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DYNAMIC STABILITY OF BEAMS ON PASTERNAK FOUNDATION UNDER TIME-VARYING AXIAL LOADS

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Abstract. This paper investigates the dynamic stability of structures subjected to periodic loads, modeled as beams on a Pasternak foundation experiencing time-varying compressive forces. The stability analysis is conducted using the Euler-Bernoulli beam theory, the Mathieu-Hill equations, and the Floquet theory. The results indicate that variations in the foundation's stiffness and shear modulus significantly influence stability regions, especially at higher frequencies. Stiffness has a more pronounced effect, reducing the unstable region, while changes in both parameters affect the minimum excitation intensity required to induce instability. These findings highlight complex interactions between stiffness and shear properties, suggesting the need for further investigation.
Keywords: dynamic stability, Pasternak foundation, Euler-Bernoulli beam theory.

1. Introduction

The dynamic stability of structures under periodic loads plays a critical role in the analysis and design of engineering systems. This stability refers to a structure's ability to maintain equilibrium when subjected to forces that vary periodically over time. In engineering contexts, such loads can manifest as sinusoidal, pulsed, or

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more complex forms and Mathieu-Hill equations are frequently employed for the mathematical modeling of structures subjected to periodic loading.

Early interest in the dynamic instability of Euler beams is well documented. For instance, [2] examined the dynamic response of simply supported Euler beams under axial loading, calculating the primary frequencies of parametric resonance and deriving the Mathieu-Hill equation for structural dynamic instability. Krylov and Bogoliubov [8, 9] extended this analysis by studying the impact of arbitrary boundary conditions on the dynamic instability of Euler beams using the Galerkin method. Then Bolotin [3] provided an in-depth exploration of the dynamic instability of structural components subjected to axial or in-plane periodic loads. The Mathieu-Hill equation was derived in [5] to model the parametric vibrations of beams under compressive dynamic forces and Nayfeh and Mook [11] employed the perturbation method to solve the Mathieu-Hill equation, analyzing the behavior of elastic systems under parametric excitation.

In various engineering fields, structural components such as homogeneous plates and beams resting on elastic foundations are commonly used. In Ref. [1], the effect of an elastic foundation on the dynamic stability of columns was explored, while [4] examined the dynamic stability of beams under axial loading resting on an elastic base with damping. It was demonstrated that increasing the damping or stiffness of the foundation raises the critical dynamic load and shifts the unstable regions to higher applied frequencies. Lee [10] studied the dynamic instability of a tapered cantilever beam on an elastic foundation, while Subba Ratnam et al. [12] explored the dynamic instability of beams on elastic foundations. Ying et al. [15], using the Floquet theory, Fourier series, and matrix eigenvalue analysis, investigated multi-mode coupled periodically supported beams under general harmonic excitation. In [6], the authors developed closed-form expressions, based on the Floquet theory, to predict the dynamic instability regions of slender Euler-Bernoulli columns.

This paper investigates the dynamic stability of structures under periodic loading, modelled as beams resting on a Pasternak foundation and subjected to time-varying compressive forces. The stability analysis is conducted using Euler-Bernoulli beam theory, the Mathieu-Hill equations, and the Floquet theory. Particular emphasis is placed on examining the influence of variations in the foundation's stiffness and shear modulus on the stability regions.

2. Formulation

The model presented in Fig. 2.1 consists of a beam of length L placed on a Pasternak foundation, where the beam is subjected to time-varying axial compressive forces. The Pasternak foundation is characterized by the stiffness of the elastic layer, defined by the coefficient K, and the shear influence of the layer, defined by the shear modulus G. The Euler-Bernoulli beam theory was applied to solve the problem

The differential equation of transverse vibrations of the system is shown in

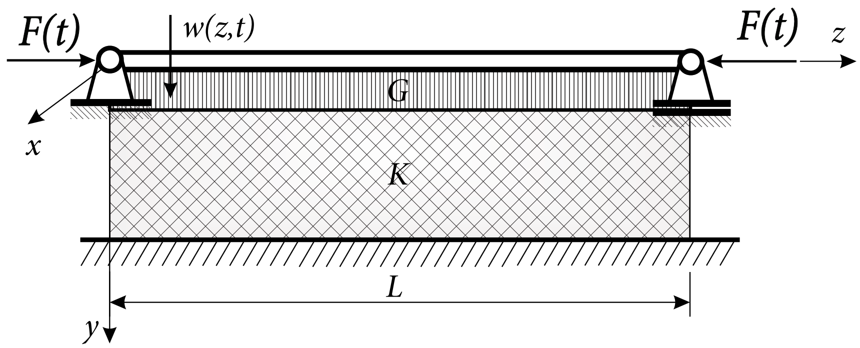


Fig. 2.1: Beam under axial loads

Fig. 2.1 is

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial z^4} + F(t) \frac{\partial^2 w}{\partial z^2} + Kw - G \frac{\partial^2 w}{\partial z^2} = 0 \tag{2.1}$$

where w represents the transverse beam deflection which is positive if downward, I and A the moment of inertia of the beam cross-section and the cross-sectional area of the beam, and E and ρ, Young's modulus and the mass density. The boundary conditions for a simply supported beam are

$$w(0,t) = w(L,t) = 0, \quad \frac{\partial^2 w(0,t)}{\partial z^2} = \frac{\partial^2 w(L,t)}{\partial z^2} = 0 \tag{2.2}$$

According to the Galerkin method, we assume the solution of equation (2.1) in the form of a product of functions

$$w(z,t) = \sum_{i=1}^{\infty} \chi_i(z) q_i(t), \quad \chi_i(z) = \sin(kz), \quad k = \frac{i\pi}{L}, \quad i = 1, 2, \ldots \tag{2.3}$$

where $q_i(t)$ are the unknown time functions and $\chi_i(z)$ are the modal functions of a simply supported beam satisfying the boundary conditions (2.2) and possessing the orthogonality property

$$\int_0^L \chi_i(z) \chi_j(z)\, dz = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{1}{2} L & \text{if } i = j. \end{cases} \tag{2.4}$$

Substituting the assumed solution (2.3) into equation (2.1), multiplying by $\chi_n(z)$, integrating with respect to z from 0 to L, and using the orthogonality of $\chi_i(z)$, we get the following equation:

$$A\rho \ddot{q}_n + (EIk^4 - F(t)k^2 + Gk^2 + K) q_n = 0, \quad n = 1, 2, \ldots$$

or

$$\ddot{q}_n + \omega_n^2(1 - P_n)q_n = 0, \quad n = 1, 2, \ldots \tag{2.5}$$

where

$$\omega_n = k^2\sqrt{\frac{EI}{A\rho}}, \quad P_n = \frac{F(t)k^2 - Gk^2 - K}{EIk^4}$$

Here, ω_n is the n-th natural frequency of a simply supported beam when $F(t) = 0$. Equation (2.6) represents the Hill equation when $F(t)$ is a periodic function of period T, i.e., $F(t) = F(t + T)$. Also, if $F(t)$ is a sinusoidal function, then the equation (2.6) is the Mathieu equation.

3. Stability of the Mathieu-Hill equation

Now using the theoretical foundations of the stability of the Mathieu-Hill equation given in [14], we investigate the stability of equation (2.6) in the case when the periodic load $F(t)$ is in the form of the rectangular pulse wave, as shown in Fig. 3.1.

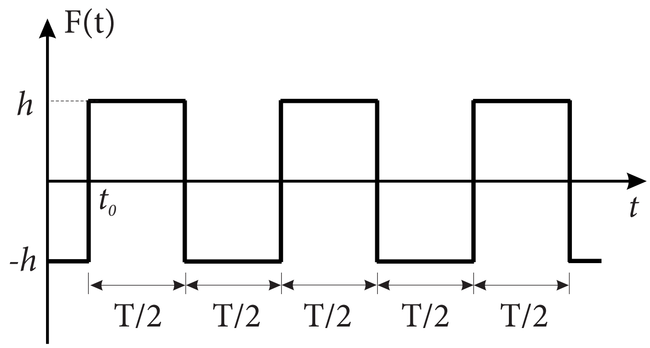


FIG. 3.1: Rectangular wave

Suppose that t_0 is the initial time and that $q(t_0)$ and $\dot{q}(t_0)$ are the initial conditions of the system.

Phase I: $t_0 \leq t \leq t_0 + \frac{1}{2}T$, $F(t) = h$, equation (2.6) becomes

$$\ddot{q} + \omega^2(1 - H_1)q = 0, \quad q(t_0) = q_0, \quad \dot{q}(t_0) = \dot{q}_0, \tag{3.1}$$

where $H_1 = \frac{hk^2 - Gk^2 - K}{EIk^4}$. Throughout this chapter, ω is used as a simplified notation for the natural frequency, focusing on the case $n = 1$, which represents the first mode of the system. This approach aligns with the analysis in the results section.

The characteristic equation for equation (3.1) is

$$m^2 + \omega^2(1 - H_1) = 0, \tag{3.2}$$

from which it follows that $m = \pm i\alpha$, $\alpha = \omega\sqrt{1 - H_1}$.

The solution of the equation (3.1) can be written in the form

$$q(t) = A\sin(\alpha t) + B\cos(\alpha t), \tag{3.3}$$

where A and B are unknown constants. From the initial conditions for $t = t_0$, we get that $A = \frac{\dot{q}_0}{\alpha}$ and $B = q_0$, so the solution (3.3) is of the following form:

$$q(t) = \frac{\dot{q}_0}{\alpha}\sin(\alpha t) + q_0\cos(\alpha t). \tag{3.4}$$

Considering the periodicity of the solution of the Mathieu-Hill equation, we can conclude that at the moment $t = t_0 + \frac{T}{2}$, the solution of equation (2.6) will be

$$q(t_0 + \frac{T}{2}) = q_{\frac{T}{2}} = \frac{\dot{q}_0}{\alpha}\sin\left(\frac{\alpha T}{2}\right) + q_0\cos\left(\frac{\alpha T}{2}\right),$$

$$\dot{q}(t_0 + \frac{T}{2}) = \dot{q}_{\frac{T}{2}} = \dot{q}_0\cos\left(\frac{\alpha T}{2}\right) - \alpha q_0\sin\left(\frac{\alpha T}{2}\right). \tag{3.5}$$

Phase II: $t_0 + \frac{1}{2}T \le t \le t_0 + T$, $F(t) = -h$, equation (2.6) becomes

$$\ddot{q} + \omega^2(1 - H_2)q = 0, \quad q(t_0) = q_0, \quad \dot{q}(t_0) = \dot{q}_0, \tag{3.6}$$

where $H_2 = \frac{-hk^2 - Gk^2 - K}{EIk^4}$.

The characteristic equation is

$$m^2 + \omega^2(1 - H_2) = 0, \tag{3.7}$$

from which it follows that $m = \pm i\beta$, $\beta = \omega\sqrt{1 - H_2}$. The solution to equation (3.6) can be written in the form

$$q(t) = C\sin(\beta t) + D\cos(\beta t), \tag{3.8}$$

where C and D are constants determined from the initial conditions $q(t_0 + \frac{T}{2}) = q_{\frac{T}{2}}$, $\dot{q}(t_0 + \frac{T}{2}) = \dot{q}_{\frac{T}{2}}$, and have the following form:

$$\begin{aligned} C &= q_{\frac{T}{2}}\sin\left(\frac{\beta T}{2}\right) + \frac{1}{\beta}\dot{q}_{\frac{T}{2}}\cos\left(\frac{\beta T}{2}\right), \\ D &= q_{\frac{T}{2}}\cos\left(\frac{\beta T}{2}\right) - \frac{1}{\beta}\dot{q}_{\frac{T}{2}}\sin\left(\frac{\beta T}{2}\right). \end{aligned} \tag{3.9}$$

Now the system solution at time $t = t_0 + T$ can be determined as

$$q(t_0 + T) = q_T = q_{\frac{T}{2}}\cos\left(\frac{\beta T}{2}\right) + \frac{1}{\beta}\dot{q}_{\frac{T}{2}}\sin\left(\frac{\beta T}{2}\right),$$

$$\dot{q}(t_0+T)=\dot{q}_T=-\beta q_{\frac{T}{2}}\sin\left(\frac{\beta T}{2}\right)+\dot{q}_{\frac{T}{2}}\cos\left(\frac{\beta T}{2}\right). \tag{3.10}$$

Substituting the expressions for $q_{\frac{T}{2}}$ and $\dot{q}_{\frac{T}{2}}$ into the expressions (3.10), we obtain the following matrix form:

$$\begin{pmatrix} q_T \\ \dot{q}_T \end{pmatrix} = A\begin{pmatrix} q_0 \\ \dot{q}_0 \end{pmatrix}, \tag{3.11}$$

where

$$\mathbf{A}=\begin{pmatrix} \cos\left(\frac{\alpha T}{2}\right)\cos\left(\frac{\beta T}{2}\right)-\frac{\alpha}{\beta}\sin\left(\frac{\alpha T}{2}\right)\sin\left(\frac{\beta T}{2}\right) & \frac{1}{\alpha}\sin\left(\frac{\alpha T}{2}\right)\cos\left(\frac{\beta T}{2}\right)+\frac{1}{\beta}\cos\left(\frac{\alpha T}{2}\right)\sin\left(\frac{\beta T}{2}\right) \\ -\beta\cos\left(\frac{\alpha T}{2}\right)\sin\left(\frac{\beta T}{2}\right)-\alpha\sin\left(\frac{\alpha T}{2}\right)\cos\left(\frac{\beta T}{2}\right) & -\frac{\beta}{\alpha}\sin\left(\frac{\alpha T}{2}\right)\sin\left(\frac{\beta T}{2}\right)+\cos\left(\frac{\alpha T}{2}\right)\cos\left(\frac{\beta T}{2}\right) \end{pmatrix}$$

The eigenvalues of matrix $\mathbf{A}$ are given by

$$\det(\mathbf{A}-\rho\mathbf{I})=\rho^2-2b\rho+c=0, \tag{3.12}$$

where

$$b=\cos\left(\frac{\alpha T}{2}\right)\cos\left(\frac{\beta T}{2}\right)-\frac{(\alpha^2+\beta^2)}{2\alpha\beta}\sin\left(\frac{\alpha T}{2}\right)\sin\left(\frac{\beta T}{2}\right), \quad c=1.$$

The characteristic roots of (3.12) are

$$\rho_1,\rho_2=b\pm\sqrt{b^2-1}.$$

As explained in [14], depending on the value of b, there are three possibilities for ρ_1 and ρ_2:

- $|b|>1$, ρ_1 and ρ_2 are real and distinct, leading to instability of the solution of equation (2.6).
- $|b|=1$, $\rho_1=\rho_2=\pm 1$. If $\rho_1=\rho_2=+1$, the solutions are periodic with period T, and if $\rho_1=\rho_2=-1$, the solution is periodic with period $2T$.
- $|b|<1$, ρ_1 and ρ_2 are complex conjugates, leading to a bounded, nearly periodic solution.

From this, we conclude that the stability condition for equation (2.6) is given by $|b|=1$. The natural frequency ω directly affects the stability condition through its role in defining α and β, which influence the matrix $\mathbf{A}$ and the critical stability parameter b. Different values of ω result in shifts in the boundaries between stable and unstable regions. By introducing the notations $\Omega = \frac{2\pi}{T}$, $\nu = \frac{\Omega}{2\omega}$, we get the stability boundaries given by

$$\begin{aligned}(3.13)\quad &\Bigg|\cos\left(\frac{\pi\sqrt{k^2(G-h+EIk^2)+K}}{2\sqrt{EI}k^2\nu}\right)\cos\left(\frac{\pi\sqrt{k^2(G+h+EIk^2)+K}}{2\sqrt{EI}k^2\nu}\right)\\ &-\frac{(Gk^2+EIk^4+K)\sin\left(\frac{\pi\sqrt{k^2(G-h+EIk^2)+K}}{2\sqrt{EI}k^2\nu}\right)\sin\left(\frac{\pi\sqrt{k^2(G+h+EIk^2)+K}}{2\sqrt{EI}k^2\nu}\right)}{\sqrt{k^2(G-h+EIk^2)+K}\cdot\sqrt{k^2(G+h+EIk^2)+K}}\Bigg| = 1.\end{aligned}$$

4. Results and discussion

According to the theory of Mathieu functions [14, 13], the nature of the solution depends on the selection of load frequency and amplitude. The frequency-amplitude domain is divided into regions that result in either stable or unstable solutions. On the border lines between these regions, the solutions are periodic with a period T or $2T$, as derived from Floquet theory [3]. Specifically, two solutions with the same periodicity define the instability region, while two solutions with different periodicity define the stability region. In this context, a stable solution indicates that the motion remains within a bounded neighborhood of the initial conditions, representing Lyapunov stability [7].

For the model shown in Fig. 2.1, the stability regions are presented by applying the stability condition (3.13) for the following parameters:

$$\begin{gathered} E = 2\times10^{10}\ \mathrm{Nm}^{-2},\quad K_0 = 2\times10^{5}\ \mathrm{Nm}^{-2},\quad G_0 = 1\times10^{5}\ \mathrm{Nm}^{-2},\\ \rho = 2\times10^{3}\ \mathrm{kgm}^{-3},\quad A = 5\times10^{-2}\ \mathrm{m}^2,\quad I = 4\times10^{-4}\ \mathrm{m}^4,\\ L = 10\ \mathrm{m},\quad b = \tfrac{1}{8}\sqrt{\tfrac{5}{3}}\ \mathrm{m},\quad h_c = \tfrac{2}{5}\sqrt{\tfrac{3}{5}}\ \mathrm{m}. \end{gathered}$$

where K_0 and G_0 represent the baseline values of the stiffness coefficient K and shear modulus G, which are varied throughout the research; L is the length of the beam, and b and h_c are the dimensions of the beam's cross-section. As a reference case, the scenario where both the shear modulus and stiffness coefficient are zero was analyzed first, i.e., the stability of Euler's simply supported beam under the influence of a periodic compressive load. The resulting stability regions are shown in Fig. 4.1 and align with the stability regions presented in references [6, 13] for the same case. The diagrams represent the dependence of the dimensionless parameters ν and $\bar{h}$, where ν represents the normalized frequency of periodic axial load $\nu = \frac{\Omega}{2\omega}$, and $\bar{h}$ represents the amplitude, or the ratio of the applied axial force to the critical buckling load of the simply supported beam $\bar{h} = \frac{h}{EIk^2}$. In Fig. 4.1, the shaded areas represent regions that lead to unstable solutions, where the first stability boundaries start from the value $\nu = 1$, the second boundaries from $\nu = 1/2$, and so on.

The left figure in Fig. 4.1 depicts the third and fourth stability regions, while the right figure illustrates the first and second regions. This separation provides a clearer visualization of the higher-order stability regions (third and fourth), which

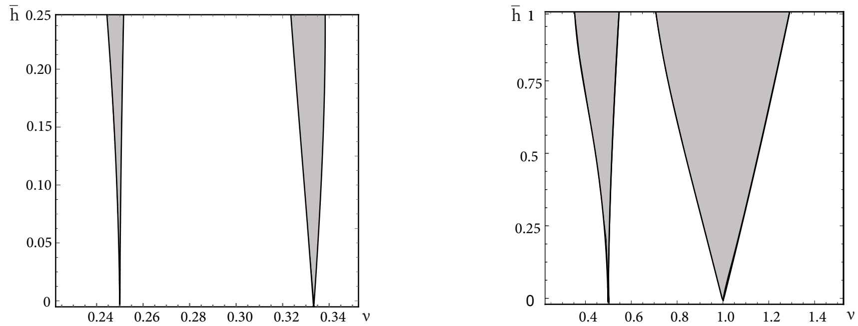


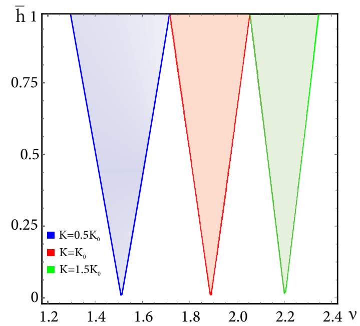
FIG. 4.1: Regions of stability for a simply supported beam

are challenging to discern in a combined plot due to scale differences. The first region of dynamic instability occupies the majority of the entire unstable area, making it highly significant in engineering practice and commonly referred to as the primary region of dynamic instability. When the frequency of external excitation is significantly higher than the system's natural frequency, the likelihood of the parameters falling within stable regions increases, thereby reducing the risk of strong instability. However, in engineering practice, the frequency of external excitation may also be lower than the system's natural frequency, making the selection of system parameters critically important.

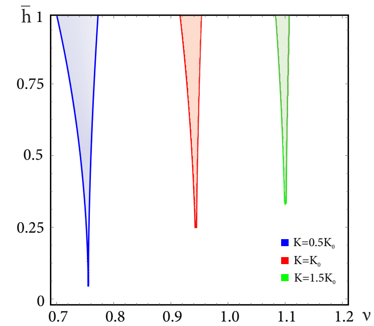
To better understand the impact of the elastic foundation on the stability of beams subjected to a rectangular wave type of load, we investigate the changes in stability regions as the stiffness coefficient K and shear modulus G vary. The first step is to analyze the effect of changing the stiffness of the layer K under the assumption that the shear modulus $G = 0$. This assumption corresponds to the case of a beam resting on a Winkler foundation. This analysis lays the groundwork for later investigations into more complex models that also take into account the effects of shear in the elastic layer, i.e., foundation. By comparing the stability boundary results obtained for the case of a simply supported beam (Fig. 4.1) and a beam resting on a Winkler foundation (Fig. 4.2), it can be observed that the existence of an elastic layer leads to the emergence of regions with unstable solutions at higher frequencies.

However, the presence of the Winkler layer generally increases the stability of the mechanical system. This indicates that a higher stiffness of the layer reduces the likelihood of instability within a certain frequency range, thereby enhancing the overall stability of the beam. Thus, variations in the layer stiffness K significantly influence the dynamics of the beam's stability, allowing for the control of stability boundaries through adjustments solely to the stiffness of the elastic layer.

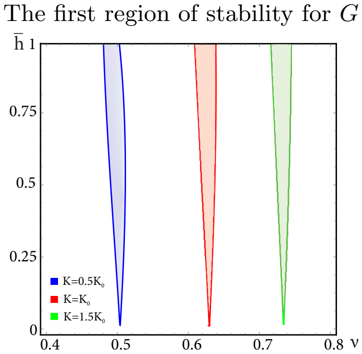
Finally, we consider the impact of the shear modulus of the Pasternak foundation on the stability regions. Fig. 4.3 presents the results obtained by varying the shear modulus while keeping the layer stiffness fixed at a specific value $K = K_0$. The first case, represented in red, corresponds to the scenario where $G = 0$, which has been



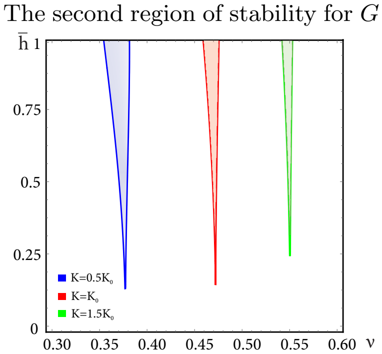
(a) The first region of stability for $G = 0$



(b) The second region of stability for $G = 0$



(c) The third region of stability for $G = 0$



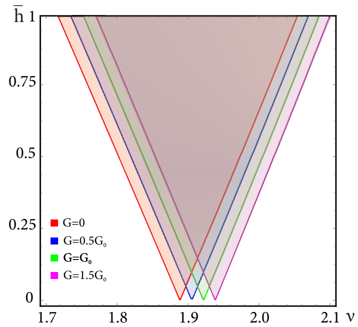
(d) The fourth region of stability for $G = 0$

FIG. 4.2: Stability regions for a beam on a Winkler foundation

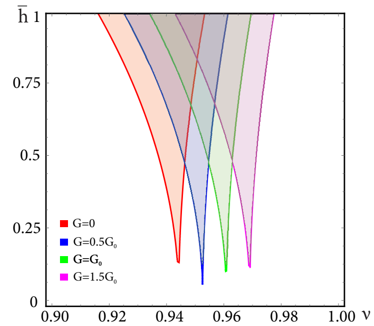
discussed previously. Subsequently, three additional cases with different values of the shear modulus G were analyzed.

Analyzing the results, we observe that as the shear modulus increases, the stability boundaries shift towards higher frequencies; however, this shift is less pronounced compared to the impact of changes in the layer stiffness K. Additionally, the change in the shear modulus G has minimal effect on the size of the regions with unstable solutions; rather, these regions simply shift to higher frequencies with an increase in the shear modulus G. From all displayed results, it is also evident that in the second and fourth stability regions, variations in the stiffness coefficient or shear modulus results in changes in the amplitude of the load required to reach the unstable region. In certain cases, the unstable region is reached at a lower or higher values of the applied load compared to other scenarios.

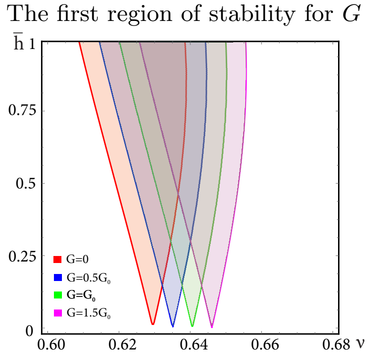
The presented results indicate complex interactions between the shear modulus and layer stiffness, highlighting the need for further research to fully understand the mechanisms affecting the stability of beams supported on elastic foundations of various types.



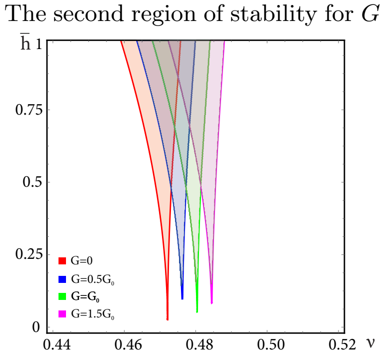
(a) The first region of stability for $G = 0$



(b) The second region of stability for $G = 0$



(c) The third region of stability for $G = 0$



(d) The fourth region of stability for $G = 0$

FIG. 4.3: Stability regions for a beam on a Pasternak foundation

5. Conclusion

To gain a deeper understanding of the influence of the Pasternak foundations parameters on the stability of beams subjected to periodic axial compressive forces, the effects of varying the stiffness of the elastic layer, K, and the shear modulus, G, were investigated. The application of Floquet theory enabled an analysis of stability and instability regions with respect to these parameter changes. Initially, the problem was formulated, and the stability conditions for the beam resting on a Pasternak foundation were defined. Subsequently, by varying the values of the layer stiffness and shear modulus, the shifts in stability boundaries were examined. The numerical analysis began by investigating the impact of varying the layer stiffness K, assuming $G = 0$, which corresponds to a beam on a Winkler foundation. The analysis revealed that the first instability regions occur at higher frequencies of the applied force as the stiffness coefficient increases. Additionally, a significant reduction in all stability regions was observed with the increase of the stiffness coefficient K. Thus, changes in layer stiffness K substantially affect the dynamic stability of the beam.

Finally, the influence of the Pasternak foundation's shear modulus on the stability regions was analyzed. Results from varying the shear modulus G while keeping the layer stiffness constant showed that instability regions also shift to higher frequencies as the shear modulus increases, although this shift is less pronounced compared to the changes in stiffness coefficient K. It was concluded that variations in the shear modulus G do not significantly alter the size of the instability regions, but rather shift them towards higher frequencies. These findings highlight the complex interactions between the shear modulus and layer stiffness, and underscore the need for further research to fully understand the mechanisms affecting the stability of beams resting on elastic foundations of different types. Future research directions may include additional analyses of factors such as beam rotational inertia, shear, as well as more complex forms of external forces and coupled beam models.

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