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HOPF REAL HYPERSURFACES IN $S^6(1)$ WHOSE STRUCTURE JACOBI OPERATOR IS OF CODAZZI TYPE

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Abstract. We prove the non-existence of Hopf real hypersurfaces in the nearly Kähler sphere $S^6(1)$ whose structure Jacobi operator is of Codazzi type.
Keywords: Hopf hypersurfaces, Kähler sphere, Codazzi type operator.

1. Introduction

It is known that the 6-dimensional unit sphere $S^6(1)$ has a nearly Kähler structure (J, g), where J is an almost complex structure defined on $S^6(1)$ using the vector cross product of purely imaginary Cayley numbers $\operatorname{Im}\mathcal{O} = \mathbb{R}^7$ and g is the induced metric on $S^6(1)$ as a hypersurface of $\mathbb{R}^7$.

Let M be a real hypersurface in $S^6(1)$ with a unit normal vector field N and let $\xi = -JN$ be the characteristic vector field on M. We say that a hypersurface M is Hopf if the vector field ξ is principal, that is, $A\xi = \alpha\xi$ for a certain function α on the submanifold, where A is the shape operator of the hypersurface. We also note that the function α is locally constant, see [2]. It was shown in [2] that a connected Hopf hypersurface of a nearly Kähler $S^6(1)$ is an open part of either a geodesic hypersphere or a tube around an almost complex curve in $S^6(1)$.

The Jacobi operator on M with respect to ξ is called the structure Jacobi operator and is denoted by $l(X) = R_\xi(X) = R(X, \xi)\xi$ for any X tangent to M, where R denotes the curvature tensor of M. Some papers devoted to studying several conditions on the structure Jacobi operator of a real hypersurface in different ambient spaces are [7, 8].

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Recently, we proved the non-existence of real hypersurfaces in $S^6(1)$ with parallel structure Jacobi operator [1]. Also, the non-existence of real hypersurfaces in $S^6(1)$ whose Lie derivative of structure Jacobi operator coincides with the covariant derivative of it is proven in [5].

The structure Jacobi operator l is of Codazzi type if $(\nabla_X l)Y = (\nabla_Y l)X$, for any X, Y tangent to M, where ∇ denotes the covariant derivative on M. Naturally, this is a weaker condition than l being parallel. In [9] the authors proved the non existence of real hypersurfaces in complex projective space whose structure Jacobi operator is of Codazzi type. See also [6].

The purpose of the present paper is to study Hopf real hypersurfaces of $S^6(1)$ whose structure Jacobi operator is of Codazzi type. Concretely we prove

Theorem 1.1. *There exist no Hopf real hypersurfaces in $S^6(1)$ with Codazzi type structure Jacobi operator.*

2. Preliminaries

Let M be a Riemannian submanifold of the nearly Kähler sphere $S^6(1)$ with nearly Kähler structure (J, g). Then the $(2,1)$-tensor field G on $S^6(1)$ defined by $G(X,Y) = (\bar{\nabla}_X J)Y$, where $\bar{\nabla}$ is the Levi-Civita connection on $S^6(1)$, is skew symmetric and also satisfies

$$G(X, JY) + JG(X,Y) = 0, \quad g(G(X,Y), Z) + g(G(X,Z), Y) = 0.$$

Moreover, see [4], we have

$$(\bar{\nabla} G)(X,Y,Z) = g(X,Z)JY - g(X,Y)JZ - g(JY,Z)X, \tag{2.1}$$

for arbitrary vector fields X, Y, Z tangent to $S^6(1)$.

Also, for $X, Y, Z, W \in TM$, we have that the following Gauss equation

$$\begin{aligned} R(X,Y,Z,W) &= g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ &+ g(h(X,W), h(Y,Z)) - g(h(X,Z), h(Y,W)), \end{aligned} \tag{2.2}$$

where we denote by R the Riemannian curvature tensor of M.

We denote by N the unit normal vector field of M and by $\xi = -JN$ the corresponding Reeb vector field with dual 1-form $\eta(X) = g(X, \xi)$ a 1-form on M. Let $\mathcal{D} = \text{Ker } \eta = \{X \in TM \mid \eta(X) = 0\}$. Then $\mathcal{D}$ is a 4-dimensional smooth distribution on M, which is J-invariant.

3. The moving frame for hypersurfaces in $S^6(1)$

Let us present one of the convenient moving frames to work with and the relations between the connection coefficients in it, for details see [3].

For each unit vector field $E_1 \in \mathcal{D}$, let $E_2 = JE_1$, $E_3 = G(E_1, \xi)$, $E_4 = JE_3$. Then the set $\{E_1, E_2, E_3, E_4, E_5 = \xi\}$ is a local orthonormal frame on M, see [3]. Moreover, the following holds.

Lemma 3.1. *([3]) For the previously defined orthonormal frame the following relations hold*

$$\begin{aligned} &G(E_1,E_2)=0, \quad G(E_1,E_3)=-\xi, \quad G(E_1,E_4)=N, \quad G(E_1,\xi)=E_3,\\ &G(E_1,N)=-E_4, \quad G(E_2,E_3)=-N, \quad G(E_2,E_4)=\xi, \quad G(E_2,\xi)=-E_4,\\ &G(E_2,N)=-E_3, \quad G(E_3,E_4)=0, \quad G(E_3,\xi)=-E_1, \quad G(E_3,N)=E_2,\\ &G(E_4,\xi)=E_2, \quad G(E_4,N)=E_1. \end{aligned} \tag{3.1}$$

Note that such a moving frame is not uniquely determined and depends on the choice of the vector field $E_1 \in \mathcal{D}$.

For one such frame, let us denote by

$$g_{ij}^k = g(D_{E_i}E_j, E_k), \quad h_{ij} = g(D_{E_i}E_j, N), \quad 1 \le i,j,k \le 5, \tag{3.2}$$

where D is Levi-Civita connection in $\mathbb{R}^7$. The connection D is metric and the second fundamental form symmetric, which gives us $g_{ij}^k = -g_{ik}^j$, and $h_{ij} = h_{ji}$.

Lemma 3.2. *([1]) For the previously defined coefficients we have*

$$\begin{aligned} &g_{12}^3=-g_{11}^4, \quad g_{12}^4=g_{11}^3, \quad h_{11}=-g_{12}^5, \quad h_{12}=g_{11}^5, \quad g_{22}^3=-g_{21}^4,\\ &g_{22}^4=g_{21}^3, \quad g_{22}^5=-g_{11}^5, \quad h_{22}=g_{21}^5, \quad g_{32}^3=-g_{31}^4, \quad g_{32}^4=g_{31}^3,\\ &h_{13}=1-g_{32}^5, \quad h_{23}=g_{31}^5, \quad g_{42}^3=-g_{41}^4, \quad g_{42}^4=g_{41}^3, \quad h_{14}=-g_{42}^5,\\ &h_{24}=-1+g_{41}^5, \quad g_{52}^3=-1-g_{51}^4, \quad g_{52}^4=g_{51}^3, \quad h_{15}=-g_{52}^5, \quad h_{25}=g_{51}^5,\\ &g_{32}^5=2+g_{14}^5, \quad g_{42}^5=-g_{13}^5, \quad g_{31}^5=-g_{24}^5, \quad g_{41}^5=2+g_{23}^5, \quad h_{33}=-g_{43}^5,\\ &h_{34}=g_{33}^5, \quad g_{44}^5=-g_{33}^5, \quad h_{44}=g_{43}^5, \quad h_{35}=-g_{54}^5, \quad h_{45}=g_{53}^5. \end{aligned} \tag{3.3}$$

Lemma 3.3. *([1]) The differentiable functions (3.2) satisfy*

$$\begin{aligned} &g_{52}^5=g_{11}^2+g_{13}^4, \quad g_{51}^5=-g_{21}^2-g_{23}^4, \quad g_{54}^5=g_{31}^2+g_{33}^4,\\ &g_{53}^5=-g_{41}^2-g_{43}^4, \quad h_{55}=-g_{51}^2-g_{53}^4. \end{aligned} \tag{3.4}$$

Since we still have a choice for $E_1 \in \mathcal{D}$, from now on let it be parallel to the projection of $A\xi$ on $\mathcal{D}$. Then there exist differentiable functions α and β such that

$$A\xi = \beta E_1 + \alpha\xi. \tag{3.5}$$

Since the components of $A\xi$ in direction of E_2, E_3, E_4 vanish, we have

$$g_{13}^4=-g_{11}^2-\beta, \quad g_{23}^4=-g_{21}^2, \quad g_{33}^4=-g_{31}^2, \quad g_{43}^4=-g_{41}^2, \quad g_{53}^4=-g_{51}^2-\alpha.$$

Now we will use the Gauss equations to obtain further relations between the coefficients.

Lemma 3.4. *For the coefficients (3.2) and β the following relations hold*

$$
\begin{aligned}
E_1(g_{21}^2) &= -1-(g_{11}^2)^2+2(g_{11}^5)^2-(g_{21}^2)^2-2g_{11}^4g_{21}^3+2g_{11}^3g_{21}^4+2g_{12}^5g_{21}^5\\
&- (g_{11}^4+g_{21}^3)g_{31}^2+g_{11}^3g_{41}^2-g_{21}^4g_{41}^2+g_{12}^5g_{51}^2-g_{21}^5g_{51}^2+E_2(g_{11}^2),\\
E_1(g_{31}^2) &= -g_{11}^5(3+2g_{14}^5)-2g_{12}^5g_{24}^5+g_{11}^4(g_{21}^2-2g_{31}^3)-g_{31}^2(g_{21}^2+g_{31}^3)\\
&+ 2g_{11}^3g_{31}^4-g_{11}^2(g_{11}^3+g_{41}^2)+(g_{13}^5+g_{24}^5)g_{51}^2-g_{41}^2(g_{31}^4+\beta)+E_3(g_{11}^2),\\
E_1(g_{41}^2) &= 3g_{12}^5+2g_{11}^5g_{13}^5-g_{11}^3g_{21}^2+2g_{12}^5g_{23}^5+g_{11}^2(-g_{11}^4+g_{31}^2)-g_{21}^2g_{41}^2-g_{31}^2g_{41}^3\\
&- 2g_{11}^4g_{41}^3+2g_{11}^3g_{41}^4-g_{41}^2g_{41}^4-2g_{51}^2+g_{14}^5g_{51}^2-g_{23}^5g_{51}^2+g_{31}^2\beta+E_4(g_{11}^2),\\
E_1(\beta) &= g_{13}^5-2g_{11}^5g_{51}^2+2g_{51}^3+2g_{14}^5g_{51}^3-2g_{13}^5g_{51}^4+g_{11}^5\alpha-\xi(g_{12}^5),\\
E_2(\beta) &= 2(g_{11}^5)^2+g_{14}^5+2g_{12}^5g_{21}^5+g_{23}^5(1+2g_{14}^5)-2g_{13}^5g_{24}^5+(g_{21}^5-g_{12}^5)\alpha+g_{11}^2\beta,\\
E_3(\beta) &= -g_{11}^5(3+2g_{14}^5)-2g_{12}^5g_{24}^5+g_{33}^5(1+2g_{14}^5)-2g_{13}^5g_{34}^5-(g_{13}^5+g_{24}^5)\alpha+g_{11}^3\beta,\\
E_4(\beta) &= 2g_{13}^5(g_{11}^5+g_{33}^5)+g_{12}^5(3+2g_{23}^5)+g_{43}^5(1+2g_{14}^5)+(g_{23}^5-g_{14}^5+2)\alpha+g_{11}^4\beta,
\end{aligned}
$$

Proof. The first three equations are obtained by taking $X=E_1$, $Y=E_i$, $Z=E_1$ and $W=E_2$, $i=2,3,4$ into (2.2), respectively.
The last four equations are obtained from (2.2), respectively, for $(X,Y,Z,W)=(E_1,\xi,E_2,\xi)$,$(X,Y,Z,W)=(E_1,E_2,E_3,E_4)$, $(X,Y,Z,W)=(E_1,E_3,E_3,E_4)$ and $(X,Y,Z,W)=(E_1,E_4,E_3,E_4)$. $\square$.

Now, further computation of the Gaussian equation for various choices of vector fields appearing in it, the covariant derivatives of some of the coefficients in the direction of the vector field ξ are obtained.

Lemma 3.5. *The functions (3.2) satisfy*

$$
\begin{aligned}
\xi(g_{11}^5) &= 1+(g_{11}^5)^2+g_{12}^5g_{21}^5-g_{13}^5g_{24}^5+g_{12}^5g_{51}^2+g_{21}^5g_{51}^2+g_{13}^5g_{51}^3-g_{24}^5g_{51}^3\\
&+ 2g_{51}^4+g_{23}^5g_{51}^4+g_{14}^5(2+g_{23}^5+g_{51}^4)-g_{12}^5\alpha+g_{11}^2\beta-\beta^2,\\
\xi(g_{13}^5) &= g_{13}^5g_{33}^5+g_{14}^5g_{43}^5-g_{14}^5g_{51}^2+g_{23}^5g_{51}^2+g_{11}^5(g_{13}^5-g_{51}^3)-\alpha+g_{33}^5g_{51}^3\\
&+ g_{43}^5g_{51}^4+g_{12}^5(1+g_{23}^5+g_{51}^4)-2g_{14}^5\alpha+g_{11}^4\beta,\\
\xi(g_{14}^5) &= -g_{14}^5g_{33}^5+g_{13}^5g_{34}^5+g_{13}^5g_{51}^2+g_{24}^5g_{51}^2+g_{12}^5(g_{24}^5-g_{51}^3)+g_{34}^5g_{51}^3\\
&+ g_{11}^5(g_{14}^5-g_{51}^4)-g_{33}^5g_{51}^4+2g_{13}^5\alpha-g_{11}^3\beta,\\
\xi(g_{21}^5) &= 2g_{51}^3+2g_{23}^5g_{51}^3+g_{24}^5(3+2g_{51}^4)+g_{11}^5(-2g_{51}^2+\alpha)+g_{21}^2\beta,\\
\xi(g_{23}^5) &= -g_{33}^5+g_{23}^5g_{33}^5+g_{24}^5g_{43}^5+g_{43}^5g_{51}^3-g_{24}^5g_{51}^2-g_{21}^5g_{51}^3-g_{33}^5g_{51}^4\\
&+ g_{13}^5(g_{21}^5-g_{51}^2)-g_{11}^5(1+g_{23}^5+g_{51}^4)-2g_{24}^5\alpha+g_{21}^4\beta,\\
\xi(g_{24}^5) &= -g_{24}^5g_{33}^5-g_{34}^5+g_{23}^5g_{34}^5+g_{14}^5(g_{21}^5-g_{51}^2)+g_{23}^5g_{51}^2-g_{33}^5g_{51}^3+\alpha\\
&+ g_{11}^5(-g_{24}^5+g_{51}^3)-g_{21}^5g_{51}^4-g_{34}^5g_{51}^4+2g_{23}^5\alpha-g_{21}^3\beta,\\
\xi(g_{33}^5) &= 3-g_{13}^5g_{24}^5+g_{34}^5g_{43}^5-g_{34}^5g_{51}^2-g_{43}^5g_{51}^2-g_{13}^5g_{51}^3+g_{24}^5g_{51}^3+(g_{33}^5)^2\\
&+ 2g_{51}^4+g_{23}^5(3+g_{51}^4)+g_{14}^5(1+g_{23}^5+g_{51}^4)-2g_{34}^5\alpha-g_{43}^5\alpha+g_{31}^4\beta,\\
\xi(g_{34}^5) &= 2g_{33}^5g_{51}^2-2g_{51}^3-2g_{14}^5g_{51}^3+g_{24}^5(3+2g_{51}^4)+3g_{33}^5\alpha-g_{31}^3\beta,\\
\xi(g_{43}^5) &= g_{13}^5+2g_{33}^5g_{51}^2-2g_{51}^3-2g_{23}^5g_{51}^3-2g_{13}^5g_{51}^4+3g_{33}^5\alpha+g_{41}^4\beta,
\end{aligned}
$$

Proof. By taking in particular $(X,Y,Z,W)=(E_1,E_5,E_j,E_5)$, $j=1,3,5$, in (2.2), respectively, as a result we obtain the first three equations from the lemma. We get the next three equations from (2.2) for $(X,Y,Z,W)=(E_2,E_5,E_k,E_5)$, $k=1,3,4$, respectively. We get the last three equations from the lemma from (2.2) for $(X,Y,Z,W)=(E_3,E_5,E_3,E_5)$, $(X,Y,Z,W)=(E_3,E_5,E_4,E_5)$ and $(X,Y,Z,W)=(E_4,E_5,E_3,E_5)$. □

4. Proof of the Main theorem

Let M be a Hopf real hypersurface with a structure Jacobi operator of Codazzi type. The condition that the structure Jacobi operator is of Codazzi type is equivalent to

$$(\nabla_{E_i}l)E_j-(\nabla_{E_j}l)E_i=0,\quad i,j=1,...,5.$$

Since M is Hopf, from (3.5) we have that $\beta=0$ so from Lemma 3.4 we get

$$\begin{aligned}
&(4.1)\quad & 0 &= E_1(\beta)=g^5_{13}-2g^5_{11}g^2_{51}+2g^3_{51}+2g^5_{14}g^3_{51}-2g^5_{13}g^4_{51}\\
&&&+\ g^5_{11}\alpha-\xi(g^5_{12}),\\
&(4.2)\quad & 0 &= E_2(\beta)=2(g^5_{11})^2+g^5_{14}+2g^5_{12}g^5_{21}+g^5_{23}(1+2g^5_{14})-2g^5_{13}g^5_{24}\\
&&&+\ (g^5_{21}-g^5_{12})\alpha,\\
&(4.3)\quad & 0 &= E_3(\beta)=-g^5_{11}(3+2g^5_{14})-2g^5_{12}g^5_{24}+g^5_{33}(1+2g^5_{14})-2g^5_{13}g^5_{34}\\
&&&-\ (g^5_{13}+g^5_{24})\alpha,\\
&(4.4)\quad & 0 &= E_4(\beta)=2g^5_{13}(g^5_{11}+g^5_{33})+g^5_{12}(3+2g^5_{23})+g^5_{43}(1+2g^5_{14})\\
&&&+\ (g^5_{23}-g^5_{14}+2)\alpha,
\end{aligned}$$

and then from (4.1) we obtain

$$\xi(g^5_{12})=g^5_{13}-2g^5_{11}g^2_{51}+2g^3_{51}+2g^5_{14}g^3_{51}-2g^5_{13}g^4_{51}+g^5_{11}\alpha. \tag{4.5}$$

Also, since α is now constant, we have $E_i(\alpha)=0$, $i=1,...,5$.

From

$$\begin{aligned}
0 &= g\left((\nabla_{E_1}l)\xi-(\nabla_\xi l)E_1,E_3\right)=g^5_{13}(1+\alpha^2),\\
0 &= g\left((\nabla_{E_2}l)\xi-(\nabla_\xi l)E_2,E_3\right)=g^5_{23}(1+\alpha^2),
\end{aligned}$$

we obtain $g^5_{13}=0$ and $g^5_{23}=0$. Next, from

$$\begin{aligned}
0 &= g\left((\nabla_{E_1}l)\xi-(\nabla_\xi l)E_1,E_2\right)=g^5_{12}(1+\alpha^2),\\
0 &= g\left((\nabla_{E_2}l)\xi-(\nabla_\xi l)E_2,E_2\right)=g^5_{11}(1+\alpha^2),
\end{aligned}$$

we get $g^5_{12}=0$ and $g^5_{11}=0$ so from (4.5) and Lemma 3.5, respectively, we obtain

$$\begin{aligned}
&(4.6)\quad & 0 &= \xi(g^5_{12})=2(1+g^5_{14})g^3_{51},\\
&(4.7)\quad & 0 &= \xi(g^5_{11})=1+g^5_{21}g^2_{51}-g^5_{24}g^3_{51}+2g^4_{51}+g^5_{14}(2+g^4_{51}).
\end{aligned}$$

Now let's look at (4.6). If we assume that $g_{14}^5 = -1$, from $0 = g((\nabla_{E_1} l)E_2 - (\nabla_{E_2} l)E_1, \xi) = -g_{21}^5 - \alpha$ we get $g_{21}^5 = -\alpha$ so (4.2) becomes $0 = -1 - \alpha^2$, which is a contradiction. Therefore, it must be $g_{51}^3 = 0$.

Further from (4.2) we get $g_{14}^5 + g_{21}^5\alpha = 0$, so $g_{14}^5 = -g_{21}^5\alpha$ so then $0 = g((\nabla_{E_1} l)E_2 - (\nabla_{E_2} l)E_1, \xi) = -g_{21}^5(1 + \alpha^2)$ gives us $g_{21}^5 = 0$, so, from Lemma 3.5, we get

$$0 = \xi(g_{21}^5) = g_{24}^5(3 + 2g_{51}^4). \tag{4.8}$$

From (4.4) we obtain $g_{43}^5 = -2\alpha$. Then from (4.7) we get $g_{51}^4 = -\frac{1}{2}$ and from (4.8) we have $g_{24}^5 = 0$. Next, (4.3) becomes $g_{33}^5 = 0$ and from $0 = g((\nabla_{E_2} l)\xi - (\nabla_\xi l)E_2, E_1) = \frac{1}{2}(g_{34}^5 - 2\alpha)$ we obtain $g_{34}^5 = 2\alpha$. Finally from

$$0 = g((\nabla_{E_3} l)\xi - (\nabla_\xi l)E_3, E_2) = 2(1 + \alpha^2)$$

we get a contradiction, which completes the proof of the theorem.

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