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NEW EXAMPLES OF GEODESIC ORBIT NILMANIFOLDS

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Abstract. In this paper we discuss properties of geodesic orbit Riemannian metrics on nilpotent Lie groups and some recent examples of such metrics. In particular, we explain the construction of continuous families of pairwise non-isomorphic connected and simply connected nilpotent Lie groups of dimension $4k+6$, $k \geq 1$, every of which admits geodesic orbit metrics.
Keywords: Lie groups, geodesic orbit metrics, geodesic orbit Riemannian metrics.

1. Introduction

A Riemannian manifold (M,g) is called *a manifold with homogeneous geodesics or a geodesic orbit manifold* (shortly, *GO-manifold*) if any geodesic γ of M is an orbit of a 1-parameter subgroup of the full isometry group of (M,g). A Riemannian manifold $(M=G/H,g)$, where H is a compact subgroup of a Lie group G and g is a G-invariant Riemannian metric, is called *a space with homogeneous geodesics* or *a geodesic orbit space* (shortly, *GO-space*) if any geodesic γ of M is an orbit of a 1-parameter subgroup of the group G. Hence, a Riemannian manifold (M,g) is a geodesic orbit Riemannian manifold, if it is a geodesic orbit space with respect to its full connected isometry group. This terminology was introduced in [38] by O. Kowalski and L. Vanhecke, who initiated a systematic study on such spaces. In the same paper, O. Kowalski and L. Vanhecke classified all GO-spaces of dimension ≤ 6. One can find many interesting results about GO-manifolds and its subclasses in [3–5, 7, 9, 15, 21, 29, 45, 52, 53, 56, 57, 62], and in the references therein.

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It is clear that any geodesic orbit space is homogeneous. All homogeneous spaces in this paper are assumed to be almost effective. Let $(G/H, g)$ be a homogeneous Riemannian space. It is well known that there is an Ad(H)-invariant decomposition (that is not unique in general)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}, \tag{1.1}$$

where $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. The Riemannian metric g is G-invariant and is determined by an Ad(H)-invariant Euclidean metric $g = (\cdot, \cdot)$ on the space $\mathfrak{p}$ which is identified with the tangent space T_oM at the initial point $o = eH$. By $[\cdot, \cdot]$ we denote the Lie bracket in $\mathfrak{g}$, and by $[\cdot, \cdot]_{\mathfrak{p}}$ its $\mathfrak{p}$-component according to (1.1). The following is a well-known criteria of GO-spaces, see other details and useful facts in [10, 47].

Lemma 1.1. [38] *A homogeneous Riemannian space* $(G/H, g)$ *with the reductive decomposition* (1.1) *is a GO-space if and only if for any* $X \in \mathfrak{p}$ *there is* $Z \in \mathfrak{h}$ *such that*

$$([X + Z, Y]_{\mathfrak{p}}, X) = 0 \textit{ for all } Y \in \mathfrak{p}.$$

Due to this lemma, the property to be geodesic orbit is related to classes of locally isomorphic homogeneous spaces. Let us recall the simplest type of GO-spaces. The metric g is called *naturally reductive* if an Ad(H)-invariant complement $\mathfrak{p}$ can be chosen in such a way that $([X, Y]_{\mathfrak{p}}, X) = 0$ for all $X, Y \in \mathfrak{p}$. In this case, we say that the (naturally reductive) metric *g is generated by the pair* $(\mathfrak{p}, (\cdot, \cdot))$. It immediately follows that any naturally reductive space is a geodesic orbit space; the converse is false when $\dim(M) \geqslant 6$ [38]. It should be noted that the property of being naturally reductive depends on the choice of the group G (the choice of the presentation $M = G/H$). Every isotropy irreducible Riemannian space is naturally reductive, and hence geodesic orbit, see e.g. [13].

The class of (Riemannian) geodesic orbit spaces includes same important classes of Riemannian manifolds: symmetric spaces, weakly symmetric spaces [5, 11, 43, 59, 62, 66], naturally reductive spaces [1, 17, 30, 37, 54, 55], normal and generalized normal homogeneous (δ-homogeneous) spaces [8–10], and Clifford – Wolf homogeneous manifolds [7, 10]. For the current state of knowledge in the theory of geodesic orbit spaces and manifolds we refer the reader to the book [10], the papers [4, 6, 14, 29, 47, 55], and the references therein.

It should be noted that GO property is a very general geometric phenomenon: it is extensively studied in Riemannian, Lorentzian and general pseudo-Riemannian settings (see [16, 18, 44, 60]), in Finsler geometry (see recent papers [19, 61, 63] and the references therein), in affine geometry [20]. It should be noted also that homogeneous geodesics exist also in homogeneous sub-Riemannian manifolds [48] and in homogeneous sub-Finsler manifolds [65].

There is no hope to obtain a complete classification of all Riemannian geodesic orbit spaces. Partial classifications are possible only for special types of geodesic orbit metrics (for instance, Clifford – Wolf homogeneous metrics [7]) or for small dimensions (for $\dim(M) \leqslant 6$ see [38] and references therein).

One of important subclasses of geodesic orbit Riemannian manifolds are connected and simply connected nilpotent Lie groups supplied with some left-invariant Riemannian geodesic orbit metrics (GO-nilmanifolds). The theory of this type of GO-manifolds, we consider in the next section.

The main goal of this paper is to discuss recent results showing that the set of nilpotent groups admitting Riemannian geodesic orbit metrics is quite extensive. To do this, we will consider (in the last section) Theorem 3.1 (obtained originally in [46]) that gives for any $k \geq 1$, a k-parameter family of pairwise non-isomorphic connected and simply connected nilpotent Lie groups of dimension $4k+6$, such that every of them admits a 3-parameter family of Riemannian geodesic orbit metrics. The minimum dimension of such groups is 10 (for $k = 1$). In the next section, we recall important results on Riemannian geodesic orbit metrics on nilpotent Lie groups. The main role here is played by C. Gordon's results on the structure of geodesic orbital nilmanifolds and on the description of GO-metrics on nilpotent Lie groups.

2. Riemannian geodesic orbit metrics on nilpotent Lie groups

We discuss some properties of GO-nilmanifolds. The foundations of the corresponding theory were developed by C. Gordon in [29]. We recall some important facts. In what follows we consider only connected and simply connected nilpotent Lie group N supplied with some left-invariant Riemannian metric g, and we call (N, g) a nilmanifold.

The book [32] is a standard source on the theory of nilpotent groups and Lie algebras. Note that the class of nilpotent Lie algebras is very wide and there is no hope of obtaining a reasonable classification of them in an arbitrary dimension. Nevertheless, the classification of nilpotent Lie algebras of small dimensions is known. The classification of complex nilpotent Lie algebras of small dimension has a long history, yet only for dimension ≤ 7 has it been completed, see e. g. [41] for a survey. It is known that there are finite numbers of isomorphism classes of complex or real nilpotent Lie algebra in dim ≤ 6. On the other hand there are six 1-parameter families of nilpotent Lie algebras of dimension 7, pairwise not isomorphic [32].

Two-step nilpotent (metabelian) Lie algebras form the first non-trivial subclass of nilpotent algebras. However even the classification of these special nilpotent Lie algebras is a rather complicated problem. This problem is completely solved in the case of 1-dimensional or 2-dimensional center [40]. Known results on small-dimensional two-step nilpotent Lie algebras (in particular, the classification of complex two-step nilpotent Lie algebra in dim ≤ 9) can be found in [28,34]. It should be noted that there are several continuous families of pairwise non-isomorphic two-step nilpotent Lie algebras in dimension 9.

Recall that the classification of complex two-step nilpotent (metabelian, in other terms) Lie algebras of dimension ≤ 9 is obtained in [28]. See also [64] and [34]. Important structure and (partial) classification results on 2-step nilpotent Lie algebra could be found in the following papers by P. Eberlein: [22–24].

In what follows, we consider only real nilpotent Lie algebras $\mathfrak{n}$. Recall that the corresponding Lie groups N are assumed connected and simply connected. This imply that N is diffeomorphic to a Euclidean space (a detailed description of GO-manifolds diffeomorphic to Euclidean spaces is obtained in [31]).

It is known that the full connected isometry group $G = \mathrm{Isom(N, g)}$ of a given nilmanifold (N, g) is such that N is the nilradical of G, in particular, N is a normal subgroup in G [58]. We denote by H the isotropy subgroup of G at the unit element $e \in N$.

For G/H as above, the Lie algebra $\mathfrak{n} = \mathrm{Lie}(N)$ is an ideal in $\mathfrak{g} = \mathrm{Lie}(G)$, hence we can write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}, \tag{2.1}$$

vector space direct sum, which is $\mathrm{Ad_G(H)}$-invariant. The Riemannian metric g corresponds to an $\mathrm{Ad_G(H)}$-invariant inner product $g_{eH} = (\cdot, \cdot)$ on $\mathfrak{n}$. Let $O(\mathfrak{n}, (\cdot, \cdot))$ be the group of orthogonal maps on the metric Lie algebra $(\mathfrak{n}, (\cdot, \cdot))$ and $D(\mathfrak{n})$ the space of skew-symmetric derivations of the metric Lie algebra $(\mathfrak{n}, (\cdot, \cdot))$.

Lemma 2.1. (E.N. Wilson [58]) Let (N, g) be a Riemannian nilmanifold and $(\cdot, \cdot)$ the associated inner product on the Lie algebra $\mathfrak{n}$. Then $\mathrm{Isom(N, g) = N \rtimes H}$ and the full isometry algebra of (N, g) is the semi-direct sum $\mathfrak{n} \rtimes \mathfrak{h}$, where $\mathfrak{h}$ is the space $D(\mathfrak{n})$ of skew-symmetric derivations of $(\mathfrak{n}, (\cdot, \cdot))$.

In particular, if Riemannian nilmanifolds (N_1, g_1) and (N_2, g_2) are isometric to each other, then the Lie group N_1 is isomorphic to N_2, as well as their Lie algebras are isomorphic to each other.

Let us recall the following important result.

Proposition 2.1. (C. Gordon [29]) If (N, g) is geodesic orbit Riemannian manifold, then the Lie algebra $\mathfrak{n} = \mathrm{Lie}(N)$ is either commutative or two-step nilpotent.

In the case when $\mathfrak{n}$ is commutative, (N, g) is Euclidean space. Hence, **in what follow we suppose that $\mathfrak{n}$ is two-step nilpotent**.

Now we recall one helpful method to represent any two-step nilpotent metric Lie algebra. Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra with an inner product $(\cdot, \cdot)$. Denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and by $\mathfrak{v}$ the $(\cdot, \cdot)$-orthogonal complement to $\mathfrak{z}$ in $\mathfrak{n}$. It is clear that $[\mathfrak{n}, \mathfrak{n}] = [\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$. We denote by $\mathfrak{so}(\mathfrak{z})$ and $\mathfrak{so}(\mathfrak{v})$ the algebras of skew symmetric transformations of $(\mathfrak{z}, (\cdot, \cdot))$ and $(\mathfrak{v}, (\cdot, \cdot))$ respectively. It is easy to see that any $D \in D(\mathfrak{n}) = \mathfrak{h}$ saves both $\mathfrak{z}$ and $\mathfrak{v}$.

For any $Z \in \mathfrak{z}$, we consider the operator

$$J_Z : \mathfrak{v} \to \mathfrak{v}, \quad \text{such that} \quad (J_Z(X), Y) = ([X, Y], Z), \quad X, Y \in \mathfrak{v}. \tag{2.2}$$

It is clear that J_Z are skew-symmetric and $J_Z(Y) = \mathrm{(adY)'(Z)}$, where $\mathrm{(adY)'}$ is adjoint to adY with respect to $(\cdot, \cdot)$. The map $J : Z \to J_Z$ is obviously linear.

The following proposition is a corollary of Lemma 1.1.

Proposition 2.2. (C. Gordon [29]) In the above notations, (N, g) is geodesic orbit Riemannian manifold if and only if for any $X \in \mathfrak{z}$ and $Y \in \mathfrak{v}$ there is $D \in D(\mathfrak{n})$ such that $[D, X] = D(X) = 0$, $[D, Y] = D(Y) = J_X(Y)$.

It is clear that $J_Z \equiv 0$ for $Z \in \mathfrak{z}$ if and only if Z is orthogonal to $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$ (recall that $(J_Z(X), Y) = ([X, Y], Z)$ for $X, Y \in \mathfrak{v}$). Therefore, $J : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$ is an injective map for any two-step nilpotent metric Lie algebra with $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{z}$, because $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}] = [\mathfrak{v}, \mathfrak{v}]$. It should be noted that the latter condition is not too restrictive (see e.g. [46, Lemma 3]).

In what follows, we suppose that $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{z}$, $m := \dim(\mathfrak{z})$, $n := \dim(\mathfrak{v}) = \dim(\mathfrak{n}) - \dim(\mathfrak{z})$. In particular, the linear map $J := \mathfrak{z} \mapsto J_Z$ is injective, $\mathcal{V} = \{J_Z \,|\, Z \in \mathfrak{z}\}$ is m-dimensional linear subspace in $\mathfrak{so}(\mathfrak{v})$.

If $\varphi : \mathfrak{h} \to \mathfrak{so}(\mathfrak{v})$ is the restriction of isotropy representation to $\mathfrak{v}$, we may reformulate the condition of Proposition 2.2 as follows. We know that $\mathcal{V} = J(\mathfrak{z})$ is a linear subspace in $\mathfrak{so}(\mathfrak{v})$. Further, for every $X \in \mathfrak{h}$ and $Z \in \mathfrak{z}$ we get $J_{[X,Z]} = [\varphi(X), J_Z]$ (it easily follows from the condition on X to be skew-symmetric derivation), hence, the subspace $\mathcal{V} = J(\mathfrak{z})$ is normalized by the subalgebra $\mathcal{N} := \varphi(\mathfrak{h})$ in $\mathfrak{so}(\mathfrak{v})$. The equality $J_{[X,Z]} = [\varphi(X), J_Z]$ implies that the representation $\varphi : \mathfrak{h} \to \mathfrak{so}(\mathfrak{v})$ is faithful (otherwise, some non-trivial $X \in \mathfrak{h}$ acts trivially both on $\mathfrak{v}$ and on $\mathfrak{z}$, hence, on $\mathfrak{n}$). Therefore, we have

a) a Lie subalgebra $\mathcal{N} \subset \mathfrak{so}(\mathfrak{v})$ (acted on $\mathfrak{v}$) and

b) an $\mathrm{ad}(\mathcal{N})$-invariant module $\mathcal{V}$ in $\mathfrak{so}(\mathfrak{v})$,

such that for every $Y \in \mathfrak{v}$ and $Z \in \mathcal{V}$ there is $X \in \mathcal{N}$ with the following properties: $[X, Z] = 0$ and $X(Y) = Z(Y)$.

Since $\dim(\mathfrak{v}) = n$, we naturally identify $\mathfrak{so}(\mathfrak{v})$ with $\mathfrak{so}(n)$.

Definition 2.1. ([29]) Let $\mathcal{V}$ be a linear subspace of $\mathfrak{so}(n)$ and $\mathcal{N}$ the normalizer of $\mathcal{V}$ in $\mathfrak{so}(n)$. We say that $\mathcal{V}$ satisfies *the transitive normalizer condition* if for every $Y \in \mathbb{R}^n$ and every $Z \in \mathcal{V}$ there is some $X \in \mathcal{N}$ such that $[X, Z] = 0$ and $X(Y) = Z(Y)$.

Proposition 2.3. (C. Gordon [29]) *Let $\mathcal{V}$ be a linear subspace of $\mathfrak{so}(n)$ with the normalizer $\mathcal{N} \subset \mathfrak{so}(n)$. Suppose that $\mathcal{V}$ satisfies the transitive normalizer condition. Then the metric Lie algebra $(\mathcal{V} \rtimes \mathbb{R}^n, (\cdot,\cdot)_1 + (\cdot,\cdot)_2)$ defines a geodesic orbit nilmanifold, where $(\cdot,\cdot)_1$ is any $\mathrm{ad}(\mathcal{N})$-invariant inner product on $\mathcal{V}$, $(\cdot,\cdot)_2$ is the standard inner product in $\mathbb{R}^n$, $[X, Y] = 0$ if $X \in \mathcal{V}$ and $Y \in \mathcal{V}$ or $Y \in \mathbb{R}^n$, and $([X, Y], Z)_1 = (Z(X), Y)_2$ for all $X, Y \in \mathbb{R}^n$ and $Z \in \mathcal{V}$. In particular, $\mathcal{V}$ is the derived algebra of $\mathcal{V} \rtimes \mathbb{R}^n$ and, moreover, if for any $Y \in \mathbb{R}^n$ there is $Z \in \mathcal{V}$ such that $0 \neq Z(Y) \in \mathbb{R}^n$, then $\mathcal{V}$ is the center of $\mathcal{V} \rtimes \mathbb{R}^n$.*

Remark 2.1. Suppose, that a linear subspace $\mathcal{V}$ of $\mathfrak{so}(n)$ with the normalizer $\mathcal{N} \subset \mathfrak{so}(n)$ satisfies the transitive normalizer condition. It is possible that there is a Lie subalgebra

$\mathcal{N}' \subset \mathcal{N}$ such that for every $Y \in \mathbb{R}^n$ and every $Z \in \mathcal{V}$ there is some $X \in \mathcal{N}'$ such that $[X, Z] = 0$ and $X(Y) = Z(Y)$. This means that even the subalgebra $\mathcal{N}'$ together with $\mathcal{V}$ generate a geodesic orbit nilmanifold. Moreover, since the condition on $(\cdot,\cdot)_1$ to be $\mathrm{ad}(\mathcal{N}')$-invariant is weaker than the condition to be $\mathrm{ad}(\mathcal{N})$-invariant, we can get more GO-metric using less extensive subgroup $\mathcal{N}'$.

Definition 2.2. We will say that $\mathcal{V}$ satisfies *the transitive normalizer condition with respect to* $\mathcal{N}'$ if $\mathcal{N}' \subset \mathcal{N}$ is as in Remark 2.1.

One obvious possibility to choose $\mathcal{V}$, satisfied the transitive normalizer condition, is the following: $\mathcal{V}$ is a Lie subalgebra of $\mathfrak{so}(n)$. The following result is valid (see Section 2 in [29]).

Proposition 2.4. *Suppose that* $\mathcal{V}$ *is a Lie subalgebra of* $\mathfrak{so}(n)$ (*in particular,* $\dim(\mathcal{V}) = 1$). *Then* $\mathcal{V} \subset \mathcal{N}$ *and we can take* $\mathcal{N}' = \mathcal{V}$ *in the notation of Remark 2.1. Any corresponding GO-nilmanifold* (N, g) (*that depends on* $\mathrm{ad}(\mathcal{V})$*-invariant inner product on* $\mathcal{V}$) *is naturally reductive. On the other hand, if a subspace* $\mathcal{V} \subset \mathfrak{so}(n)$ *determined a naturally reductive GO-manifold, then* $\mathcal{V}$ *is a subalgebra of* $\mathfrak{so}(n)$.

Proof. Since $[\mathcal{V},\mathcal{V}] \subset \mathcal{V}$ and $\mathcal{N} = \{U \in \mathfrak{so}(n) \,|\, [U,\mathcal{V}] \subset \mathcal{V}\}$, then $\mathcal{V} \subset \mathcal{N}$. All other assertions easily follows from Proposition 2.3, Remark 2.1, and Theorem 2.8 in [29]. □

Remark 2.2. If $\dim(\mathcal{V}) = 1$, then the structure of the corresponding nilpotent Lie algebra depend of one operator J_Z, where $Z \in \mathcal{V}$ and has norm 1. The simplest non-trivial example is $J_Z = \mathrm{diag}\,(J, J, \cdots, J)$, where we have n numbers of blocks $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, corresponds to a simply-connected Heisenberg group of dimension $2n+1$ for every $n \geq 1$. In the general case, J_Z could be any skew-symmetric, then the corresponding GO-nilmanifolds constructed from operators J_{Z_1} and J_{Z_2} with distinct sets of eigenvalues (counted with multiplicities) are not isometric each to other.

Remark 2.3. It is possible that $\mathcal{V}$ is a Lie subalgebra of $\mathfrak{so}(n)$, but its normalizer N contains a Lie subalgebra $\mathcal{V} \oplus \mathcal{N}'$ such that $\mathcal{V}$ satisfies the transitive normalizer condition with respect to $\mathcal{N}'$. In this case $\mathcal{V}$ can be supplied with an arbitrary inner product, since any such inner product is $\mathrm{ad}(\mathcal{N}')$-invariant ($[\mathcal{V},\mathcal{N}'] = 0$). If $\mathcal{V}$ is not abelian, then there is an inner product on $\mathcal{V}$, that is not $\mathrm{ad}(\mathcal{V})$-invariant. Therefore, the corresponding GO-nilmanifold is not naturally reductive. The simplest example is $\mathcal{V} \oplus \mathcal{N}' = \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{so}(4)$, see details and other examples in Section 3.

In what follows, we will work with GO-nilmanifolds that are not naturally reductive. We know that the relation $[\mathcal{V},\mathcal{V}] \not\subset \mathcal{V}$ is sufficient for this.

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra with the center $\mathfrak{z}$. Then $\mathfrak{n}$ is called *non-singular* (often called also *regular* or *fat*) if the operator $\mathrm{ad(X)} : \mathfrak{n} \to \mathfrak{z}$ is surjective for all $X \in \mathfrak{n} \setminus \mathfrak{z}$.

It is obvious that $[\mathfrak{n},\mathfrak{n}] = \mathfrak{z}$ for any two-step nilpotent non-singular Lie algebra.

Let $\mathfrak{n}$ be a two-step nilpotent non-singular Lie algebra supplied with an inner product $(\cdot,\cdot)$, $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $\mathfrak{v}$ is an $(\cdot,\cdot)$-orthogonal complement to $\mathfrak{z}$ in $\mathfrak{n}$. Since $\mathfrak{n}$ is non-singular, then all operators J_Z are bijective for nontrivial $Z \in \mathfrak{z}$. Indeed, if $J_Z(X) = 0$ for some non-trivial $X \in \mathfrak{v}$, then $0 = (J_Z(X), Y) = ([X,Y], Z)$ for all $Y \in \mathfrak{v}$. Hence, the image of the operator $\operatorname{ad}(X) : \mathfrak{n} \to \mathfrak{z}$ is not whole $\mathfrak{z}$, that is impossible. Therefore, the operator $J_Z : \mathfrak{v} \to \mathfrak{v}$ is bijective for any $Z \neq 0$. Since J_Z is skew-symmetric, then $n = \dim(\mathfrak{v})$ is even.

Let us consider the unit sphere $S = \{X \in \mathfrak{v} \,|\, (X,X) = 1\}$ in $\mathfrak{v}$. Any $Z \in \mathfrak{z}$ determines a tangent vector fields on S as follows: $J_Z(X)$ is a tangent vector to S at the point $X \in S$. Therefore, the sphere S admits m linear independent tangent vector fields, where $m = \dim(\mathfrak{z})$. It is known that $0 \leq \dim(\mathfrak{z}) = m < \rho(n)$, where ρ is the function defined by $\dim(\mathfrak{v}) = n = (2a+1)\cdot 2^{4b+c} \mapsto \rho(n) = 8b + 2^{c}$, where $a, b, c \in \mathbb{N}$, $0 \leq c \leq 3$, see e. g. [33, Theorem 8.2]. In particular, we get that n even for $m = 1$, $n = 4k$ for $m \in \{2,3\}$, $n = 8k$ for $m \in \{4,5,6,7\}$, where $k \in \mathbb{N}$.

Important examples of non-singular two-step nilpotent Lie algebra are so called H-type algebras, which generalize Heisenberg algebras.

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra. We say that $\mathfrak{n}$ is an H-type algebra if there exists an inner product $(\cdot,\cdot)$ such that $J_Z^2 = -(Z,Z)\mathrm{Id}$ for every $Z \in \mathfrak{z}$. We note that Heisenberg algebras are exactly H-type algebras with one-dimensional centers $\mathfrak{z}$.

For any H-type algebra, the orthogonal complement $\mathfrak{v} = \mathfrak{z}^{\perp}$ is a Cliffold module over the Clifford algebra $C(\mathfrak{z})$ generated by $\mathfrak{z}$ and 1 modulo relation $J_Z^2 + (Z,Z)\cdot 1 = 0$, $Z \in \mathfrak{z}$. Indeed, the linear map J from $\mathfrak{z}$ to $\mathfrak{v}$ extends to a representation of $C(\mathfrak{z})$, and $\mathfrak{v}$ is $C(\mathfrak{z})$-module. Moreover, every Clifford module arises in this way [36]. The Clifford modules are completely classified, see details e.g. in [33, 36].

We have one useful isomorphism invariant for two-step Lie algebras $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ with even $n = \dim \mathfrak{v}$, called *the Pfaffian form*, which is the projective equivalence class of the homogeneous polynomial $f_{\mathfrak{n}}$ of degree $n/2$ in $m = \dim(\mathfrak{z})$ variables defined by

$$\Big(f_{\mathfrak{n}}(Z)\Big)^2 = \det(J_Z), \qquad Z \in \mathfrak{z}.$$

It is known that $\mathfrak{n}$ is non-singular if and only if $f_{\mathfrak{n}}(Z)$ is a positive polynomial, i.e., $f_{\mathfrak{n}}(Z) > 0$ for any non-zero $Z \in \mathfrak{z}$, see details in [50].

3. Some natural examples and Theorem 3.1

At first, we consider the following two 3-parameter families of matrices from $\mathfrak{so}(4)$:

$$L(\beta_1,\beta_2,\beta_3) = \begin{pmatrix} 0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & 0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & 0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & 0 \end{pmatrix}, \quad \beta_1,\beta_2,\beta_3 \in \mathbb{R}, \tag{3.1}$$

$$R(\gamma_1,\gamma_2,\gamma_3)=\begin{pmatrix} 0 & -\gamma_1 & -\gamma_2 & -\gamma_3 \\ \gamma_1 & 0 & \gamma_3 & -\gamma_2 \\ \gamma_2 & -\gamma_3 & 0 & \gamma_1 \\ \gamma_3 & \gamma_2 & -\gamma_1 & 0 \end{pmatrix}, \quad \gamma_1,\gamma_2,\gamma_3\in\mathbb{R}. \tag{3.2}$$

The following results are well known (and they are easy to prove): Every matrix $U\in\mathfrak{so}(4)$ can be uniquely presented as $L(\beta_1,\beta_2,\beta_3)+R(\gamma_1,\gamma_2,\gamma_3)$ for suitable β_i and γ_i, $i=1,2,3$. Moreover, $[L(\beta_1,\beta_2,\beta_3),R(\gamma_1,\gamma_2,\gamma_3)]=0$ for any values of β_i and γ_i, $i=1,2,3$. The matrices $L(\beta_1,\beta_2,\beta_3)$ for various values of β_i, $i=1,2,3$, constitutes a Lie algebra isomorphic to $\mathfrak{so}(3)$. The same we can say about the matrices $R(\gamma_1,\gamma_2,\gamma_3)$ for various values of γ_i, $i=1,2,3$. Hence, we deal with the decomposition $\mathfrak{so}(4)=\mathfrak{so}(3)\oplus\mathfrak{so}(3)$ of $\mathfrak{so}(4)$ into the direct sum of its three-dimensional ideals (we may assume the first summand is determined by matrices of the form $L(\beta_1,\beta_2,\beta_3)$, while the second summand is determined by matrices of the form $R(\gamma_1,\gamma_2,\gamma_3)$.

For a given $r>0$, we consider $S^3_r=\{X=(x_1,x_2,x_3,x_4)\in\mathbb{R}^4\,|\,x_1^2+x_2^2+x_3^2+x_4^2=r^2\}$, the sphere of radius r with center at the origin. Note that, the tangent plane to S^3_r at the point $U\in S^3_r$ is naturally identified with $\{W(U)\,|\,W\in\mathfrak{so}(4)\}$. The following result is well known, but we consider an outline of its proof for completeness.

Lemma 3.1. *For any tangent vector V to S^3_r at given point $U\in S^3_r$, there is a triple of β_1,β_2,β_3, as well a triple of $\gamma_1,\gamma_2,\gamma_3$, such that $L(U)=R(U)=V$, where $L=L(\beta_1,\beta_2,\beta_3)$ and $R=R(\gamma_1,\gamma_2,\gamma_3)$.*

Proof. Note that the three-dimensional sphere S^3 topologically is the Lie group $Sp(1)$, the group of unit quaternions. We have a natural action of $Sp(1)\times Sp(1)$ on $S^3=Sp(1)$ as follows: $(q_1,q_2):q\mapsto q_1\cdot q\cdot q_2^{-1}$. In particular, we have a surjective homomorphism $\psi: Sp(1)\times Sp(1)\mapsto SO(4)$ with the ineffective kernel $\mathbb{Z}_2=\{(1,1),(-1,-1)\}$. The Lie algebras of the images of the first and the second multiples in $Sp(1)\times Sp(1)$ under ψ are the first and the second ideals in the Lie algebra $\mathfrak{so}(4)=\mathfrak{so}(3)\oplus\mathfrak{so}(3)$, that are determined by the matrices $L(\beta_1,\beta_2,\beta_3)$ and $R(\gamma_1,\gamma_2,\gamma_3)$ respectively, see (3.1) and (3.2).

Both these images of $Sp(1)$ under ψ act transitively on $S^3\subset\mathbb{R}^4$, see e. g. [42]. It implies the following observation: If $U=(u_1,u_2,u_3,u_4)\in\mathbb{R}^4$ and $r=\left(u_1^2+u_2^2+u_3^2+u_4^2\right)^{1/2}$, then for any V in the tangent plane to the sphere S^3_r at the point U, there is an element W in the chosen ideal $\mathfrak{so}(3)$ such that $W(U)=V$. □

Lemma 3.1 and the discussion before it imply the following observation: If $\mathcal{V}$ is a linear subspace in one copy of $\mathfrak{so}(3)$ for the decomposition $\mathfrak{so}(4)=\mathfrak{so}(3)\oplus\mathfrak{so}(3)$, then the second copy of $\mathfrak{so}(3)$, which will be denoted as $\mathcal{Z}$, centralizes $\mathcal{V}$. Moreover, $\mathcal{V}$ satisfies the transitive normalizer condition with respect to $\mathcal{Z}$, hence every inner product on $\mathcal{V}$ determines a GO-nilmanifold of dimension $\dim(\mathcal{V})+4$. Up to isomorphism, there is only one choice of $\mathcal{V}$ of dimension 1, 2, or 3.

The above arguments lead also to the proof of the following theorem.

Theorem 3.1. ([46]) For any $k \geq 1$, there is a k-parameter family of pairwise non-isomorphic connected and simply connected nilpotent Lie groups $N_{t_1,t_2,\dots,t_k}$ of dimension $4k+6$, such that every of them admits a 3-parameter family of Riemannian geodesic orbit metrics.

Proof. Let us define a family of two-dimensional subspaces in $\mathfrak{so}(4k+4)$ as follows. At first we fix pairwise distinct real numbers $1 = t_0 < t_1 < t_2 < \cdots < t_k$ and define the matrices

$$C_j = \begin{pmatrix} 0 & 0 & -t_j \cdot x & y \\ 0 & 0 & y & t_j \cdot x \\ t_j \cdot x & -y & 0 & 0 \\ -y & -t_j \cdot x & 0 & 0 \end{pmatrix}, \qquad x, y \in \mathbb{R}, \quad 0 \leq j \leq k.$$

Now, we define

$$\mathcal{V}_{t_1,t_2,\dots,t_k} = \operatorname{diag}(C_0, C_1, C_2, \cdots, C_k).$$

It is clear that $\mathcal{V}_{t_1,t_2,\dots,t_k}$ forms a two-dimensional subspace in $\mathfrak{so}(4k+4)$ ($x, y \in \mathbb{R}$ are arbitrary) for any fixed k-tuple $\{t_i\}$, $1 \leq i \leq k$.

The Pfaffian forms for the corresponding nilpotent Lie algebras $\mathfrak{n}_{t_1,t_2,\dots,t_k}$ with operators $J_Z = \mathcal{V}_{t_1,t_2,\dots,t_k} \subset \mathfrak{so}(4k+4)$ are

$$(x^2+y^2)(t_1^2 \cdot x^2 + y^2) \times \cdots \times (t_k^2 \cdot x^2 + y^2) = \prod_{j=0}^{k} (t_i^2 \cdot x^2 + y^2).$$

Hence, for distinct k-tuples of $\{t_j\}$, $j = 1, \dots, k$, we get non-isomorphic non-singular two-step nilpotent Lie algebra $\mathfrak{n}_{t_1,t_2,\dots,t_k}$, see [50].

Now, we note that the normalizer of $\mathcal{V}_{t_1,t_2,\dots,t_k}$ in $\mathfrak{so}(4k+4)$ contains a subalgebra (that is the centralizer of $\mathcal{V}_{t_1,t_2,\dots,t_k}$ in $\mathfrak{so}(4k+4)$)

$$\operatorname{diag}(D, D, \dots, D) \subset \mathfrak{so}(4k+4),$$

where

$$D = D(\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} 0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & 0 & \alpha_3 & -\alpha_2 \\ \alpha_2 & -\alpha_3 & 0 & \alpha_1 \\ \alpha_3 & \alpha_2 & -\alpha_1 & 0 \end{pmatrix}, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}.$$

Taking any $X \in \mathbb{R}^{4k+4}$, that we can represent it as $X = (X_0, X_1, \dots, X_k)$, where $X_j \in \mathbb{R}^4$, $j = 0, 1, 2, \dots, k$. Now, take any $Y = Y^{x,y} \in \mathcal{V}_{t_1,t_2,\dots,t_k}$ (we should fix x and y for this goal). We consider also $C_j = C_j^{x,y}$ with the same fixed x and y for all $j = 0, 1, \dots, k$, i. e. $Y^{x,y} = \operatorname{diag}(C_0^{x,y}, C_1^{x,y}, C_2^{x,y}, \cdots, C_k^{x,y})$.

Further, for any $0 \leq j \leq k$, we can find $\alpha_1^j, \alpha_2^j, \alpha_3^j$ such that $D^j := D(\alpha_1^j, \alpha_2^j, \alpha_3^j)(X_j) = C_j^{x,y}(X_j)$. Note that any C_j is of type (3.2) and any D^j is of type (3.1), hence, it suffices to apply Lemma 3.1. Therefore, $D^{X,x,y} := \operatorname{diag}(D^0, D^1, \cdots, D^k) \in \mathcal{Z}$ is such that $D^{X,x,y}(X) = Y^{x,y}(X)$.

According to Remark 2.1, we can take $\mathcal{Z}$ as $\mathcal{N}' \subset \mathcal{N}$ in order to apply Proposition 2.3. Since $[\mathcal{Z}, \mathcal{V}_{t_1,t_2,\dots,t_k}] = 0$ then any inner product on $\mathcal{V}_{t_1,t_2,\dots,t_k}$ is $\mathrm{ad}(\mathcal{Z})$-invariant. Hence, there is a 3-parameter family of suitable inner product on $\mathcal{V}_{t_1,t_2,\dots,t_k}$ (since $\dim(\mathcal{V}_{t_1,t_2,\dots,t_k}) = 2$). Therefore, the theorem is completely proved. $\square$

Let us consider a more general situation, related to the transitive normalizer condition. Suppose that $\mathcal{A} \oplus \mathcal{B}$ is a Lie subalgebra of $\mathfrak{so}(n)$ such that the corresponding Lie group $B \subset SO(n)$, where $\mathcal{B} = \mathrm{Lie(B)}$, acts transitively on the unit sphere S of $\mathbb{R}^n = \mathfrak{v}$. All such Lie groups classified in [42]. Note that for any $X \in S$ and any $U \in \mathfrak{so}(n)$, $U(X)$ is a tangent vector to S at the point X. Let us consider any linear subspace $\mathcal{V} \subset \mathcal{A}$. Then $\mathcal{V}$ satisfies the transitive normalizer condition. Indeed, $\mathcal{B}$ is a subset of the normalizer $\mathcal{N}$ of $\mathcal{V}$ in $\mathfrak{so}(n)$. If we fix $Z \in \mathcal{V}$ and $X \in S \subset \mathbb{R}^n$, then we can find $Y \in \mathcal{B}$ such that $Y(X) = Z(X)$. It follows from the fact that $\mathrm{Lie}(\mathcal{B})$ acts transitively on the unit sphere S, hence, $\mathcal{B}(X)$ coincides with the tangent space to S at the point X. Moreover, by our assumptions, $[Y, Z] = 0$. Hence, $\mathcal{V}$ satisfies the transitive normalizer condition. By Proposition 2.3 we get a family of geodesic orbit nilmanifolds, corresponding to the metric Lie algebras $(\mathcal{V} \rtimes \mathbb{R}^n, (\cdot,\cdot)_1 + (\cdot,\cdot)_2)$, where $(\cdot,\cdot)_1$ is any $\mathrm{ad}(\mathcal{B})$-invariant inner product on $\mathcal{V}$ and $(\cdot,\cdot)_2$ is the standard inner product in $\mathbb{R}^n$. Since $[\mathcal{V}, \mathcal{B}] = 0$, $(\cdot,\cdot)_1$ is any invariant inner product on $\mathcal{V}$ (here we take $\mathcal{B}$ as $\mathcal{N}'$ in the notations of Remark 2.1).

In particular, we know that $U(1) \cdot SU(n) \subset SO(2n)$ and $SU(n)$ acts transitively on the unit sphere S of $\mathbb{R}^{2n}$. Therefore, we can consider $\mathcal{V} = \mathcal{A} = \mathfrak{u}(1)$ and $\mathcal{B} = \mathfrak{su}(n)$. The corresponding GO-nilmanifolds are the Heisenberg groups with suitable Riemannian metrics.

We know also that $Sp(1) \cdot Sp(n) \subset SO(4n)$ and $Sp(n)$ acts transitively on the unit sphere S of $\mathbb{R}^{4n}$. Hence, we can consider $\mathcal{V} = \mathcal{A} = \mathfrak{sp}(1)$ and $\mathcal{B} = \mathfrak{sp}(n)$. The corresponding GO-nilmanifold are the quaternionic Heisenberg groups [2]. In these two partial cases we get all possible naturally reductive H-type group, see [35, Proposition 1]. Note that the first examples of commutative spaces which are not weakly symmetric (which provides an answer to Selberg's question about the existence of such examples [51]) is modeled as the quaternionic Heisenberg group, endowed with certain special naturally reductive metrics [39].

Instead of $Sp(1)$ we can take also $\mathcal{V} = U(1) \subset Sp(1) = \mathcal{A}$ and any two-dimensional subspace $\mathcal{V} \subset Sp(1) = \mathcal{A}$. In the first case we again obtain the Heisenberg groups (only of dimension $4n+1$), in the second case we get H-type groups of dimension $4n+2$, that are not naturally reductive, but are geodesic orbit, see [35, Proposition 3].

The complete classification of geodesic orbit H-type groups was obtained by C. Riehm in [49]: a given H-type group is geodesic orbit if and only if $m = 1, 2, 3$; or $m = 5, 6$ and $n = 8$; or $m = 7$, $n = 8, 16, 24$ and $\mathfrak{v}$ is an isotypic Clifford module (in this case it is equivalent to the following property: if $Z_1, Z_2, \dots, Z_7$ is an orthonormal basis of $\mathfrak{z}$, the linear transformation $T : X \mapsto J_{Z_1}(J_{Z_2}(\cdots J_{Z_7}(X)\cdots))$ of $\mathfrak{v}$ is either Id or $-$Id). The classification of weakly symmetric H-type group was obtained in [12]: a given H-type group is weakly symmetric if and only if $m = 1, 2, 3$;

or $m = 5, 6, 7$ and $n = 8$; or $m = 7$, $n = 16$ and $\mathfrak{v}$ is an isotypic Clifford module.

A natural and topical problem is to obtain the complete classification of geodesic orbit H-type groups in the pseudo-Riemannian case. Some important ingredients are obtained in [26, 27] (the complete classification of pseudo H-type algebras) and [25] (the group of automorphisms of pseudo H-type Lie algebras).

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