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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR GENERALIZED CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, we establish some new generalized Hermite-Hadamard type inequalities for local fractional integrals on fractal sets R^α $(0<\alpha\le 1)$ of real line numbers. We also give some applications.
Keywords: Hermite-Hadamard inequality, fractional integration, generalized convex function

1. Preliminaries

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral. (see [14, 15])

Recently, the theory of Yang's fractional sets [14] was introduced as follows.

For $0<\alpha\le 1$, we have the following α-type set of element sets:

Z^α : The α-type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, \dots, \pm n^\alpha, \dots\}$.

Q^α : The α-type set of rational numbers is defined as the set $\{m^\alpha = \left(\frac{p}{q}\right)^\alpha : p,q\in Z,\ q\neq 0\}$.

J^α : The α-type set of irrational numbers is defined as the set $\{m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p,q\in Z,\ q\neq 0\}$.

R^α : The α-type set of real line numbers is defined as the set $R^\alpha = Q^\alpha \cup J^\alpha$.

If a^α, b^α and c^α belong to the set R^α of real line numbers, then

(1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belong to the set R^α;

(2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;

(3) $a^\alpha + (b^\alpha + c^\alpha) = (a+b)^\alpha + c^\alpha$;

(4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;

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(5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;

(6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;

(7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 1.1. [14] A non-differentiable function $f : R \to R^\alpha$, $x \to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local continuous on the interval (a, b), we denote $f(x) \in C_\alpha(a, b)$.

Definition 1.2. [14] The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(\alpha + 1) (f(x) - f(x_0))$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \ldots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denote $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \ldots$

Definition 1.3. [14] Let $f(x) \in C_\alpha [a, b]$. Then the local fractional integral is defined by,

$${}_aI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t\to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha,$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max \{\Delta t_1, \Delta t_2, \ldots, \Delta t_{N-1}\}$, where $[t_j, t_{j+1}]$, $j = 0, \ldots, N - 1$ and $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ is partition of interval $[a, b]$.

Here, it follows that ${}_aI_b^\alpha f(x) = 0$ if $a = b$ and ${}_aI_b^\alpha f(x) = -{}_bI_a^\alpha f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_aI_x^\alpha f(x)$, then we denote by $f(x) \in I_x^\alpha [a, b]$.

Lemma 1.1. *[14]*

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha [a, b]$, *then we have*

$${}_aI_b^\alpha f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha [a, b]$ *and* $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha [a, b]$, *then we have*

$${}_aI_b^\alpha f(x)g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_aI_b^\alpha f^{(\alpha)}(x)g(x).$$

Lemma 1.2. *[14]*

$$\frac{d^{\alpha}x^{k\alpha}}{dx^{\alpha}} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha};$$

$$\frac{1}{\Gamma(\alpha+1)}\int_a^b x^{k\alpha}(dx)^{\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha}-a^{(k+1)\alpha}\right),\ k\in R.$$

More detailed information on local fractional calculus can be found in ([14]-[18]).

2. Introduction

A function $f : I \subseteq R \to R$ is said to be convex if

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y) \tag{2.1}$$

holds for every $x, y \in I$ and $t \in [0,1]$.

Let $f : I \subseteq R \to R$ be a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{2.2}$$

is known as the Hermite–Hadamard inequality.

The Hermite–Hadamard inequality (2.2) has become an important cornerstone in probability and optimization. An account on the history of this inequality can be found in [4]. Surveys on various generalizations and developments can be found in [9].

Definition 2.1. [14] Let $f : I \subseteq R \to R^{\alpha}$. For any $x_1, x_2 \in I$ and $\lambda \in [0,1]$, if the following inequality

$$f(\lambda x_1+(1-\lambda)x_2) \le \lambda^{\alpha} f(x_1)+(1-\lambda)^{\alpha} f(x_2) \tag{2.3}$$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:

(1) $f(x) = x^{\alpha p},\ p > 1$;

(2) $f(x) = E_{\alpha}(x^{\alpha}),\ x \in R$ where $E_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffer function.

In recent years, the fractal theory has received a significant attention [1, 2, 6, 7, 8, 10, 11, 12, 13]. In one of these papers [5], Mo *et al*. proved the following generalized Hermite-Hadamard inequality for generalized convex functions:

Let $f(x) \in I_x^{\alpha}[a,b]$ be a generalized convex function on $[a,b]$ with $a<b$. Then,

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha} f(x) \leq \frac{f(a)+f(b)}{2^{\alpha}}. \tag{2.4}$$

Lemma 2.1. (Generalized Hölder's inequality) *[14] Let $f,g \in C_{\alpha}[a,b]$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)}\int_a^b |f(x)g(x)|\,(dx)^{\alpha}\\ &\leq \left(\frac{1}{\Gamma(\alpha+1)}\int_a^b |f(x)|^p\,(dx)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\alpha+1)}\int_a^b |g(x)|^q\,(dx)^{\alpha}\right)^{\frac{1}{q}}.\end{aligned}$$

In this paper, inspired by the papers [3] and [5], we establish some new generalized Hermite-Hadamard type inequalities for local fractional integrals on fractal sets R^{α} $(0<\alpha\leq 1)$ of real line numbers.

3. Main Results

Lemma 3.1. *Let $I \subseteq R$ be an interval, $f: I^{\circ} \to R^{\alpha}$ (I° is interior of I) such that $f \in D_{\alpha}(I^{\circ})$ and $f^{(\alpha)} \in C_{\alpha}[a,b]$ for $a,b \in I^{\circ}$ with $a<b$. Then, for all $x \in [a,b]$, we have the identity*

$$\begin{aligned}&\frac{(b-x)^{\alpha} f(b)+(x-a)^{\alpha} f(a)}{(b-a)^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha} f(t) \\ =\ &\frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_0^1 (t-1)^{\alpha} f^{(\alpha)}(tx+(1-t)a)\,(dt)^{\alpha} \\ +\ &\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_0^1 (t-1)^{\alpha} f^{(\alpha)}(tx+(1-t)b)\,(dt)^{\alpha}.\end{aligned} \tag{3.1}$$

Proof. Using the local fractional integration by parts, we have

$$\begin{aligned} I_1 &= \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (t-1)^{\alpha} f^{(\alpha)}(tx+(1-t)a)(dt)^{\alpha} \\ &= \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \left[\left. \frac{(t-1)^{\alpha} f(tx+(1-t)a)}{(x-a)^{\alpha}} \right|_0^1 \right. \\ & \left. -\frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{\Gamma(1+\alpha) f(tx+(1-t)a)(dt)^{\alpha}}{(x-a)^{\alpha}} \right] \\ &= \frac{(x-a)^{\alpha} f(a)}{(b-a)^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \, {}_aI_x^{\alpha} f(t) \end{aligned}$$

Similarly, we have

$$\begin{aligned} I_2 &= \frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (t-1)^{\alpha} f^{(\alpha)}(tx+(1-t)b)(dt)^{\alpha} \\ &= \frac{(b-x)^{\alpha} f(b)}{(b-a)^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \, {}_xI_b^{\alpha} f(t). \end{aligned}$$

If we add I_1 and I_2 , then we obtain the desired identity. □

Theorem 3.1. *The assumptions of Lemma 3.1 are satisfied. If $\left|f^{(\alpha)}\right|$ is generalized convex, we have the inequality*

$$\begin{aligned} &\left| \frac{(b-x)^{\alpha} f(b) + (x-a)^{\alpha} f(a)}{(b-a)^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \, {}_aI_b^{\alpha} f(t) \right| \\ \le & \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \left[A \left| f^{(\alpha)}(x) \right| + B \left| f^{(\alpha)}(a) \right| \right] \\ & + \frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \left[A \left| f^{(\alpha)}(x) \right| + B \left| f^{(\alpha)}(b) \right| \right], \end{aligned} \tag{3.2}$$

where $A = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}$ *and* $B = \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}$.

Proof. Using Lemma 3.1 and taking the modulus, we have

$$\begin{aligned} &\left| \frac{(b-x)^{\alpha} f(b) + (x-a)^{\alpha} f(a)}{(b-a)^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \, {}_aI_b^{\alpha} f(t) \right| \\ \le & \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha} \left| f^{(\alpha)}(tx+(1-t)a) \right| (dt)^{\alpha} \\ & + \frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha} \left| f^{(\alpha)}(tx+(1-t)b) \right| (dt)^{\alpha}. \end{aligned}$$

Since $\left|f^{(\alpha)}\right|$ is generalized convex, then we have

$$\begin{aligned}
&\left|\frac{(b-x)^{\alpha} f(b)+(x-a)^{\alpha} f(a)}{(b-a)^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha} f(t)\right| \\
\leq\ & \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left[t^{\alpha}\left|f^{(\alpha)}(x)\right|+(1-t)^{\alpha}\left|f^{(\alpha)}(a)\right|\right](dt)^{\alpha} \\
&+\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left[t^{\alpha}\left|f^{(\alpha)}(x)\right|+(1-t)^{\alpha}\left|f^{(\alpha)}(b)\right|\right](dt)^{\alpha}
\end{aligned}$$

Then, we have

$$\begin{aligned}
J_1 &= \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left[t^{\alpha}\left|f^{(\alpha)}(x)\right|+(1-t)^{\alpha}\left|f^{(\alpha)}(a)\right|\right](dt)^{\alpha} \\
&= \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\left[\frac{\left|f^{(\alpha)}(x)\right|}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha} t^{\alpha}(dt)^{\alpha}+\frac{\left|f^{(\alpha)}(a)\right|}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{2\alpha}(dt)^{\alpha}\right]
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= \frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left[t^{\alpha}\left|f^{(\alpha)}(x)\right|+(1-t)^{\alpha}\left|f^{(\alpha)}(b)\right|\right](dt)^{\alpha} \\
&= \frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}}\left[\frac{\left|f^{(\alpha)}(x)\right|}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha} t^{\alpha}(dt)^{\alpha}+\frac{\left|f^{(\alpha)}(b)\right|}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{2\alpha}(dt)^{\alpha}\right].
\end{aligned}$$

Using Lemma 1.2, we have

$$\frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-t)^{\alpha} t^{\alpha}(dt)^{\alpha}=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \tag{3.3}$$

and

$$\frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-t)^{2\alpha}(dt)^{\alpha}=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}. \tag{3.4}$$

Substituting the equalities (3.3) and (3.4) in J_1 and J_2, we obtain desired inequality, which completes the proof. □

Corollary 3.1. *In Theorem 3.1, if we choose $x=\frac{a+b}{2}$, we obtain*

$$\begin{aligned}
&\left|\frac{f(b)+f(a)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha} f(t)\right| \\
\leq\ & \frac{(b-a)^{\alpha}}{4^{\alpha}}\left[2^{\alpha} A\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|+B\left(\left|f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)\right|\right)\right].
\end{aligned}$$

Remark 3.1. In Corollary 3.1, since $\left|f^{(\alpha)}\right|$ is generalized convex, then we obtain

$$\begin{aligned}&\left|\frac{f(b)+f(a)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(t)\right|\\ \leq\;& \frac{(b-a)^{\alpha}}{4^{\alpha}}(A+B)\left(\left|f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)\right|\right).\end{aligned}$$

Theorem 3.2. *The assumptions of Lemma 3.1 are satisfied. If* $\left|f^{(\alpha)}\right|^q$ *is generalized convex, then we have the inequality*

$$\begin{aligned}&\left|\frac{(b-x)^{\alpha}f(b)+(x-a)^{\alpha}f(a)}{(b-a)^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(t)\right|\\ \leq\;& \left(\frac{1}{2}\right)^{\frac{\alpha}{q}}\left(\frac{\Gamma(1+\alpha p)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\\ \times\;& \left[\frac{(x-a)^{2\alpha}\left(\left|f^{(\alpha)}(a)\right|^q+\left|f^{(\alpha)}(x)\right|^q\right)^{\frac{1}{q}}+(b-x)^{2\alpha}\left(\left|f^{(\alpha)}(b)\right|^q+\left|f^{(\alpha)}(x)\right|^q\right)^{\frac{1}{q}}}{(b-a)^{\alpha}}\right]\end{aligned}$$

for $x\in[a,b]$ *and* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.1 and Hölder inequality, we have

$$\begin{aligned}&\left|\frac{(b-x)^{\alpha}f(b)+(x-a)^{\alpha}f(a)}{(b-a)^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(t)\right|\\ \leq\;& \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-t)^{\alpha}\left|f^{(\alpha)}(tx+(1-t)a)\right|(dt)^{\alpha}\\ &+\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-t)^{\alpha}\left|f^{(\alpha)}(tx+(1-t)b)\right|(dt)^{\alpha}\\ \leq\;& \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-t)^{\alpha p}(dt)^{\alpha}\right)^{\frac{1}{p}}\\ &\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}(tx+(1-t)a)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}}\\ &+\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-t)^{\alpha p}(dt)^{\alpha}\right)^{\frac{1}{p}}\\ &\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}(tx+(1-t)b)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}}.\end{aligned}$$

Also, since $\left|f^{(\alpha)}\right|^q$ is generalized convex and using the generalized Hermite-Hadamard inequality, we have

$$\int_0^1\left|f^{(\alpha)}(tx+(1-t)a)\right|^q(dt)^{\alpha}\leq\left[\frac{\left|f^{(\alpha)}(a)\right|^q+\left|f^{(\alpha)}(x)\right|^q}{2^{\alpha}}\right]\tag{3.5}$$

and

$$\int_0^1 \left|f^{(\alpha)}(tx+(1-t)a)\right|^q (dt)^\alpha \le \left[\frac{\left|f^{(\alpha)}(b)\right|^q+\left|f^{(\alpha)}(x)\right|^q}{2^\alpha}\right]. \tag{3.6}$$

So

$$\begin{aligned}&\left|\frac{(b-x)^\alpha f(b)+(x-a)^\alpha f(a)}{(b-a)^\alpha}-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)\right|\\ \le\ & \left(\frac{1}{2}\right)^{\frac{\alpha}{q}}\left(\frac{\Gamma(1+\alpha p)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\\ \times\ & \left[\frac{(x-a)^{2\alpha}\left(\left|f^{(\alpha)}(a)\right|^q+\left|f^{(\alpha)}(x)\right|^q\right)^{\frac{1}{q}}+(b-x)^{2\alpha}\left(\left|f^{(\alpha)}(b)\right|^q+\left|f^{(\alpha)}(x)\right|^q\right)^{\frac{1}{q}}}{(b-a)^\alpha}\right]\end{aligned}$$

which completes the proof. □

Corollary 3.2. *In Theorem 3.2, if we choose* $x=\frac{a+b}{2}$, *we obtain*

$$\begin{aligned}&\left|\frac{f(b)+f(a)}{2^\alpha}-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)\right|\\ \le\ & \frac{(b-a)^\alpha}{4^\alpha}\left(\frac{1}{2}\right)^{\frac{\alpha}{q}}\left(\frac{\Gamma(1+\alpha p)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\\ & \times\left[\left(\left|f^{(\alpha)}(a)\right|^q+\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f^{(\alpha)}(b)\right|^q+\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]\\ \le\ & \frac{(b-a)^\alpha}{2^\alpha}\left(\frac{1}{2}\right)^{\frac{\alpha}{q}}\left(\frac{\Gamma(1+\alpha p)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left[\left|f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)\right|\right].\end{aligned}$$

In the last part of this inequality, we first used the fact that $\sum_{k=1}^n (a_k+b_k)^s \le \sum_{k=1}^n (a_k)^s+\sum_{k=1}^n (b_k)^s$, $(0\le s<1)$ *and* $a_i, b_i \ge 0$ *for* $i=1,2,..,n$. *Finally, we use the fact that* $\left|f^{(\alpha)}\right|$ *is generalized convex.*

Theorem 3.3. *The assumptions of Lemma 3.1 are satisfied. If* $|f^{(\alpha)}|^q$ *is generalized concave for* $q>1$

$$\begin{aligned}&\left|\frac{(b-x)^\alpha f(b)+(x-a)^\alpha f(a)}{(b-a)^\alpha}-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)\right|\\ \le\ & \left[\frac{\Gamma(1+\frac{q}{q-1}\alpha)}{\Gamma(1+\frac{2q-1}{q-1}\alpha)}\right]^{\frac{q-1}{q}}\\ \times\ & \left[\frac{(x-a)^{2\alpha}\left|f^{(\alpha)}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(a+x)\right)\right|+(b-x)^{2\alpha}\left|f^{(\alpha)}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(b+x)\right)\right|}{(b-a)^\alpha}\right]\end{aligned}$$

for each $x\in[a,b]$ *and* $q=\frac{p}{p-1}$.

Proof. As in the previous theorem and using Lemma 3.1 and generalized Hölder inequality for $q > 1$, we have

$$
\begin{aligned}
& \left| \frac{(b-x)^{\alpha} f(b) + (x-a)^{\alpha} f(a)}{(b-a)^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \, {}_{a}I_{b}^{\alpha} f(t) \right| \\
\leq\ & \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha} \left| f^{(\alpha)}(tx + (1-t)a) \right| (dt)^{\alpha} \\
& + \frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha} \left| f^{(\alpha)}(tx + (1-t)b) \right| (dt)^{\alpha} \\
\leq\ & \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\frac{\alpha q}{q-1}} (dt)^{\alpha} \right)^{\frac{q-1}{q}} \\
\times\ & \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 \left| f^{(\alpha)}(tx + (1-t)a) \right|^{q} (dt)^{\alpha} \right)^{\frac{1}{q}} \\
& + \frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\frac{\alpha q}{q-1}} (dt)^{\alpha} \right)^{\frac{q-1}{q}} \\
\times\ & \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 \left| f^{(\alpha)}(tx + (1-t)b) \right|^{q} (dt)^{\alpha} \right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $\left| f^{(\alpha)} \right|$ is generalized concave, we can use the generalized Jensen's integral inequality to obtain:

$$
\begin{aligned}
& \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left| f^{(\alpha)}(tx + (1-t)a) \right|^{q} (dt)^{\alpha} \\
=\ & \frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{0\alpha} \left| f^{(\alpha)}(tx + (1-t)a) \right|^{q} (dt)^{\alpha} \\
\leq\ & \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{0\alpha} (dt)^{\alpha} \right) \\
& \times \left| f^{(\alpha)} \left(\frac{1}{\frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{0\alpha} (dt)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (tx + (1-t)a)^{\alpha} (dt)^{\alpha} \right) \right|^{q} \\
=\ & \left| f^{(\alpha)} \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} (a+x) \right) \right|^{q}.
\end{aligned}
$$

Similarly,

$$
\frac{1}{\Gamma(1+\alpha)} \int_0^1 \left| f^{(\alpha)}(tx - (1-t)b) \right|^{q} (dt)^{\alpha} \leq \left| f^{(\alpha)} \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} (b+x) \right) \right|^{q}. \tag{3.7}
$$

Combining all the obtained inequalities, we get

$$\begin{aligned}
&\left|\frac{(b-x)^{\alpha} f(b)+(x-a)^{\alpha} f(a)}{(b-a)^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha} f(t)\right| \\
\leq\ & \left[\frac{\Gamma(1+\frac{q}{q-1}\alpha)}{\Gamma(1+\frac{2q-1}{q-1}\alpha)}\right]^{\frac{q-1}{q}} \\
\times\ & \left[\frac{(x-a)^{2\alpha}\left|f^{(\alpha)}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(a+x)\right)\right|+(b-x)^{2\alpha}\left|f^{(\alpha)}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(b+x)\right)\right|}{(b-a)^{\alpha}}\right]
\end{aligned}$$

which completes the proof. □

Remark 3.2. In Theorem 3.3, if we choose $x=\frac{a+b}{2}$, we obtain

$$\begin{aligned}
&\left|\frac{f(b)+f(a)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha} f(t)\right| \\
\leq\ & \frac{(b-a)^{\alpha}}{4^{\alpha}}\left[\frac{\Gamma(1+\frac{q}{q-1}\alpha)}{\Gamma(1+\frac{2q-1}{q-1}\alpha)}\right]^{\frac{q-1}{q}} \\
&\times\left[\left|f^{(\alpha)}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{3a+b}{2}\right)\right)\right|+\left|f^{(\alpha)}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{3b+a}{2}\right)\right)\right|\right]
\end{aligned}$$

Theorem 3.4. *The assumptions of Lemma 3.1 are satisfied. If $|f^{(\alpha)}|^{q}$ is generalized convex for $q\geq 1$, then we have the inequality*

$$\begin{aligned}
&\left|\frac{(b-x)^{\alpha} f(b)+(x-a)^{\alpha} f(a)}{(b-a)^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha} f(t)\right| \\
\leq\ & C\left[\frac{(x-a)^{2\alpha}\left(A\left|f^{(\alpha)}(x)\right|^{q}+B\left|f^{(\alpha)}(a)\right|^{q}\right)^{\frac{1}{q}}}{(b-a)^{\alpha}}\right. \\
+\ & \left.\frac{(b-x)^{2\alpha}\left(A\left|f^{(\alpha)}(x)\right|^{q}+B\left|f^{(\alpha)}(b)\right|^{q}\right)^{\frac{1}{q}}}{(b-a)^{\alpha}}\right],
\end{aligned}$$

where $A=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}$, $B=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}$ *and* $C=\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{1-\frac{1}{q}}$.

Proof. From Lemma 3.1 and the well-known power-mean inequality, we have

$$
\begin{aligned}
(3.8)\qquad & \left|\frac{(b-x)^{\alpha} f(b)+(x-a)^{\alpha} f(a)}{(b-a)^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\, {}_aI_b^{\alpha} f(t)\right| \\
\leq\ & \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left|f^{(\alpha)}(tx+(1-t)a)\right|(dt)^{\alpha} \\
& +\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left|f^{(\alpha)}(tx+(1-t)b)\right|(dt)^{\alpha} \\
\leq\ & \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}(dt)^{\alpha}\right)^{1-\frac{1}{q}} \\
& \times\left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left|f^{(\alpha)}(tx+(1-t)a)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}} \\
& +\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}(dt)^{\alpha}\right)^{1-\frac{1}{q}} \\
& \times\left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left|f^{(\alpha)}(tx+(1-t)b)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $\left|f^{(\alpha)}\right|^q$ is generalized convex, we get

$$
\begin{aligned}
(3.9)\qquad & \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left|f^{(\alpha)}(tx+(1-t)a)\right|^q(dt)^{\alpha} \\
\leq\ & \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left[t^{\alpha}\left|f^{(\alpha)}(x)\right|^q+(1-t)^{\alpha}\left|f^{(\alpha)}(a)\right|^q\right](dt)^{\alpha} \\
=\ & A\left|f^{(\alpha)}(x)\right|^q+B\left|f^{(\alpha)}(a)\right|^q.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(3.10)\qquad & \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left|f^{(\alpha)}(tx+(1-t)b)\right|^q(dt)^{\alpha} \\
\leq\ & A\left|f^{(\alpha)}(x)\right|^q+B\left|f^{(\alpha)}(b)\right|^q.
\end{aligned}
$$

If we substitute the inequalities (3.9) and (3.10) in (3.8), then we can easily see the desired inequality. This completes the proof of this theorem. □

Corollary 3.3. *In Theorem 3.4, if we choose $x=\frac{a+b}{2}$, we obtain*

$$\begin{aligned}
&\left|\frac{f(b)+f(a)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(t)\right|\\
\leq\;& C\frac{(b-a)^{\alpha}}{4^{\alpha}}\times\left[A^{\frac{1}{q}}\left(\left|f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)\right|\right)+2^{\alpha}B^{\frac{1}{q}}\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|\right]\\
\leq\;& C\frac{(b-a)^{\alpha}}{4^{\alpha}}\times\left[\left(A\left|f^{(\alpha)}(a)\right|^{q}+B\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(A\left|f^{(\alpha)}(b)\right|^{q}+B\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right]\\
\leq\;& C\frac{(b-a)^{\alpha}}{4^{\alpha}}\left(A^{\frac{1}{q}}+B^{\frac{1}{q}}\right)\left(\left|f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)\right|\right).
\end{aligned}$$

Theorem 3.5. *The assumptions of Lemma 3.1 are satisfied. If $\left|f^{(\alpha)}\right|^{q}$ is generalized concave for $q\geq 1$, then we have the inequality*

$$\begin{aligned}
&\left|\frac{(b-x)^{\alpha}f(b)+(x-a)^{\alpha}f(a)}{(b-a)^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(t)\right|\\
\leq\;& C\left[\frac{(x-a)^{2\alpha}\left(f^{(\alpha)}(Ax+Ba)\right)+(b-x)^{2\alpha}\left(f^{(\alpha)}(Ax+Bb)\right)}{(b-a)^{\alpha}}\right]
\end{aligned}$$

where $A=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}$, $B=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}$ and $C=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}$.

Proof. Suppose that $\left|f^{(\alpha)}\right|^{q}$ is generalized concave. Then,

$$\left|f^{(\alpha)}(tx+(1-t)a)\right|^{q}\geq t^{\alpha}\left|f^{(\alpha)}(x)\right|^{q}+(1-t)^{\alpha}\left|f^{(\alpha)}(x)\right|^{q}. \tag{3.11}$$

Since $q\geq 1$, also we have

$$\left|f^{(\alpha)}(tx+(1-t)a)\right|\geq t^{\alpha}\left|f^{(\alpha)}(x)\right|+(1-t)^{\alpha}\left|f^{(\alpha)}(x)\right|. \tag{3.12}$$

That is, $\left|f^{(\alpha)}\right|$ is generalized concave.

Now, taking Lemma 3.1 and Jensen inequality for local fractional integrals, we have

$$\begin{aligned}
&\left|\frac{(b-x)^{\alpha} f(b)+(x-a)^{\alpha} f(a)}{(b-a)^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha} f(t)\right| \\
\leq\ & \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left|f^{(\alpha)}(tx+(1-t)a)\right|(dt)^{\alpha} \\
&+\frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}\left|f^{(\alpha)}(tx+(1-t)b)\right|(dt)^{\alpha} \\
\leq\ & \frac{(x-a)^{2\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}(dt)^{\alpha}\right) \\
\times & \left(\left|f^{(\alpha)}\left(\frac{\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}(tx+(1-t)a)^{\alpha}(dt)^{\alpha}}{\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}(dt)^{\alpha}}\right)\right|\right) \\
+\ & \frac{(b-x)^{2\alpha}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}(dt)^{\alpha}\right) \\
\times & \left(\left|f^{(\alpha)}\left(\frac{\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}(tx+(1-t)b)^{\alpha}(dt)^{\alpha}}{\frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-t)^{\alpha}(dt)^{\alpha}}\right)\right|\right) \\
\leq\ & \left(1-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left[\frac{(x-a)^{2\alpha}\left(f^{(\alpha)}(Ax+Ba)\right)+(b-x)^{2\alpha}\left(f^{(\alpha)}(Ax+Bb)\right)}{(b-a)^{\alpha}}\right].
\end{aligned}$$

which completes the proof of this theorem. □

Remark 3.3. In Theorem 3.4, if we choose $x=\frac{a+b}{2}$, we obtain

$$\begin{aligned}
&\left|\frac{f(b)+f(a)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha} f(t)\right| \\
\leq\ & \frac{(b-a)^{\alpha}}{8^{\alpha}}\left[\left|f^{(\alpha)}\left(\frac{5a+b}{6}\right)\right|+\left|f^{(\alpha)}\left(\frac{5b+a}{6}\right)\right|\right].
\end{aligned}$$

4. Applications to Special Means

Let us recall some generalized means:

(1) The generalized arithmetic mean:

$$A\left(a^{\alpha}, b^{\alpha}\right)=\frac{a^{\alpha}+b^{\alpha}}{2^{\alpha}};$$

(2) The generalized logarithmic mean:

$$L_n\left(a^\alpha,b^\alpha\right)=\left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)}\left(\frac{b^{(n+1)\alpha}-a^{(n+1)\alpha}}{(b-a)^\alpha}\right)\right]^{\frac{1}{n}};$$

$n\in Z\setminus\{-1,0\}$, $a,b\in R$, $a\neq b$.

Proposition 4.1. *Let $a,b\in R$, $a<b$, $0\notin[a,b]$ and $n\in Z$, $|n|\geq 2$. Then, for all $p>1$*

(a)

$$\begin{aligned}&\left|A\left(a^{n\alpha},b^{n\alpha}\right)-\Gamma(1+\alpha)L_n\left(a^\alpha,b^\alpha\right)\right|\\ \leq\;&(b-a)^\alpha\left(\frac{1}{2}\right)^{\frac{\alpha}{q}}\left(\frac{\Gamma(1+\alpha p)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}A\left(\left|a^{(n-1)\alpha}\right|,\left|b^{(n-1)\alpha}\right|\right)\end{aligned}$$

and

(b)

$$\begin{aligned}&\left|A\left(a^{n\alpha},b^{n\alpha}\right)-\Gamma(1+\alpha)L_n^n\left(a^\alpha,b^\alpha\right)\right|\\ \leq\;&C\frac{(b-a)^\alpha}{2^\alpha}\left(A^{\frac{1}{q}}+B^{\frac{1}{q}}\right)\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}A\left(\left|a^{(n-1)\alpha}\right|,\left|b^{(n-1)\alpha}\right|\right).\end{aligned}$$

Proof. The assertion follows from Corollary 3.2 and Corollary 3.3 for $f(x)=x^{n\alpha}$, $x\in[a,b]$, $n\in Z$, $|n|\geq 2$. □

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