FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 1 (2016), 237–244

GENERALIZATION OF CERTAIN RESULTS ON PROJECTIVE MOTION IN A FINSLER SPACE

P. N. Pandey and Sweta Kumari*

Abstract. The paper contains a study of an infinitesimal projective motion in a Finsler space $F^n(n > 2)$, which leaves invariant the skew-symmetric part of the covariant derivative of projective deviation tensor and it is proved that either the Finsler space $F^n(n > 2)$ admitting such projective motion is a space of scalar curvature or the infinitesimal projective motion is necessarily an affine motion. It is established that an infinitesimal projective motion in a projectively flat as well as in a non-Riemannian symmetric Finsler space of dimension greater than 2, is necessarily an affine motion while a symmetric Finsler space $F^n(n > 2)$ admitting a non-affine projective motion is a Riemannian space of constant Riemannian curvature. An infinitesimal projective motion in a Finsler space $F^n(n > 2)$ of recurrent projective deviation tensor is an affine motion if the projective motion leaves invariant the recurrence vector of the space. It is further proved that such result also holds in case of projective recurrent and recurrent Finsler space of dimension greater than 2.
Keywords: Finsler space, Projective motion, projectively symmetric space, recurrent space, projective recurrent space.

1. Introduction

K. Yano and T. Nagano [1] discussed an infinitesimal projective motion in an n-dimensional Riemannian space $V^n(n > 2)$ which leaves invariant the covariant derivative of Weyl's projective curvature tensor. It was proved by them that if the space is not of constant curvature then the projective motion is necessarily an affine motion. They also discussed an infinitesimal projective motion in a symmetric Riemannian space $V^n(n > 2)$ and proved that such space is of constant curvature if the projective motion is non-affine. P. N. Pandey [2] extended these results to an n-dimensional Finsler space F^n which leaves invariant the covariant derivative of Weyl's projective tensor W^i_{jkh}. He also generalized these results by considering an infinitesimal projective motion which leaves invariant the covariant derivative of the projective deviation tensor. This condition is certainly weaker than the earlier.

Received October 12, 2015; Accepted December 11, 2015
2010 *Mathematics Subject Classification.* 53B40
*Corresponding author.

Similar results were also found by P. N. Pandey [2] for a projectively symmetric Finsler space which is more general than a symmetric Finsler space. The aim of the present paper is to generalize the results of the first author (P. N. Pandey [2]) by considering an infinitesimal projective motion which leaves invariant the skew-symmetric part of the covariant derivation tensor. We also aim to prove certain results for recurrent and projective recurrent Finsler spaces of dimension greater than 2, admitting an infinitesimal projective motion.

2. Preliminaries

Let us consider a Finsler space $F^n (n > 2)$ of dimension n having F as a metric function satisfying the requisite conditions [3]. Let us denote the components of corresponding metric tensor, Berwald's connection parameters, components of Berwald's curvature tensor and components of projective curvature tensor by $g_{ij}, G^i_{jk}, H^i_{jkh}$ *and* W^i_{jkh} respectively. H^i_{jkh} *and* W^i_{jkh} are positively homogeneous of degree 0 in y^i and are skew-symmetric in their last two lower indices. Following tensors are obtained after transvection of curvature and projective curvature tensors by the directional arguments y^i of the line element (x^i, y^i):

$$\text{(2.1)} \qquad \begin{aligned} &(a)\ H^i_{kh} = H^i_{jkh} y^j, \quad (b)\ H^i_h = H^i_{kh} y^k, \\ &(c) W^i_{kh} = W^i_{jkh} y^j, \quad (d)\ W^i_h = W^i_{kh} y^k. \end{aligned}$$

The tensors H^i_h and W^i_h are called deviation tensor and projective deviation tensor respectively.

The above tensors are also related by

$$\text{(2.2)} \qquad \begin{aligned} &(a) H^i_{jkh} = \dot{\partial}_j H^i_{kh}, \qquad (b) H^i_{kh} = \frac{1}{3}(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k), \\ &(c) W^i_{jkh} = \dot{\partial}_j W^i_{kh}, \qquad (d) W^i_{kh} = \frac{1}{3}(\dot{\partial}_k W^i_h - \dot{\partial}_h W^i_k) \\ &(e) W^i_h = H^i_h - H\delta^i_h - \frac{y^i}{n+1}(\dot{\partial}_r H^r_h - \dot{\partial}_h H) \end{aligned}$$

where H is scalar curvature defined by $H = \frac{1}{(n-1)} H^i_i$ and $\dot{\partial}_r \equiv \frac{\partial}{\partial y^r}$.

The directional differential operator $\dot{\partial}_k$ and Berwald covariant differential operator $\mathfrak{B}_\mathfrak{h}$ satisfy the commutation formula:

$$\text{(2.3)} \qquad \dot{\partial}_k \mathfrak{B}_h T^i_j - \mathfrak{B}_h \dot{\partial}_k T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj}\,,$$

where T^i_j denote the components of an arbitrary tensor of type $(1,1)$ and $G^i_{khr} = \dot{\partial}_r G^i_{kh}$ are components of a symmetric tensor i.e. $G^i_{jkh} = G^i_{khj} = G^i_{hjk}$. A Finsler space F^n is called symmetric or projectively symmetric according as [4]

$$\text{(2.4)} \qquad \mathfrak{B}_m H^i_{jkh} = 0, \quad H^i_{jkh} \neq 0.$$

or

$$\mathfrak{B}_m W^i_{jkh} = 0, \quad W^i_{jkh} \neq 0. \tag{2.5}$$

A Finsler space F^n is called recurrent or projectively recurrent according as

$$\mathfrak{B}_m H^i_{jkh} = \lambda_m H^i_{jkh}, \quad H^i_{jkh} \neq 0. \tag{2.6}$$

or

$$\mathfrak{B}_m W^i_{jkh} = \lambda_m W^i_{jkh}, \quad W^i_{jkh} \neq 0. \tag{2.7}$$

The vector $\lambda_m (\neq 0)$ is called recurrence vector. It was established by P. N. Pandey [5] that the recurrence vector of a recurrent space is independent of directional arguments provided the scalar curvature, $H \neq 0$. The tensors W^i_{kh} and W^i_h of a recurrent as well as a projectively recurrent Finsler space are recurrent, i.e.

$$(a) \mathfrak{B}_m W^i_{kh} = \lambda_m W^i_{kh}, \qquad (b) \mathfrak{B}_m W^i_h = \lambda_m W^i_h. \tag{2.8}$$

Let us consider an infinitesimal transformation $T : F^n \to F^n$ such that $T(x^i) = \widetilde{x}^i$ and

$$\widetilde{x}^i = x^i + \epsilon v^i(x^j) \tag{2.9}$$

where ϵ is an infinitesimal constant. Let us denote the operator of Lie- differentiation with respect to the above transformation by £. The commutation formula for the operators £ and $\mathfrak{B}_m$ is given by

$$£\mathfrak{B}_m T^i_j - \mathfrak{B}_m £ T^i_j = T^r_j £ G^i_{rm} - T^i_r £ G^r_{jm} - (\dot{\partial}_r T^i_j) £ G^r_{mh} y^h \tag{2.10}$$

where T^i_j are components of an arbitrary tensor of type $(1, 1)$.
The transformation (2.9) is known as an affine motion or a projective motion if it preserves the parallelism of vectors or geodesics respectively. The necessary and sufficient condition for the transformation (2.9) to be an affine motion is

$$£ G^i_{jk} = 0 \tag{2.11}$$

while the necessary and sufficient condition for the transformation (2.9) to be a projective motion is

$$£ G^i_{jk} = y^i p_{jk} + p_j \delta^i_k + p_k \delta^i_j, \tag{2.12}$$

where

$$(a)\ p_j = \dot{\partial}_j p, \qquad (b)\ p_{jk} = \dot{\partial}_j \dot{\partial}_k p, \tag{2.13}$$

and p is a scalar invariant which is positively homogeneous in y^i of degree 1. The vector p_j and the tensor p_{jk} satisfy

$$(a)\ p_j y^j = p, \qquad (b)\ p_{jk} y^k = 0. \tag{2.14}$$

Every affine motion is a projective motion trivially. A non-affine projective motion is characterized by (2.12) and $p \neq 0$.

3. Projective Motion in Symmetric and Projectively Symmetric Finsler Spaces

Let a Finsler space $F^n(n > 2)$ admits an infinitesimal projective motion. Then, we have (2.12) together with (2.13). The integrability condition [3] of (2.12) is given by

$$\pounds W^i_{jkh} = 0. \tag{3.1}$$

Transvecting (3.1) by y^j and using (2.1c), we get

$$\pounds W^i_{kh} = 0. \tag{3.2}$$

Again transvecting (3.2) by y^k and using (2.1d), we get

$$\pounds W^i_h = 0. \tag{3.3}$$

Replacing T^i_j in the commutation formula (2.10) by W^i_j and using (3.3), we have

$$\pounds \mathfrak{B}_m W^i_j = W^r_j \pounds G^i_{rm} - W^i_r \pounds G^r_{jm} - (\dot{\partial} r W^i_j) \pounds G^r_{ms} y^s \tag{3.4}$$

which in view of (2.12) yields

$$\pounds \mathfrak{B}_m W^i_j = y^i W^r_j p_{rm} + p_r W^r_j \delta^i_m - p_j W^i_m - 2p_m W^i_j - (\dot{\partial}_m W^i_j) p. \tag{3.5}$$

Taking skew-symmetric part of (3.5) with respect to the indices m and j, we obtain

$$\begin{aligned} \pounds(\mathfrak{B}_m W^i_j - \mathfrak{B}_j W^i_m) = {} & y^i W^r_j p_{rm} - y^i W^r_m p_{rj} + p_r (W^r_j \delta^i_m - W^r_m \delta^i_j) \\ & + (p_j W^i_m - p_m W^i_j) - p(\dot{\partial}_m W^i_j - \dot{\partial}_j W^i_m). \end{aligned} \tag{3.6}$$

Under the assumption that the left hand side of (3.6) equals to zero, i.e. $\pounds(\mathfrak{B}_m W^i_k - \mathfrak{B}_k W^i_m) = 0$, it follows that

$$y^i W^r_j p_{rm} - y^i W^r_m p_{rj} + p_r (W^r_j \delta^i_m - W^r_m \delta^i_j) + (p_j W^i_m - p_m W^i_j) - 3p W^i_{mj} = 0. \tag{3.7}$$

Transvecting (3.7) by y^m and using (2.14a), we conclude

$$y^i p_r W^r_j - 4p W^i_j = 0. \tag{3.8}$$

Transvecting (3.8) by p_i and using (2.14a), we get

$$3p p_r W^r_j = 0. \tag{3.9}$$

This implies at least one of the following conditions:

$$(a)\ \ p = 0, \qquad (b)\ \ p_r W^r_j = 0. \tag{3.10}$$

If (3.10a) holds, the projective motion is an affine motion. If (3.10a) does not hold, (3.10b) must hold. Using (3.10b) in (3.8), we get $pW^i_j = 0$, which implies $W^i_j = 0$ for $p \neq 0$. Z. I. Szabo [6] and P. N. Pandey [7] proved that a Finsler space $F^n(n > 2)$ whose projective deviation tensor vanishes identically, is a Finsler space of scalar curvature. Therefore, the space considered is a Finsler space of scalar curvature. Thus, the next statement holds.

Theorem 3.1. *A Finsler space $F^n(n > 2)$ admitting an infinitesimal projective motion which leaves invariant the skew-symmetric part of the covariant derivative of projective deviation tensor, i.e. $\pounds(\mathfrak{B}_m W^i_k - \mathfrak{B}_k W^i_m) = 0$, is a space of scalar curvature or the projective motion is an affine motion.*

Let us consider a Finsler space $F^n(n > 2)$ which satisfies $\mathfrak{B}_m W^i_k = 0, W^i_k \neq 0$. Then, the condition $\pounds(\mathfrak{B}_m W^i_k - \mathfrak{B}_k W^i_m) = 0$ is trivially satisfied. The Finsler space considered cannot be of scalar curvature because a Finsler space of scalar curvature is projectively flat, which gives $W^i_h = 0$. Therefore, in view of the above theorem, the next result can be stated.

Theorem 3.2. *If the projective deviation tensor W^i_h of a Finsler space $F^n(n > 2)$ satisfies $\mathfrak{B}_m W^i_h = 0$, an infinitesimal projective motion in such space is necessarily an affine motion.*

Since, the projective deviation tensor of a projective symmetric space satisfies $\mathfrak{B}_m W^i_h = 0$, it is possible to conclude:

Theorem 3.3. *An infinitesimal projective motion in a projectively symmetric space $F^n(n > 2)$ is necessarily an affine motion.*

Let $F^n(n > 2)$ be a symmetric Finsler space characterised by (2.4). If it admits an infinitesimal projective motion, then we have (2.12). On transvecting (2.4) by y^j and using (2.1*a*), we get

$$\mathfrak{B}_m H^i_{kh} = 0. \tag{3.11}$$

Differentiating (3.11) partially with respect to y^j and using (2.3) , we get

$$H^r_{kh} G^i_{jmr} - H^i_{rh} G^r_{jmk} - H^i_{kr} G^r_{jmh} = 0. \tag{3.12}$$

Transvecting (3.12) by y^k and using (2.1*a*) and (2.1*b*), we get

$$H^r_h G^i_{jmr} - H^i_r G^r_{jmh} = 0. \tag{3.13}$$

Transvecting (3.11) by y^k and using (2.1*b*), we get

$$\mathfrak{B}_m H^i_h = 0. \tag{3.14}$$

Differentiating (3.14) partially with respect to y^k and using (2.3) and (3.13), we have

$$\mathfrak{B}_m \dot{\partial}_k H^i_h = 0. \tag{3.15}$$

Contracting the indices i and h in equations (3.14) and (3.15), one can verify

$$\mathfrak{B}_m H = 0 \quad \textit{and} \quad \mathfrak{B}_m \dot{\partial}_k H = 0. \tag{3.16}$$

Contracting the indices i and h in equation (3.15), we conclude

$$\mathfrak{B}_m \dot{\partial}_r H^r_h = 0. \tag{3.17}$$

Differentiating (2.2 e) covariantly with respect to x^m and using (3.14), (3.15), (3.16) and (3.17), we get

$$\mathfrak{B}_m W^i_h = 0. \tag{3.18}$$

which implies

$$£\mathfrak{B}_m W^i_h = 0. \tag{3.19}$$

Taking skew-symmetric part of (3.19) with respect to the indices m and h, the next follows:

$$£(\mathfrak{B}_m W^i_h - \mathfrak{B}_h W^i_m) = 0. \tag{3.20}$$

In view of Theorem 3.1, either the projective motion is an affine motion or the space is of scalar curvature. P. N. Pandey [8] established that a symmetric Finsler space F^n of scalar curvature is a Riemannian space of constant Riemannian curvature. Hence, we conclude:

Theorem 3.4. *An infinitesimal projective motion in a non-Riemannian symmetric Finsler space $F^n(n > 2)$ is necessarily an affine motion.*

Theorem 3.5. *A symmetric Finsler space $F^n(n > 2)$ admitting a non-affine projective motion is a Riemannian space of constant Riemannian curvature.*

4. Projective Motion in Recurrent and Projective Recurrent Finsler Spaces

Let us consider a Finsler space $F^n(n > 2)$ whose projective deviation tensor W^i_h is recurrent. Such Finsler space is characterized by (2.8b) and $W^i_h \neq 0$. Suppose that it admits an infinitesimal projective motion (2.9). Then, we have (2.12) together with (2.13). Operating equation (2.8b) by the Lie differential operator £, we find $£\mathfrak{B}_m W^i_h = £(\lambda_m W^i_h)$. This implies

$$£\mathfrak{B}_m W^i_h = (£\lambda_m) W^i_h. \tag{4.1}$$

If $£\lambda_m = 0$, equation (4.1) implies $£\mathfrak{B}_m W^i_h = 0$. Therefore, $£(\mathfrak{B}_m W^i_h - \mathfrak{B}_h W^i_m) = 0$. In view of Theorem 3.1, this implies that either the infinitesimal projective motion is an affine motion or the space is of scalar curvature. The space considered can not be of scalar curvature for the projective deviation tensor W^i_h of a space of scalar curvature necessarily vanishes which contradicts our assumption $W^i_h \neq 0$. Therefore, the projective motion is necessarily affine.

This leads to:

Theorem 4.1. *An infinitesimal projective motion in a Finsler space $F^n(n > 2)$ of recurrent projective deviation tensor, which leaves the recurrence vector invariant, is necessarily an affine motion.*

A projective recurrent Finsler space $F^n(n > 2)$ necessarily satisfies (2.8*b*) [9]. Therefore, in view of Theorem 4.1, we have:

Corollary 4.1. *An infinitesimal projective motion in a projective recurrent Finsler space $F^n(n > 2)$ is an affine motion if the infinitesimal projective motion leaves the recurrence vector of the projective recurrent space invariant.*

Let us consider a recurrent Finsler space F^n characterised by (2.6). Transvecting (2.6) by y^j and using (2.1*a*), we have

$$\mathfrak{B}_m H^i_{kh} = \lambda_m H^i_{kh} \tag{4.2}$$

Transvecting (4.2) by y^k and using (2.1*b*), we get

$$\mathfrak{B}_m H^i_h = \lambda_m H^i_h. \tag{4.3}$$

Contracting the indices i and h in (4.3), we get

$$\mathfrak{B}_m H = \lambda_m H \tag{4.4}$$

where $H^r_r = (n-1)H$. Differentiating (2.2*e*) covariantly with respect to y^m and using (4.3) and (4.4), we have

$$\mathfrak{B}_m W^i_h = \lambda_m (H^i_h - H\delta^i_h) - \frac{y^i}{n+1}(\mathfrak{B}_m \dot{\partial}_r H^r_h - \mathfrak{B}_m \dot{\partial}_r H) \tag{4.5}$$

Using the commutation formula exhibited by (2.3) and using (4.3), (4.4) and (2.2*e*), we get

$$\mathfrak{B}_m W^i_h = \lambda_m W^i_h - \frac{y^i}{n+1}(H^r_s G^s_{hmr} - H^s_h G^r_{rms}). \tag{4.6}$$

P. N. Pandey [5] proved that a recurrent space admits the identity (3.13). Thus, equation (4.6) gives (2.8 b). Therefore, in view of Theorem 4.1, we have the next result.

Theorem 4.2. *An infinitesimal projective motion in a recurrent Finsler space $F^n(n > 2)$ which leaves the recurrence vector invariant is necessarily an affine motion.*

REFERENCES

[1] K. Yano and T. Nagano, Some theorems on projective and conformal transformations, *Nederl. Akad. Wetensch. Proc. Ser. A. 60= Indag. Math.*, **19** (1957), 451-458.

[2] P. N. Pandey, Projective motion in a symmetric and projectively symmetric Finsler manifold , *Proc. Nat. Acad. Sci.*, **54(A)** (1984), 274-278.

[3] H. Rund, The differential Geometry of Finsler spaces, *Springer- Verlag*, (1959).

[4] R. B. Misra, A projective symmetric Finsler space, *Math. Z.*, **126** (1972), 143-153.

[5] P. N. Pandey, A note on recurrence vector, *Proc. Nat. Acad. Sci.*, **50 (A)** (1980), 6-8.

[6] Z. I. Szabo, Ein Finslerscher Raum ist gerade dann von skalarer Krümmung, wenn seine Weylsche Projektivkrümmung verschwindet, *Acta Sci. Math. Szeged*, **39** (1977), 163-168.

[7] P. N. Pandey, On a Finsler space of zero projective curvature,*Acta. Math. acad. Sci. Hungar*, **39(4)**(1982), 387-388.

[8] P. N. Pandey, On Some Finsler spaces of scalar curvature, *Prog. of Maths.*, **18(1)** (1984), 41-48.

[9] P. N. Pandey and R. B. Misra, Projective recurrent Finsler manifolds I , *Publications Mathematicae Debrecen*, **28(3-4)** (1981), 191-198.

S. Kumari
Department of Mathematics
University of Allahabad
Allahabad- 211002, India
jhasweta735@gmail.com

P. N. Pandey
Department of Mathematics
University of Allahabad
Allahabad- 211002, India
pnpiaps@gmail.com