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DEFORMATIONS PRESERVING DUAL ARC LENGTH IN DUAL 3-SPACE

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Abstract. In this paper we study infinitesimal bending of dual spherical curves using the Blaschke frame. We give the necessary and sufficient conditions for the infinitesimal bending field. Also, we consider the hyperbolic paraboloid as a ruled surface corresponding to a dual spherical curve.
Keywords: dual spherical curves, infinitesimal bending field, Blaschke frame.

1. Introduction

Based on the E. Study's map [7], the set of all oriented lines in Euclidean 3-space $\mathbb{E}^3$ is in one to one correspondence with the set of points of dual unit sphere in dual 3-space $\mathbb{D}^3$ and a differentiable curve on dual unit sphere represents a ruled surface. Consequently, observing the deformations of dual spherical curves we actually study the deformations of the corresponding ruled surfaces. In paper [5], a deformation of dual curves preserving the dual arc length was defined. Precisely, the following definition was introduced.

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Definition 1.1. [5] Let us consider a dual curve of class C^3

$$\tilde{C}:\ \tilde{\mathbf{r}}(u)=\mathbf{r}(u)+\epsilon\mathbf{r}^*(u),\quad u\in J\subseteq\mathbb{R} \tag{1.1}$$

included in a family of dual curves

$$\tilde{C}_t:\tilde{\mathbf{r}}_t(u)=\tilde{\mathbf{r}}(u)+t\tilde{\mathbf{z}}(u),\quad u\in J,\, t\geq 0,\, t\to 0, \tag{1.2}$$

where u is a real parameter, ϵ is a dual unit and we get $\tilde{C}$ for $t=0$ ($\tilde{C}=\tilde{C}_0$). The family of dual curves $\tilde{C}_t$ is an **infinitesimal bending of the dual curve** $\tilde{C}$ if

$$d\tilde{s}_t^2-d\tilde{s}^2=\tilde{o}(t)=o(t)+\epsilon o^*(t), \tag{1.3}$$

where $\tilde{\mathbf{z}}(u)=\mathbf{z}(u)+\epsilon\mathbf{z}^*(u)$, $\tilde{\mathbf{z}}\in C^3$, is the **dual infinitesimal bending field** of the dual curve $\tilde{C}$, $\tilde{s}$ and $\tilde{s}_t$ are the dual arc lengths of $\tilde{C}$ and $\tilde{C}_t$, respectively.

Theorem 1.1. [5] *The necessary and sufficient condition for $\tilde{\mathbf{z}}(u)$ to be a dual infinitesimal bending field of the dual curve $\tilde{C}$ is to be*

$$d\tilde{\mathbf{r}}\cdot d\tilde{\mathbf{z}}=0, \tag{1.4}$$

or, equivalently,

$$d\mathbf{r}\cdot d\mathbf{z}=0\quad\wedge\quad d\mathbf{r}\cdot d\mathbf{z}^*+d\mathbf{z}\cdot d\mathbf{r}^*=0.\quad\Box \tag{1.5}$$

In this paper we will study the infinitesimal bending of dual spherical curves. For the basic concepts of dual curves and their characterization, we refer to papers [1–10].

2. Dual bending field for dual spherical curves using the Blaschke frame

Let a differentiable curve $\tilde{C}$ be given on the dual unit sphere $\mathbb{S}^2$ of the dual space $\mathbb{D}^3$

$$\tilde{C}(u):\ \tilde{\mathbf{r}}(u)=\mathbf{r}(u)+\epsilon\mathbf{r}^*(u). \tag{2.1}$$

Let us introduce the orthonormal Blaschke frame

$$\tilde{\mathbf{a}}_1=\tilde{\mathbf{r}}(u),\quad \tilde{\mathbf{a}}_2=\frac{\dot{\tilde{\mathbf{a}}}_1(u)}{\sqrt{\dot{\tilde{\mathbf{a}}}_1^2(u)}},\quad \tilde{\mathbf{a}}_3(u)=\tilde{\mathbf{a}}_1\times\tilde{\mathbf{a}}_2. \tag{2.2}$$

It is necessary to suppose that $\mathbf{r}(u)$ is not a constant vector, i.e. that the ruled surface does not contain a cylinder. The corresponding Blaschke formula is

$$\frac{d}{du}\begin{pmatrix}\tilde{\mathbf{a}}_1\\ \tilde{\mathbf{a}}_2\\ \tilde{\mathbf{a}}_3\end{pmatrix}=\begin{pmatrix}0 & \tilde{p} & 0\\ -\tilde{p} & 0 & \tilde{q}\\ 0 & -\tilde{q} & 0\end{pmatrix}\begin{pmatrix}\tilde{\mathbf{a}}_1\\ \tilde{\mathbf{a}}_2\\ \tilde{\mathbf{a}}_3\end{pmatrix}, \tag{2.3}$$

where $\tilde{p} = p+\epsilon p^* = \|\dot{\tilde{\mathbf{a}}}_1\|$, $\tilde{q} = q+\epsilon q^* = \frac{[\tilde{\mathbf{a}}_1, \dot{\tilde{\mathbf{a}}}_1, \ddot{\tilde{\mathbf{a}}}_1]}{\tilde{p}^2}$ are the Blaschke integral invariants and the integrals $\int \tilde{p}\, du$ and $\int \tilde{q}\, du$ are the dual arc lengths of the dual curves $\tilde{\mathbf{a}}_1(u)$ and $\tilde{\mathbf{a}}_3(u)$, respectively (see [1,3]).

Let the dual vector field $\tilde{\mathbf{z}}(u)$ be decomposed into dual vectors of the Blaschke frame

$$\tilde{\mathbf{z}}(u) = \tilde{\lambda}(u)\tilde{\mathbf{a}}_1(u) + \tilde{\mu}(u)\tilde{\mathbf{a}}_2(u) + \tilde{\nu}(u)\tilde{\mathbf{a}}_3(u), \tag{2.4}$$

whereas $\tilde{\lambda}(u) = \lambda(u) + \epsilon\lambda^*(u)$, $\tilde{\mu}(u) = \mu(u) + \epsilon\mu^*(u)$, $\tilde{\nu}(u) = \nu(u) + \epsilon\nu^*(u)$.

Let us determine the dual infinitesimal bending field so that all dual bent curves are on the initial dual sphere $\mathbb{S}^2$ with a given precision, i.e. let be valid

$$\|\tilde{\mathbf{r}}_t\|^2 = 1 + \tilde{o}(t). \tag{2.5}$$

We have the following theorem.

Theorem 2.1. *The necessary and sufficient condition for $\tilde{\mathbf{z}}(u)$ to be the dual infinitesimal bending field of the dual spherical curve $\tilde{C}$ which leaves $\tilde{C}$ on the the dual sphere $\mathbb{S}^2$ with a given precision is*

$$\dot{\tilde{\mu}} - \tilde{\nu}\tilde{q} = 0, \tag{2.6}$$

or, equivalently,

$$\dot{\mu} - \nu q = 0 \quad \wedge \quad \dot{\mu}^* - \nu^* q - \nu q^* = 0, \tag{2.7}$$

where $\tilde{q} = q + \epsilon q^*$ is the Blaschke integral invariant.

Proof. According to Theorem 1.1, the following condition should be valid

$$d\tilde{\mathbf{r}} \cdot d\tilde{\mathbf{z}} = 0. \tag{2.8}$$

Also, based on paper [5], the necessary and sufficient condition for the infinitesimal bending of the dual spherical curve $\tilde{C}$ to be on the dual unit sphere $\mathbb{S}^2$ with a given precision is that the dual field $\tilde{\mathbf{z}}$ satisfies the condition

$$\tilde{\mathbf{r}} \cdot \tilde{\mathbf{z}} = 0. \tag{2.9}$$

Based on the decomposition of the dual vector field $\tilde{\mathbf{z}}(u)$ given in (2.4) and applying scalar product with $\tilde{\mathbf{r}}(u)$ we obtain $\tilde{\lambda}(u) = 0$. Therefore,

$$\tilde{\mathbf{z}}(u) = \tilde{\mu}(u)\tilde{\mathbf{a}}_2(u) + \tilde{\nu}(u)\tilde{\mathbf{a}}_3(u).$$

Differentiating the previous equation, applying Eq. (2.8) and the Blaschke formula we obtain

$$\dot{\tilde{\mu}}(u) - \tilde{\nu}(u)\tilde{q} = 0.$$

By separating the real and dual parts in the previous equation we get equations (2.7). □

3. Analysis of the hyperbolic paraboloid related to dual spherical curves

Since the dual curve $\tilde{C}$ on the dual unit sphere $\mathbb{S}^2$, according to the E. Study's map, corresponds to the ruled surface in $\mathbb{E}^3$

$$\mathbf{R}(u,v) = \mathbf{r}(u) \times \mathbf{r}^*(u) + v\mathbf{r}(u),$$

with the directrix $\mathbf{r} \times \mathbf{r}^*$, an infinitesimal bending of $\tilde{C}$ determines a family of surfaces in $\mathbb{E}^3$

$$\mathbf{R}(u,v,t) = \mathbf{R}_t(u,v) = \mathbf{r}_t(u) \times \mathbf{r}_t^*(u) + v\mathbf{r}_t(u).$$

Let us consider a hyperbolic paraboloid as a ruled surface

$$x_3 = x_1^2 - x_2^2$$

generated by the family of lines (rulings)

$$x_1 + x_2 = u, \quad x_1 - x_2 = \frac{x_3}{u}.$$

Let us take two arbitrary points from the ruling l of this surface for an arbitrary parameter u, for instance $\mathbf{x} = (\frac{u}{2}, \frac{u}{2}, 0)$, $\mathbf{y} = (\frac{u^2}{2u}, \frac{u^2-1}{2u}, 1)$, and denote

$$\mathbf{r} = \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|}, \quad \mathbf{r}^* = \frac{\mathbf{x}\times\mathbf{y}}{\|\mathbf{y}-\mathbf{x}\|}.$$

We have

$$\mathbf{r} = \frac{1}{\sqrt{2+4u^2}}(1,-1,2u), \quad \mathbf{r}^* = \frac{1}{\sqrt{2+4u^2}}(u^2,-u^2,-u).$$

Obviously, the following relationships are valid

$$\mathbf{r}\cdot\mathbf{r} = 1, \quad \mathbf{r}\cdot\mathbf{r}^* = 0.$$

The two vectors $\mathbf{r}$ and $\mathbf{r}^*$ determine the oriented line l with the equation $\mathbf{z}\times\mathbf{r} = \mathbf{r}^*$, where $\mathbf{z}$ is an arbitrary point on the line l. The components of $\mathbf{r}$ and $\mathbf{r}^*$ are the normalized Plučker's coordinates of the line l (see [3]). According to E. Study's map, the oriented line l is in one-to-one correspondence with the points of the dual unit sphere

$$\tilde{\mathbf{r}} = \mathbf{r} + \epsilon\mathbf{r}^*.$$

Thus, the differentiable dual curve $\tilde{C}:\ \tilde{\mathbf{r}} = \tilde{\mathbf{r}}(u)$ on the dual unit sphere, depending on a real parameter u, represents the differentiable family of straight lines in Euclidean 3-space $\mathbb{E}^3$, i.e. the ruled surface, which is the hyperbolic paraboloid. The lines

$$\tilde{\mathbf{r}} = \mathbf{r} + \epsilon\mathbf{r}^* = \frac{1}{\sqrt{2+4u^2}}[(1,-1,2u) + \epsilon(u^2,-u^2,-u)]$$

are generators (rulings) of the surface.

Let us determine the orthonormal moving Blaschke frame along the dual curve $\tilde{C}: \mathbf{r} = \tilde{\mathbf{r}}(u)$:

$$\tilde{\mathbf{a}}_1 = \tilde{\mathbf{r}}(u), \quad \tilde{\mathbf{a}}_2 = \frac{\dot{\tilde{\mathbf{a}}}_1}{\|\dot{\tilde{\mathbf{a}}}_1\|}, \quad \tilde{\mathbf{a}}_3 = \tilde{\mathbf{a}}_1 \times \tilde{\mathbf{a}}_2.$$

We have

$$\tilde{\mathbf{a}}_1 = \frac{1}{\sqrt{2+4u^2}}[(1,-1,2u) + \epsilon(u^2,-u^2,-u)], \tag{3.1}$$

$$\dot{\tilde{\mathbf{a}}}_1 = (2+4u^2)^{-3/2}[(-4u,4u,4) + \epsilon(4u+4u^3,-4u-4u^3,-2)],$$

$$\|\dot{\tilde{\mathbf{a}}}_1\| = \sqrt{\dot{\tilde{\mathbf{a}}}_1 \cdot \dot{\tilde{\mathbf{a}}}_1} = \frac{\sqrt{2}}{1+2u^2} - \frac{\epsilon\sqrt{2}}{2},$$

$$\tilde{\mathbf{a}}_2 = \frac{1}{2\sqrt{1+2u^2}}[(-2u,2u,2) + \epsilon(u,-u,2u^2)], \tag{3.2}$$

$$\tilde{\mathbf{a}}_3 = -\frac{\sqrt{2}}{2}(1,1,0). \tag{3.3}$$

The Blaschke integral invariants are respectively

$$\tilde{p} = \|\dot{\tilde{\mathbf{a}}}_1\| = p + \epsilon p^*, \quad p = \frac{\sqrt{2}}{1+2u^2}, \quad p^* = -\frac{\sqrt{2}}{2}, \tag{3.4}$$

$$\tilde{q} = \frac{[\tilde{\mathbf{a}}_1, \dot{\tilde{\mathbf{a}}}_1, \ddot{\tilde{\mathbf{a}}}_1]}{\tilde{p}^2} = q + \epsilon q^*, \quad q = q^* = 0. \tag{3.5}$$

Let us determine the curve of striction $\mathbf{s} = \mathbf{s}(u)$ on the hyperbolic paraboloid. According to [3], since

$$\frac{d\mathbf{s}}{du} = q^*\mathbf{a}_1 + p^*\mathbf{a}_3 = \frac{1}{2}(1,1,0),$$

we have $\mathbf{s}(u) = \frac{1}{2}(u,u,0)$, wherefrom we obtain the equation of the curve of striction

$$x_3 = 0, \quad x_1 = x_2,$$

which is the straight line.

Based on Theorem 2.1 we will determine the dual infinitesimal bending field

$$\tilde{\mathbf{z}}(u) = \tilde{\mu}(u)\tilde{\mathbf{a}}_2(u) + \tilde{\nu}(u)\tilde{\mathbf{a}}_3(u), \tag{3.6}$$

of the dual curve $\tilde{C}$, which satisfies the condition

$$\dot{\tilde{\mu}}(u) - \tilde{q}\tilde{\nu}(u) = 0. \tag{3.7}$$

Since $\tilde{q} = 0$, we conclude that $\tilde{\mu}(u) = \tilde{c}$ is a dual constant, $\tilde{\nu}(u)$ is arbitrary. In this way we obtain

$$\tilde{\mathbf{z}}(u) = \tilde{c}\frac{1}{2\sqrt{1+2u^2}}\Big[(-2u,2u,2) + \epsilon(u,-u,2u^2)\Big] - \frac{\sqrt{2}}{2}\tilde{\nu}(u)(1,0,0). \tag{3.8}$$

In particular, for $\tilde{c} = \sqrt{2}$, $\tilde{\nu}(u) = \frac{1}{\sqrt{1+2u^2}}(1+\epsilon)$, the dual infinitesimal bending field reduces to

$$\tilde{\mathbf{z}}(u) = \frac{1}{\sqrt{2+4u^2}}\Big[(-2u-1, 2u-1, 2) + \epsilon(u-1, -u-1, 2u^2)\Big]. \tag{3.9}$$

In Figure 3.1 we can see the family of indicatrices, the family of dual parts and the deformed surface for $t = 0.1$, respectively, which correspond to the infinitesimal bending of the dual spherical curve $\tilde{C}$ under the dual field $\tilde{\mathbf{z}}(u)$.

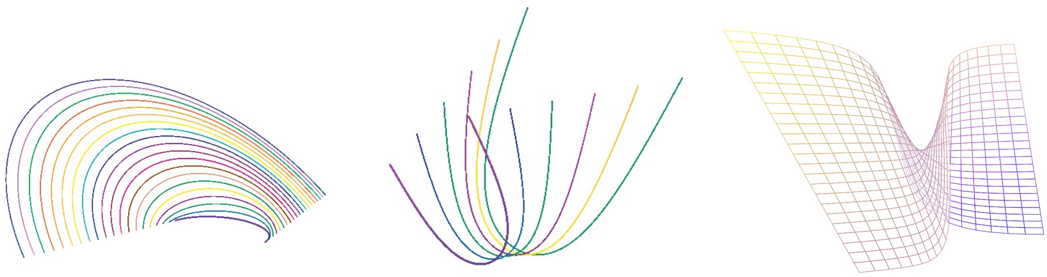

Fig. 3.1: Family of indicatrices, family of dual parts and deformed surface for $t = 0.1$, respectively

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