FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 5 (2024), 929–936
https://doi.org/10.22190/FUMI240919063C
Original Scientific Paper

THE WILLMORE ENERGY VARIATIONS AND ENERGY EFFICIENT ARCHITECTURE

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Abstract. Since the shape is an important feature of objects and can be immensely useful in characterizing objects, it should point out the shape analysis considering the variation of magnitudes that characterize the shape itself. One of the fundamental functionals that measure the bending of a surface is the Willmore energy. In view of the meaning of ruled surfaces in aesthetics, statics, scale and manufacturing technologies, we point out the possibility of a mathematical analysis in the case of infinitesimal deformations by considering the variations of the Willmore energy on Gaudi surface under infinitesimal bending in $\mathbb{R}^3$. Application could be connected with energy efficient building design.
Keywords: Willmore energy, Gaudi surface, infinitesimal bending.

1. Introduction

There are two rudimentary ways to characterize the shape of a surface S: to consider how the unit normal ν behaves as we move around (*Shape Operator*) and to compare S to a sphere (*Willmore energy*).

The Willmore energy is a quantitative measure of how much a given surface deviates from a round sphere.

Received September 19, 2024, accepted: September 30, 2024
Communicated by Mića Stanković
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2020 *Mathematics Subject Classification.* Primary 53A05; Secondary 51M15, 80A05

Definition 1.1. *Let H and K be the mean and the Gaussian curvature, respectively, of the surface S. **Willmore energy at a surface point** $p \in S$ is given by:*

$$W(p) = H(p)^2 - K(p). \tag{1.1}$$

Ruled surfaces have wide application in civil engineering and architecture, but in many other sciences, too. The simplicity of production and very rich spectrum of shapes are the main reason for application of this kind of surfaces. Ideal for surface's modeling, computer graphic and animation are ruled surfaces, Gaudi surfaces and developable surfaces. Ruled surfaces and conoids were a subject of investigation of numerous books and papers from different point of view: [1, 6, 9, 12, 13, 16–19].

It is known that the magnitudes depending on the first fundamental form are stationary under infinitesimal bending. Infinitesimal bending field of a Gaudi surface was determined in [11]. Variation of some geometric magnitudes under infinitesimal bending was considered in [8] and [7]. Curvature based functions variations were expressed in [2]. Variation of the Willmore energy under infinitesimal bending of a surface is studied in [15]. Variation of the shape operator was considered in [14] and [3]. Variation of curvatures of helicoid was expressed in [4].

1.1. Parametrization of Gaudi Surfaces

Antoni Gaudi (1852-1926) was a brilliant Catalan architect, mathematician and artist. He studied architecture, with a strong grounding in mathe-matics, especially calculus and descriptive geometry. Gaudi surfaces are ruled surfaces - an important class of surfaces that contain straight lines.

Definition 1.2. **Gaudi surface** *or sinusoidal conoid is a surface defined as*

$$r(u,v) = (u, v, ku \sin \frac{v}{a}), \tag{1.2}$$

where **k** *and* **a** *are arbitrary constants.*

For $k = 0.25$, $a = 2$, see Figure 1.1.

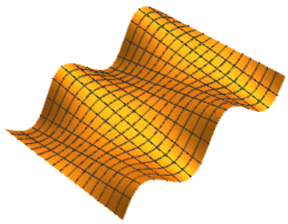

FIG. 1.1: Gaudi surface, $(k = 0.25\ ,\ a = 2)$

The directrix of this surface is sinusoid with **yz**-plane as plane of parallelity and x-axes is its axes, (see Figure 1.2).

Functions u and v in case for $k = a = 1$ are visualized in Figure 1.3.

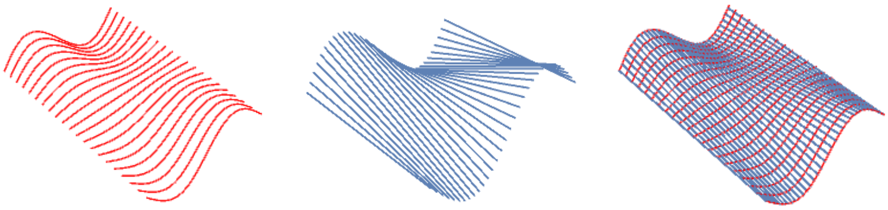

FIG. 1.2: The rulings of Gaudi surface

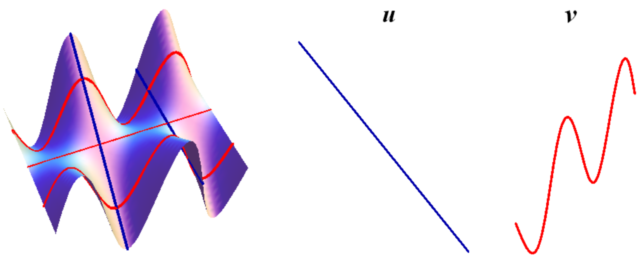

FIG. 1.3: Gaudi surface, u and v functions $(k = a = 1)$

1.2. Willmore energy of Gaudi surface $(k = a = 1)$ at the surface point

The Gaussian and the mean curvature of Gaudi surface $(k = a = 1)$ were expressed in [1] and it is valid that Gaussian curvature is everywhere non-positive:

(1.3)
$$K = -\frac{cos^2 v}{(1 + sin^2 v + u^2 cos^2 v)^2} ,$$

and

(1.4)
$$H = \frac{-u\ sinv\ (1 + sin^2 v + 2cos^2 v)}{2\ (1 + sin^2 v + u^2 cos^2 v)^{\frac{3}{2}}} .$$

Since the Willmore energy at the surface point is expressed with equation (1.1) and $H^2(p) > 0$, $-K(p) > 0$, it implies that $W(p) > 0$ for the Gaudi surface ($k = 1$, $a = 1$). The Willmore energy at the Gaudi surface point, in case $k = a = 1$, is equal to:

(1.5)
$$W = \frac{4u^2 + \cos^2 v\ (u^2 \sin^2 v \cos^2 v - 4\cos^2 v + 8)}{4\ (1 + \sin^2 v + u^2 \cos^2 v)^3} .$$

For arbitrary constants k and a the Willmore energy at the Gaudi surface point is expressed with:

$$\begin{aligned} W(p) &= \frac{k^2}{4a^2\Big(a^2+a^2k^2\sin^2\frac{v}{a}+k^2u^2\cos^2\frac{v}{a}\Big)^3}\cdot\Big(u^2\sin^2\frac{v}{a}\Big(1+k^2(1+\cos^2\frac{v}{a})\Big)^2+ \\ &+ 4a^4k^2\cos^2\frac{v}{a}\Big(a^2+a^2k^2\sin^2\frac{v}{a}+k^2u^2\cos^2\frac{v}{a}\Big)\,, \end{aligned}$$

and it will be minimal when k tends to zero:

$$k=\frac{1}{n},\ n\to\infty.$$

2. Variation of Curvatures Under Infinitesimal Bending of Surface

Geometric quantities change by infinitesimal bending and those changes can be measured with variation of geometric magnitudes. It is known that variations of some geometric magnitudes that depend on coefficients of the first fundamental form of the surface are zero under infinitesimal bending of the surface in $\mathbb{R}^3$. For instance, Christoffel's symbols, the first fundamental form, the determinant of the first and the second fundamental form, the area of a region on the surface, the Gaussian and the geodesic curvature are stationary under infinitesimal bending of a surface.

Let the surface S be a regular surface, parameterized by:

$$\mathbf{r}(u,v)=(u,v,f(u,v))\,, \tag{2.1}$$

and infinitesimal bending field by:

$$\mathbf{z}(u,v)=(\xi(u,v),\eta(u,v),\zeta(u,v)). \tag{2.2}$$

Then, the infinitesimal bending of a surface S is given with:

$$\begin{aligned} S_\epsilon:\tilde{\mathbf{r}}(u,v,\epsilon) &= \mathbf{r}(u,v)+\epsilon\mathbf{z}(u,v)= \\ &= (u+\epsilon\xi(u,v),v+\epsilon\eta(u,v),f(u,v)+\epsilon\zeta(u,v))\,. \end{aligned} \tag{2.3}$$

According to [5], the coefficients of the first and the second fundamental form of surface S_ϵ can be expressed with equations (2.4) and (2.5):

$$\begin{aligned} \tilde{E} &= \tilde{\mathbf{r}}_u\cdot\tilde{\mathbf{r}}_u=1+f_u^2+\epsilon^2(\xi_u^2+\eta_u^2+\zeta_u^2)\,, \\ \tilde{F} &= \tilde{\mathbf{r}}_u\cdot\tilde{\mathbf{r}}_v=f_uf_v+\epsilon^2(\xi_u\xi_v+\eta_u\eta_v+\zeta_u\zeta_v)\,, \\ \tilde{G} &= \tilde{\mathbf{r}}_v\cdot\tilde{\mathbf{r}}_v=1+f_v^2+\epsilon^2(\xi_v^2+\eta_v^2+\zeta_v^2)\,, \end{aligned} \tag{2.4}$$

$$
\begin{aligned}
\tilde{L} &= \tfrac{1}{\sqrt{\tilde{g}}}[\tilde{\mathbf{r}}_{uu}, \tilde{\mathbf{r}}_u, \tilde{\mathbf{r}}_v] = \tfrac{1}{\sqrt{\tilde{g}}}[f_{uu} + \epsilon\zeta_{uu}(1 + f_u^2 + f_v^2) + \epsilon^2 A_1 + \epsilon^3 A_2]\ ,\\
\tilde{M} &= \tfrac{1}{\sqrt{\tilde{g}}}[\tilde{\mathbf{r}}_{uv}, \tilde{\mathbf{r}}_u, \tilde{\mathbf{r}}_v] = \tfrac{1}{\sqrt{\tilde{g}}}[f_{uv} + \epsilon\zeta_{uv}(1 + f_u^2 + f_v^2) + \epsilon^2 B_1 + \epsilon^3 B_2]\ ,\\
\tilde{N} &= \tfrac{1}{\sqrt{\tilde{g}}}[\tilde{\mathbf{r}}_{vv}, \tilde{\mathbf{r}}_u, \tilde{\mathbf{r}}_v] = \tfrac{1}{\sqrt{\tilde{g}}}[f_{vv} + \epsilon\zeta_{vv}(1 + f_u^2 + f_v^2) + \epsilon^2 C_1 + \epsilon^3 C_2]\ .
\end{aligned}
\tag{2.5}
$$

Functions: A_i, B_i, C_i, $i = 1, 2$, are obtained in development of corresponding determinants and

$$\tilde{g} = \tilde{E}\tilde{G} - \tilde{F}^2 = 1 + f_u^2 + f_v^2 + \epsilon^2 \ldots + \epsilon^4 \ldots . \tag{2.6}$$

2.1. Variation of the Willmore energy at the surface point

Using the coefficients of the first and the second fundamental form of surface S_ϵ, we can consider the variation of curvatures under infinitesimal bending of surface.

Variation of the Willmore energy at the surface point was expressed in paper [15] with next Lemma:

Lemma 2.1. *Variation of the Willmore energy at the surface point* (2.1) *under infinitesimal bending* (2.3) *is given with equation:*

$$\delta W(p) = 2H(p)\delta H(p)\ , \tag{2.7}$$

where $H(p)$ is the mean curvature and $\delta H(p)$ its variation.

If we denote:

$$\mathbf{a} = \mathbf{a}(u, v) = (1 + f_u^2, -\sqrt{2} f_u f_v, 1 + f_v^2), \tag{2.8}$$

$$\mathbf{b} = \mathbf{b}(u, v) = (f_{vv}, \sqrt{2} f_{uv}, f_{uu}), \tag{2.9}$$

$$\mathbf{c} = \mathbf{c}(u, v) = (\zeta_{vv}, \sqrt{2}\zeta_{uv}, \zeta_{uu}), \tag{2.10}$$

variation of the Willmore energy at the surface point (2.1) under infinitesimal bending (2.3) also can be considered as (see [15]):

Theorem 2.1. *Variation of Willmore energy at the surface point (2.1) under infinitesimal bending (2.3) is given with equation:*

$$\delta W(u, v) = \frac{(\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c})}{2(\|\mathbf{a}\|^2 - 1)}\ , \tag{2.11}$$

where **a**, **b** *and* **c** *are defined with (2.8), (2.9) i (2.10).*

Next Theorem gives the necessary and sufficient condition of variation of Willmore energy at the surface point when it will be vanish.

Theorem 2.2. *Variation of Willmore energy at the surface point (2.1) under infinitesimal bending (2.3) will be equal zero if and only if holds:*

$$\mathbf{b} \cdot \mathbf{c} = \|\mathbf{b}\| \|\mathbf{c}\| \sin(\mathbf{a}, \mathbf{b}) \sin(\mathbf{a}, \mathbf{c}) \ , \tag{2.12}$$

where **a**, **b** *and* **c** *are defined with (2.8), (2.9) i (2.10).*

2.2. Variation of Willmore energy of Gaudi surface ($k = a = 1$) at the surface point

Using the coefficients of the first and the second fundamental form for surface (1.2) when it holds $k = a = 1$, the bending field found in [11] and Gaudi curvatures calculated in papers [12] and [10], it can be expressed next Lemma.

Lemma 2.2. *Variation of Willmore energy at the surface point under infinitesimal bending of Gaudi surface ($a = k = 1$) is given with:*

$$\delta W = \frac{-u \sin v (2 + \cos^2 v)\Big(3u(u + \sin^2 v) + 2\cos^2 v\Big(u^2(2\cos^2 v - 3) - 1\Big)\Big)}{4 \ln 10 \ u \cos^2 v \Big(1 + \sin^2 v + u^2 \cos^2 v\Big)^2} \ .$$

3. Geometry in energy efficient architecture

Throughout history, the human species has created shelters that were in harmony with nature and their environment. The base of this approach includes:

- Influence of the sun,
- Influence of the wind,
- Relief of the natural and created environment.

The basic principle was harmonization with the environment and that means regionality, climate compliance, material availability, construction technology, comfort and durability/changeability/maintenance. It can be noticed throughout the history that the most frequently primitive traditional objects were spherical.

Advantages of spherical geometry in energy efficient architecture are maximum use of solar energy, minimization of direct wind blows and the impact of the shape factor (the ratio between the outside surface area of the thermal insulation in the building envelope (A) and the heated volume (V)) given with ((3.1)),

$$f_o = \frac{A}{V} \ . \tag{3.1}$$

3.1. Willmore energy application

In energy efficient theory, the lower shape factor is more favorable. According to impact of the shape factor, ((3.1)), spherical shape is ideal in energy efficient building design. Since the Willmore energy at the surface point is a quantitative measure of how much a given surface deviates from a round sphere, it could be a useful tool in energy efficient architecture.

Example 3.1. *Gaudi constructions with less coefficient k are more energy efficient, on steep sides, there is less sunlight, the wind blows are direct, there is a large area of the envelope and the excess height of the floors is achieved, i.e. on the facades, (see Figure 3.1).*

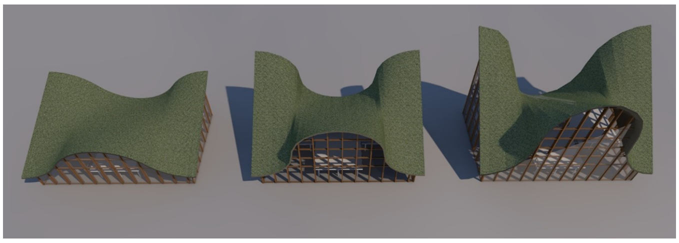

FIG. 3.1: Energy efficient of Gaudi roofs for different values k

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