h_{ij} - \overset{g}{\Gamma}{}^h_{ij}\right) g^{ij} = 0, \tag{4.1}$$

where $\overset{\overline{g}}{\Gamma}{}^h_{ij}$ and $\overset{g}{\Gamma}{}^h_{ij}$ are Christoffel symbols of the Riemannian metrics $\overline{g}$ and g, respectively.

Let us assume that there exists a conformal mapping $f_1 : M \to \widehat{M}$ and a projective mapping $f_2 : \widehat{M} \to \overline{M}$ such that $f_3 = f_1 \circ f_2 : M \to \overline{M}$ is a harmonic mapping in the sense that (4.1) holds. In this case there exists the relation (see [4], p. 58, Eq. (5.1))

$$\overset{\overline{g}}{\Gamma}{}^h_{ij}(x) = \overset{g}{\Gamma}{}^h_{ij}(x) + \varphi_i \delta^h_j + \varphi_j \delta^h_i - \frac{2}{n}\varphi^h g_{ij},$$

where $\varphi_i = \partial_i \varphi(x)$ is a gradient-like vector and $\varphi^h = g^{hp}\varphi_p$.

By using the relation (1.2) and the the corresponding relation between the curvature tensors $\overline{R}^h_{ijk}$ and R^h_{ijk} with respect to the conformally-projective harmonic mapping f_3, given in Eq. (5.3) on page 58 of [4], we easily get the relation for the Riemannian curvature tensors $\overset{\overline{g}}{R}{}^h_{ijk}$ and $\overset{g}{R}{}^h_{ijk}$ with respect to the conformally-projective harmonic mapping f_3 and furthermore the relation between the curvature tensors $\underset{\theta}{\overline{R}}{}^h_{ijk}$ and $\underset{\theta}{R}{}^h_{ijk}$, which is given by

$$\begin{aligned}
\underset{\theta}{\overline{R}}{}^h_{ijk} = \underset{\theta}{R}{}^h_{ijk} &- \delta^h_k \left(\overset{g}{\nabla}_j \varphi_i - \varphi_i\varphi_j\right) + \delta^h_j \left(\overset{g}{\nabla}_k \varphi_i - \varphi_i\varphi_k\right) \\
&- \frac{2}{n}\left(\nabla_k \varphi^h - \frac{2}{n}\varphi^h\varphi_k + \varphi_p\varphi^p\delta^h_k\right) g_{ij} \\
&+ \frac{2}{n}\left(\nabla_j \varphi^h - \frac{2}{n}\varphi^h\varphi_j + \varphi_p\varphi^p\delta^h_j\right) g_{ik} \\
&- \underset{\theta}{\mu}\left(\overset{\overline{g}}{\nabla}_k \overline{T}^h_{ij} - \overset{g}{\nabla}_k T^h_{ij}\right) - \underset{\theta}{\nu}\left(\overset{\overline{g}}{\nabla}_j \overline{T}^h_{ik} - \overset{g}{\nabla}_j T^h_{ik}\right) - \underset{\theta}{\alpha}\left(\overline{T}^h_{pk}\overline{T}^p_{ij} - T^h_{pk}T^p_{ij}\right) \\
&- \underset{\theta}{\beta}\left(\overline{T}^h_{pj}\overline{T}^p_{ik} - T^h_{pj}T^p_{ik}\right) - \underset{\theta}{\gamma}\left(\overline{T}^h_{pi}\overline{T}^p_{jk} - T^h_{pi}T^p_{jk}\right), \quad \theta = 1, \ldots, 5.
\end{aligned}$$

The last relation can be rewritten as

$$\begin{aligned}
\underset{\theta}{\overline{R}}(\partial_k, \partial_j)\partial_i = \underset{\theta}{R}(\partial_k, \partial_j)\partial_i &+ P(\partial_k, \partial_j, \partial_i) - P(\partial_j, \partial_k, \partial_i) \\
&+ A(\partial_k, \partial_j, \partial_i)[\underset{\theta}{\mu}, \underset{\theta}{\nu}] - \overline{A}(\partial_k, \partial_j, \partial_i)[\underset{\theta}{\mu}, \underset{\theta}{\nu}] \\
&+ B(\partial_k, \partial_j, \partial_i)[\underset{\theta}{\alpha}, \underset{\theta}{\beta}, \underset{\theta}{\gamma}] - \overline{B}(\partial_k, \partial_j, \partial_i)[\underset{\theta}{\alpha}, \underset{\theta}{\beta}, \underset{\theta}{\gamma}],
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned} P(\partial_k,\partial_j,\partial_i) =& (\overset{g}{\nabla}_{\partial_k}P)(\partial_i,\partial_j)+P(P(\partial_i,\partial_j),\partial_k),\\ =&\delta^h_j\left(\overset{g}{\nabla}_k\varphi_i-\varphi_i\varphi_k\right)-\frac{2}{n}\left(\overset{g}{\nabla}_k\varphi^h-\frac{2}{n}\varphi^h\varphi_k+\varphi_p\varphi^p\delta^h_k\right)g_{ij}\\ A(\partial_k,\partial_j,\partial_i)[\underset{\theta}{\mu},\underset{\theta}{\nu}] =&\underset{\theta}{\mu}(\overset{g}{\nabla}_{\partial_k}T)(\partial_i,\partial_j)+\underset{\theta}{\nu}(\overset{g}{\nabla}_{\partial_j}T)(\partial_i,\partial_k)\\ =&\underset{\theta}{\mu}\overset{g}{\nabla}_kT^h_{ij}+\underset{\theta}{\nu}\overset{g}{\nabla}_jT^h_{ik},\\ \overline{A}(\partial_k,\partial_j,\partial_i)[\underset{\theta}{\mu},\underset{\theta}{\nu}] =&\underset{\theta}{\mu}(\overset{\overline{g}}{\nabla}_{\partial_k}T)(\partial_i,\partial_j)+\underset{\theta}{\nu}(\overset{\overline{g}}{\nabla}_{\partial_j}T)(\partial_i,\partial_k)\\ =&\underset{\theta}{\mu}\overset{\overline{g}}{\nabla}_k\overline{T}^h_{ij}+\underset{\theta}{\nu}\overset{\overline{g}}{\nabla}_j\overline{T}^h_{ik} \end{aligned}$$

and

$$\begin{aligned} B(\partial_k,\partial_j,\partial_i)[\underset{\theta}{\alpha},\underset{\theta}{\beta},\underset{\theta}{\gamma}] =&\underset{\theta}{\alpha}T(T(\partial_i,\partial_j),\partial_k)+\underset{\theta}{\beta}T(T(\partial_i,\partial_k),\partial_j)+\underset{\theta}{\gamma}T(T(\partial_j,\partial_k),\partial_i)\\ =&\underset{\theta}{\alpha}T^h_{pk}T^p_{ij}+\underset{\theta}{\beta}T^h_{pj}T^p_{ik}+\underset{\theta}{\gamma}T^h_{pi}T^p_{jk},\\ \overline{B}(\partial_i,\partial_k,\partial_j)[\underset{\theta}{\alpha},\underset{\theta}{\beta},\underset{\theta}{\gamma}] =&\underset{\theta}{\alpha}\overline{T}(\overline{T}(\partial_i,\partial_j),\partial_k)+\underset{\theta}{\beta}\overline{T}(\overline{T}(\partial_i,\partial_k),\partial_j)+\underset{\theta}{\gamma}\overline{T}(\overline{T}(\partial_j,\partial_k),\partial_i)\\ =&\underset{\theta}{\alpha}\overline{T}^h_{pk}\overline{T}^p_{ij}+\underset{\theta}{\beta}\overline{T}^h_{pj}\overline{T}^p_{ik}+\underset{\theta}{\gamma}\overline{T}^h_{pi}\overline{T}^p_{jk}, \end{aligned}$$

for

$$\begin{aligned} (\theta,\underset{\theta}{\mu},\underset{\theta}{\nu},\underset{\theta}{\alpha},\underset{\theta}{\beta},\underset{\theta}{\gamma})\in\Big\{&\left(1,\frac{1}{2},-\frac{1}{2},\frac{1}{4},-\frac{1}{4},0\right),\left(2,-\frac{1}{2},\frac{1}{2},\frac{1}{4},-\frac{1}{4},0\right),\\ &\left(3,\frac{1}{2},\frac{1}{2},-\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right),\left(4,\frac{1}{2},\frac{1}{2},-\frac{1}{4},\frac{1}{4},\frac{1}{2}\right),\\ &\left(5,0,0,\frac{1}{4},\frac{1}{4},0\right)\Big\}. \end{aligned} \tag{4.3}$$

Let us consider the following linear combination based on the relation (4.2)

$$\begin{aligned} &\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\overline{R}}{}^h_{ijk}=\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{R}{}^h_{ijk}+\sum_{\theta=1}^{5}\underset{\theta}{k}\Big(P(\partial_k,\partial_j,\partial_i)-P(\partial_j,\partial_k,\partial_i)\Big)\\ &+A(\partial_k,\partial_j,\partial_i)\left[\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\mu},\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\nu}\right]-\overline{A}(\partial_k,\partial_j,\partial_i)\left[\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\mu},\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\nu}\right]\\ &+B(\partial_k,\partial_j,\partial_i)\left[\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\alpha},\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\beta},\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\gamma}\right]-\overline{B}(\partial_k,\partial_j,\partial_i)\left[\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\alpha},\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\beta},\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\gamma}\right], \end{aligned}$$

Based on (4.3) we can easily find the following sums from the last relation

$$
\begin{aligned}
&\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\mu}=\frac{\underset{1}{k}-\underset{2}{k}+\underset{3}{k}+\underset{4}{k}}{2}, \qquad \sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\nu}=\frac{-\underset{1}{k}+\underset{2}{k}+\underset{3}{k}+\underset{4}{k}}{2},\\
&\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\alpha}=\frac{\underset{1}{k}+\underset{2}{k}-\underset{3}{k}-\underset{4}{k}+\underset{5}{k}}{4}, \quad \sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\beta}=\frac{-\underset{1}{k}-\underset{2}{k}+\underset{3}{k}+\underset{4}{k}+\underset{5}{k}}{4},\\
&\sum_{\theta=1}^{5}\underset{\theta}{k}\underset{\theta}{\gamma}=\frac{-\underset{3}{k}+\underset{4}{k}}{2}.
\end{aligned}
$$

Theorem 4.2. *Let $(M,\mathcal{G}=g+\omega)$, $(\widehat{M},\widehat{\mathcal{G}}=\widehat{g}+\widehat{\omega})$ and $(\overline{M},\overline{\mathcal{G}}=\overline{g}+\overline{\omega})$ be generalized Riemannian spaces in Eisenhart's sense of dimension $n>2$. If we assume that there exist a conformal mapping $f_1:M\to\widehat{M}$ and a projective mapping $f_2:\widehat{M}\to\overline{M}$ such that $f_3=f_1\circ f_2:M\to\overline{M}$ is a harmonic mapping in the sense that the condition (4.1) holds, then for arbitrary functions $k,\underset{1}{k},\underset{2}{k},\dots,\underset{5}{k}$ the linear combination*

$$
k\overset{g}{R}{}^h_{ijk}+\underset{1}{k}\underset{1}{R}{}^h_{ijk}+\underset{2}{k}\underset{2}{R}{}^h_{ijk}+\underset{3}{k}\underset{3}{R}{}^h_{ijk}+\underset{4}{k}\underset{4}{R}{}^h_{ijk}+\underset{5}{k}\underset{5}{R}{}^h_{ijk},
$$

is preserved with respect to the mapping f_3 if and only if

$$
\begin{aligned}
(k+\underset{1}{k}+\underset{2}{k}+\underset{3}{k}+\underset{4}{k}+\underset{5}{k})\big(&-\delta^h_k\left(\overset{g}{\nabla}_j\varphi_i-\varphi_i\varphi_j\right)+\delta^h_j\left(\overset{g}{\nabla}_k\varphi_i-\varphi_i\varphi_k\right)\\
&-\frac{2}{n}\left(\overset{g}{\nabla}_k\varphi^h-\frac{2}{n}\varphi^h\varphi_k+\varphi_p\varphi^p\delta^h_k\right)g_{ij}\\
&+\frac{2}{n}\left(\overset{g}{\nabla}_j\varphi^h-\frac{2}{n}\varphi^h\varphi_j+\varphi_p\varphi^p\delta^h_j\right)g_{ik}\big)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(\underset{1}{k}-\underset{2}{k}+\underset{3}{k}+\underset{4}{k})}{2}\left(\overset{\overline{g}}{\nabla}_k\overline{T}{}^h_{ij}-\overset{g}{\nabla}_kT^h_{ij}\right)\\
&-\frac{(-\underset{1}{k}+\underset{2}{k}+\underset{3}{k}+\underset{4}{k})}{2}\left(\overset{\overline{g}}{\nabla}_j\overline{T}{}^h_{ik}-\overset{g}{\nabla}_jT^h_{ik}\right)\\
&-\frac{(\underset{1}{k}+\underset{2}{k}-\underset{3}{k}-\underset{4}{k}+\underset{5}{k})}{4}\left(\overline{T}{}^h_{pk}\overline{T}{}^p_{ij}-T^h_{pk}T^p_{ij}\right)\\
&-\frac{(-\underset{1}{k}-\underset{2}{k}+\underset{3}{k}+\underset{4}{k}+\underset{5}{k})}{4}\left(\overline{T}{}^h_{pj}\overline{T}{}^p_{ik}-T^h_{pj}T^p_{ik}\right)\\
&-\frac{(-\underset{3}{k}+\underset{4}{k})}{2}\left(\overline{T}{}^h_{pi}\overline{T}{}^p_{jk}-T^h_{pi}T^p_{jk}\right)=0.
\end{aligned}
$$

Clearly, when in Theorem 4.2 instead of the generalized Riemannian spaces $(M,\mathcal{G}=g+\omega)$, $(\widehat{M},\widehat{\mathcal{G}}=\widehat{g}+\widehat{\omega})$ and $(\overline{M},\overline{\mathcal{G}}=\overline{g}+\overline{\omega})$ we consider the Riemannian spaces (M,g), $(\widehat{M},\widehat{g})$ and $(\overline{M},\overline{g})$ then the curvature tensors $\underset{\theta}{R}{}^h_{ijk}$, $\theta = 1,\dots,5$ reduce to the Riemannian curvature tensor $\overset{g}{R}{}^h_{ijk}$ and the corresponding statement is given in Corollary 4.2.

Corollary 4.2. *Let (M,g), $(\widehat{M},\widehat{g})$ and $(\overline{M},\overline{g})$ be Riemannian spaces of dimension $n>2$. If there exist a conformal mapping $f_1 : M \to \widehat{M}$ and a projective mapping $f_2 : \widehat{M} \to \overline{M}$ such that $f_3 = f_1 \circ f_2 : M \to \overline{M}$ is harmonic then the Riemannian curvature tensor $\overset{g}{R}{}^h_{ijk}$ is preserved with respect to f_3 if and only if*

$$\begin{aligned}
&-\delta^h_k \left(\overset{g}{\nabla}_j \varphi_i - \varphi_i\varphi_j\right) + \delta^h_j \left(\overset{g}{\nabla}_k \varphi_i - \varphi_i\varphi_k\right)\\
&-\frac{2}{n}\left(\overset{g}{\nabla}_k \varphi^h - \frac{2}{n}\varphi^h\varphi_k + \varphi_p\varphi^p\delta^h_k\right) g_{ij}\\
&+\frac{2}{n}\left(\overset{g}{\nabla}_j \varphi^h - \frac{2}{n}\varphi^h\varphi_j + \varphi_p\varphi^p\delta^h_j\right) g_{ik} = 0.
\end{aligned}$$

Proof. We choose $k=1, \underset{1}{k}=0, \underset{2}{k}=0,\dots,\underset{5}{k}=0$ and the skew-symmetric bilinear forms ω, $\widehat{\omega}$ and $\overline{\omega}$ to vanish identically in Theorem 4.2. This completes the proof. □

4.2. Composition of geodesic and conformal mappings between generalized Riemannian spaces

If we consider a composition of geodesic and conformal mappings preserving the tensor $\underset{\theta}{R}ic_{ij} - \frac{1}{n}\underset{\theta}{S}g_{ij}$, for arbitrary $\theta \in \{1,\dots,5\}$, then we obtain that the tensor $\underset{\theta}{Y}{}^h_{ijk}$ is invariant with respect to this composition.

Theorem 4.3. *Let $(M,\mathcal{G} = g+\omega)$, $(\widehat{M},\widehat{\mathcal{G}} = \widehat{g}+\widehat{\omega})$ and $(\overline{M},\overline{\mathcal{G}} = \overline{g}+\overline{\omega})$ be generalized Riemannian spaces of dimension $n>2$. Let f_1 be a geodesic mapping and f_2 be a conformal mapping such that*

$$M \xrightarrow{f_1} \widehat{M} \xrightarrow{f_2} \overline{M}.$$

Let us denote $f_3 = f_1 \circ f_2 : M \to \overline{M}$ and choose $\theta \in \{1,\dots,5\}$. If the mapping f_1 preserves the tensor $\underset{\theta}{R}ic_{ij} - \frac{1}{n}\underset{\theta}{S}g_{ij}$ and the mapping f_2 preserves the tensor $\underset{\theta}{\widehat{R}}ic_{ij} - \frac{1}{n}\underset{\theta}{\widehat{S}}g_{ij}$ then the mapping f_3 preserves the tensor $\underset{\theta}{Y}{}^h_{ijk}$.

From Theorem 4.3 we get Corollary 4.3.

Corollary 4.3. *Let (M,g), $(\widehat{M},\widehat{g})$ and $(\overline{M},\overline{g})$ be Riemannian spaces of dimension $n>2$. Let us assume that there exist a geodesic mapping $f_1 : M \to \widehat{M}$ that preserves the traceless Ricci tensor $\overset{g}{R}ic_{ij} - \frac{1}{n}\overset{g}{S}g_{ij}$ and a conformal mapping $f_2 : \widehat{M} \to \overline{M}$ that* *preserves the traceless Ricci tensor* $\overset{\widehat{g}}{Ric}_{ij} - \frac{1}{n}\overset{\widehat{g}}{S}\widehat{g}_{ij}$ *then the Yano tensor of concircular curvature* $\overset{g}{Y}{}^h_{ijk}$ *is invariant with respect to the mapping* $f_3 = f_1 \circ f_2 : M \to \overline{M}$.

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