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RULED SURFACES WITH CONSTANT CURVATURES IN A STRICT WALKER 3-MANIFOLD

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Abstract. In this paper, we give explicit descriptions of a family of ruled surfaces in a strict Walker 3-manifold with constant Gaussian or mean curvature according to the causal character of the surfaces (timelike, spacelike, and lightlike). For each causal character of the considered family of ruled surface, we start with the constant Gaussian curvature case and we end with the constant mean curvature situation.
Keywords: ruled surfaces, Walker 3-manifolds.

1. Introduction

Ruled surfaces have a great effect on differential geometry and they are used in many fields, such as architecture, robotics, design and so on [8]. In real life, ruled surfaces can be seen everywhere, such as in most of the cooling tower structures of thermal power plants, the famous Mobius ring, and saddle-shaped potato chips. Ruled surfaces can be formed by a moving line in continuous motion. In this paper we present two special classes of ruled surfaces in a Lorentzian three manifold which look like ruled surfaces in the Euclidean space $\mathbb{E}^3$ and semi-Euclidean $\mathbb{E}^3_1$. These ruled surfaces are made by a one-parameter family of affine straight lines which are

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the geodesics of $\mathbb{E}^3$ (resp. $\mathbb{E}^3_1$). The study of ruled surfaces of a given ambient space is a natural and interesting problem. A surface Σ in M is said to be ruled if every point of Σ is on (an open geodesic segment) in M that lies in Σ (see [12]). Locally, a ruled surface is made by a one-parameter family of geodesic segments [5]. Several authors have studied problems on ruled surfaces (see [9,13]). The generalized ruled surface is the notion of 2-ruled hypersurfaces. In [6], the authors define three types of 2-ruled hypersurfaces in a Walker 4-manifold. They obtain the Gaussian and mean curvatures of the 2-ruled hypersurfaces of type-1, type-2 and type-3. They also give some characterizations about its minimality.
In this work, the ambient space we will consider is a Lorentzian three-manifold admitting a parallel null vector field called strict Walker manifold. It is known that Walker metrics have served as a powerful tool for constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics.

Three-dimensional geometry plays a central role in the investigation of many problems in Riemannian and Lorentzian geometry. The fact that the Ricci operator completely determines the curvature tensor is crucial to these investigations ([1]). The strict Walker manifolds are described in terms of suitable coordinates (x, y, z) of the manifolds $\mathbb{R}^3$ and their metric depends on an arbitrary function of two variables $f = f(y, z)$ and their metric tensor is given by

$$g^{\epsilon}_f = \epsilon dy^2 + 2dxdz + fdz^2 \tag{1.1}$$

where $\epsilon = \pm 1$. Curvature properties and complete characterization of locally symmetric or locally conformally flat three-dimensional Walker manifolds have been studied in [3]. Also, in [2] the authors obtained a complete classification of parallel surfaces in a Lorenztian three strict Walker manifold (i.e. admitting a parallel null vector field) as the ambient space. Some results on minimal graphs on three dimensional Walker manifolds can be found in [4]. In [10], Athoumane et al. construct two special families of ruled surfaces in a three dimensional strict Walker manifold. They show that the local degeneracy (resp. non-degeneracy) to one of these families has a strong consequence on the geometry of the ambient Walker manifold. In [11], the same authors study the geometry of minimal translation surfaces in a strict Walker 3-manifold. Based on the existence of two isometries, they classify minimal translation surfaces on this class of manifold.
Motivated by this work, we study the properties of constant Gauss curvature and mean curvature of a family of ruled surfaces in a strict Walker 3-manifold. We consider cases where our surfaces are time-like, space-like or light-like. The paper is organized as follows: in Section 2, we give some basic tools for understanding the main results. In section 3, we discuss the geometry of constant Gauss and mean curvatures of the family of ruled surfaces constructed by the geodesic equations.

2. Preliminaries

2.1. Three Dimensional Walker Spaces

General Walker manifolds are pseudo-Riemannian manifolds (M, g, D) with a distribution D on which g is zero (a lightlike distribution) and that is parallel with respect to the Levi-Civita connection of g. In the 3-dimensional case, Walker 3-manifolds have the specific feature that all geometric data is encoded in a single function. As a general fact, a canonical form for a $(2r+1)$-dimensional pseudo-Riemannian manifold M admitting a parallel field of null r-dimensional planes D is given by the metric tensor in matrix form:

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & I_r \\ 0 & \epsilon & 0 \\ I_r & 0 & B \end{pmatrix}$$

where I_r is the $r \times r$ identity matrix, B is a symmetric $r \times r$ matrix whose entries are functions of the coordinates $x_1, \ldots, x_{2r+1}$, and $\epsilon = \pm 1$ [1]. Therefore, any 3-dimensional Walker manifold is locally isometric to the manifold M_f^3 whose metric has the matrix:

$$g_f^{\varepsilon} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix}$$

with respect to the natural frame $\{\partial_x, \partial_y, \partial_z\}$, where $f = f(x, y, z)$ is a smooth function defined on an open subset $O \subseteq \mathbb{R}^3$. Thus,

$$g_f^{\epsilon} = 2\, dx\, dz + \epsilon\, dy^2 + f(x, y, z)\, dz^2. \tag{2.1}$$

The manifold M_f^3 has signature $(1, 2)$ if $\epsilon = 1$ and $(2, 1)$ if $\epsilon = -1$. If $f(x, y, z) = f(y, z)$, then M_f^3 is called a strict Walker manifold. The Levi-Civita connection ∇ is well-known, and the non-zero components of the Christoffel symbols are:

$$\nabla_{\partial_x} \partial_z = \frac{1}{2} f_x\, \partial_x, \quad \nabla_{\partial_y} \partial_z = \frac{1}{2} f_y\, \partial_x, \quad \nabla_{\partial_z} \partial_z = \frac{1}{2}(f f_x + f_z)\, \partial_x - \frac{\epsilon}{2} f_y\, \partial_y - \frac{1}{2} f_x\, \partial_z.$$

The cross product $\times$, with the property $g_f^{\epsilon}(U \times V, W) = \det(U, V, W)$ for $U, V, W \in \mathbb{R}^3$, is given by:

$$U \times V = (u_1 v_2 - u_2 v_1 - (u_2 v_3 - u_3 v_2) f, -\epsilon(u_1 v_3 - u_3 v_1), u_2 v_3 - u_3 v_2).$$

2.2. Geometry of Surfaces in a Walker 3-Manifold

In this section, we study the differential geometry of surfaces in a Walker manifold. Let U be an open subset of the plane $\mathbb{R}^2$ where horizontal or vertical lines intersect U in intervals (if at all). A two-parameter map is a smooth map $\varphi : U \to M$. Thus, φ is composed of two interwoven families of parameter curves:

1. The u-parameter curves $v = v_0$ of φ is $u \mapsto \varphi(u, v_0)$.
2. The v-parameter curves $u = u_0$ of φ is $v \mapsto \varphi(u_0, v)$.

The partial velocities $\varphi_u = d\varphi(\partial_u)$ and $\varphi_v = d\varphi(\partial_v)$ are vector fields on φ. If φ lies in the domain of a coordinate system $(x_1, \dots, x_n)$, then its coordinate functions $x_i \circ \varphi(1 \leq i \leq n)$ are real-valued functions on U, and:

$$\varphi_u = \sum_i \frac{\partial x_i}{\partial u} \partial x_i, \quad \varphi_v = \sum_i \frac{\partial x_i}{\partial v} \partial x_i.$$

Assume now that M is a pseudo-Riemannian manifold. If Z is a smooth vector field on φ, its partial covariant derivatives are:

$$Z_u = \nabla_{\partial_u} Z, \quad Z_v = \nabla_{\partial_v} Z,$$

where $Z_u(u_0, v_0)$ is the covariant derivative at u_0 of the vector field $u \mapsto Z(u, v_0)$ on the curve $u \mapsto \varphi(u, v_0)$. In coordinates, $Z = \sum_i Z^i \partial x_i$, where each $Z^i = Z(x_i)$ is a real-valued function. Then:

$$Z_u = \sum_k \left(\frac{\partial Z^k}{\partial u} + \sum_{i,j} \Gamma_{ij}^k Z^i \frac{\partial x^j}{\partial u} \right) \partial x^k. \tag{2.2}$$

In the case $Z = \varphi_u$, the derivative $Z_u = \varphi_{uu}$ gives the accelerations of the u-parameter curves, while φ_{vv} gives the v-parameter accelerations. In coordinates:

$$\varphi_{uv} = \sum_k \left(\frac{\partial^2 x^k}{\partial u \partial v} + \sum_{i,j} \Gamma_{ij}^k \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} \right) \partial x^k. \tag{2.3}$$

Next, assume that φ is an isometric immersion. The first fundamental form of the immersion φ is given by:

$$E = g_f^\epsilon(\varphi_u, \varphi_u), \quad F = g_f^\epsilon(\varphi_u, \varphi_v), \quad G = g_f^\epsilon(\phi_v, \varphi_v). \tag{2.4}$$

The coefficients of the second fundamental form of φ are:

$$\begin{cases} L &= \epsilon_1 g_f^\epsilon(\varphi_{uu}, \eta), \\ M &= \epsilon_1 g_f^\epsilon(\varphi_{uv}, \eta), \\ N &= \epsilon_1 g_f^\epsilon(\varphi_{vv}, \eta) \end{cases} \tag{2.5}$$

where $\epsilon_1 = g_f^\epsilon(\eta, \eta)$ is the sign of the unit normal η along φ. Finally, the mean curvature H of the surface φ is given by:

$$H = \frac{\epsilon_1}{2} \frac{LG - 2MF + NE}{EG - F^2}. \tag{2.6}$$

For a surface Σ in (M, g_f^ϵ), the Gauss equation relates the sectional curvature $K(\partial_u, \partial_v)$ of Σ to the sectional curvature of (M, g_f^ϵ) as:

$$K(\partial_u, \partial_v) = K(\partial_u, \partial_v) + \epsilon_1 \frac{LN - M^2}{EG - F^2}. \tag{2.7}$$

3. Ruled Surfaces with Constant Curvature in a Walker 3-manifold

3.1. Ruled Surfaces in M_f^3

In this subsection, we construct the family of ruled surfaces in (M, g_f^ϵ) which are used in the main result. From (1.6), a curve $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ is a geodesic of (M, g_f^ϵ) if the following relations are satisfied:

$$\begin{cases} \frac{d^2\gamma_1(t)}{dt^2} &= f_y \frac{d\gamma_2}{dt}\frac{d\gamma_3}{dt} + \frac{1}{2} f_z \left(\frac{d\gamma_3}{dt}\right)^2 \\ \frac{d^2\gamma_2(t)}{dt^2} &= -\frac{\epsilon}{2} f_y \left(\frac{d\gamma_3}{dt}\right)^2 \\ \frac{d^2\gamma_3(t)}{dt^2} &= 0. \end{cases} \tag{3.1}$$

These equations have the following trivial solutions: $\gamma_1(t) = a_1 t + b_1$, $\gamma_2(t) = a_2 t + b_2$, and $\gamma_3(t) = b_3$, where $a_1, a_2, b_1, b_2, b_3 \in \mathbb{R}$. From these solutions, one gets the following ruled surfaces made by affine straight lines.

Let $r \in \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ be a smooth function. We denote by $\Sigma_1(r, b)$ the surface in M defined by the equation:

$$x + \epsilon r y - \epsilon r^2 z - b(z) = 0.$$

The surface $\Sigma_1(r, b)$ can be parametrized by the map:

$$\begin{aligned} \varphi : \mathbb{R} \times \mathbb{R} &\to M \\ (y, z) &\mapsto y(-\epsilon r, 1, 0) + (b(z), rz, z). \end{aligned} \tag{3.2}$$

We put in the following $y = u$ et $z = v$
Hereafter, we denote such a surface in M_f^3 by $\Sigma(r,b)$. The tangent plane of $\Sigma(r,b)$ is spanned by:

$$\varphi_u = -\epsilon r\partial_x + \partial_u. \tag{3.3}$$

$$\varphi_v = b'\partial_x + r\partial_u + \partial_v. \tag{3.4}$$

where $b'(v) = \frac{\partial b}{\partial v}$.
The local expression of the induced metric on $\Sigma(r,b) \subset M_f^3$ is:

$$\begin{cases} E = g_f^\epsilon(\varphi_u, \varphi_v) = \epsilon, \\ F = g_f^\epsilon(\varphi_u, \varphi_v) = 0, \\ G = g_f^\epsilon(\varphi_v, \varphi_v) = 2b' + \epsilon r^2 + f \end{cases}. \tag{3.5}$$

Thus, $\Sigma(r,b)$ is non-degenerate if and only if $2b' + \epsilon r^2 + f \neq 0$. The unit normal associated with this parametrization is:

$$\eta = \frac{\varphi_u \times \varphi_v}{\|\varphi_u \times \varphi_v\|} = \frac{((-\epsilon r^2 - b') - f, r, 1)}{\sqrt{|2b' + \epsilon r^2 + f|}}. \tag{3.6}$$

The second derivatives of φ are:

$$\varphi_{uu} = 0,$$
$$\varphi_{uv} = \frac{1}{2} f_u \partial_x,$$
$$\varphi_{vv} = (b'' + \frac{1}{2} r f_u + \frac{1}{2} f_v)\partial_x - \frac{\epsilon}{2} f_u \partial_u.$$

The second fundamental form with components L, M, and N is given by:

$$\begin{cases} L = 0. \\ M = \dfrac{\frac{1}{2} f_u}{\sqrt{|2b' + \epsilon r^2 + f|}} \\ N = \dfrac{b'' + \frac{1}{2} f_v}{\sqrt{|2b' + \epsilon r^2 + f|}} \end{cases}. \tag{3.7}$$

Using the formula in (3.8) we get the mean curvature H as:

$$H = \frac{\epsilon_1}{2} \frac{\epsilon(b'' + \frac{1}{2} f_v)}{\epsilon(2b' + \epsilon r^2 + f)^{3/2}} \tag{3.8}$$

where $\epsilon_1 = g_f^\epsilon(\eta, \eta) = \pm 1$ determines the causal character of the unit normal η. The surface is timelike if $\epsilon_1 = 1$ and spacelike if $\epsilon_1 = -1$. By using (2.7) we obtain also the Gaussian curvature K of $\Sigma(r,b)$

$$K = \frac{g_f^\epsilon(R(\varphi_u, \varphi_v)\varphi_v, \varphi_u)}{EG - F^2} + \frac{\epsilon_1(LN - M^2)}{EG - F^2}, \tag{3.9}$$

which simplifies to:

$$K = -\frac{1}{2} \frac{f_{uu}}{\epsilon(2b' + \epsilon r^2 + f)} + \epsilon_1 \frac{\frac{1}{4} f_u^2}{\epsilon(2b' + \epsilon r^2 + f)|\epsilon(2b' + \epsilon r^2 + f)|}. \tag{3.10}$$

3.2. Timelike Ruled Surfaces with Constant Gaussian or Mean Curvature

In this section, we consider the case where the family of ruled surfaces $\Sigma(r,b)$ is timelike, with parametrization $\varphi(u,v) = (-\epsilon ru + b(v), u + rv, v)$, and $r \in \mathbb{R}$. We will discuss both the constant Gaussian curvature K_0 and the constant mean curvature H_0.

Case 1: Constant Gaussian Curvature K_0.

From the expression for Gaussian curvature K in (3.10), the condition $K = K_0$ gives the following equation:

$$K_0 = \frac{-\frac{1}{2}f_{uu}(\epsilon(2b' + \epsilon r^2 + f)) + \frac{1}{4}f_u^2}{(\epsilon(2b' + \epsilon r^2 + f)^2}. \tag{3.11}$$

We have the following theorem:

Theorem 3.1. *The family $\Sigma(r,b))$ of time-like ruled surfaces (3.2.) in a strict Walker 3-manifold has constant Gaussian curvature K_0 if and only if the following holds:*

1. *If $K_0 = 0$ then we have two possibilities:*
   (a) *If $f(u,v) = f_0 + f_v(v)$ then $b(z)$ is any smooth function where f_0 constant.*
   (b) *If $f_u \neq 0$ then $b(v) = \frac{1}{2}\int(\frac{\epsilon}{2}\frac{f_u^2}{f_{uu}} - r^2\epsilon f)dv$*
2. *If $K_0 \neq 0$ then $b(v) = \frac{1}{2}\int(\frac{-\frac{1}{2}f_{uu} \pm \sqrt{\frac{1}{4}f_{uu}^2 + K_0 f_u^2}}{2\epsilon K_0} - \epsilon r^2 - f)dv$ where $\frac{1}{4}f_{uu}^2 + K_0 f_u^2 \geq 0$.*

Proof.

First, we will distinguish the following cases

Case $K_0 = 0$. From (3.11) the condition reads

$$-\frac{1}{2}f_{uu}(\epsilon(2b' + \epsilon r^2 + f)) + \frac{1}{4}f_u^2 = 0.$$

If $f_u = 0$ then any smooth function is the solution of this equation.
If $f_u \neq 0$ then the equation reads to

$$\frac{1}{2}\frac{f_u^2}{f_{uu}} = \epsilon(2b' + \epsilon r^2 + f).$$

Then we have

$$\frac{1}{4\epsilon}\frac{f_u^2}{f_{uu}} - \epsilon r^2 - f = b'(v)$$

Hense

$$b(v) = \frac{1}{4\epsilon}\int(\frac{f_u^2}{f_{uu}} - \epsilon r^2 - f)dv$$

Case $K_0 \neq 0$. From (3.11) we obtain

$$K_0 = \frac{-\frac{1}{2}f_{uu}(\epsilon(2b' + \epsilon r^2 + f)) + \frac{1}{4}f_u^2}{(\epsilon(2b' + \epsilon r^2 + f)^2}$$

Let us put

$$\beta(v) = \epsilon(2b' + \epsilon r^2 + f)$$

then

$$b'(v) = \beta(v) - r^2 - \epsilon f$$

and we have

$$K_0\beta^2(v) + \frac{1}{2}f_{uu}\beta(v) - r^2 - \frac{1}{4}f_u^2 = 0.$$

For $\frac{1}{4}f_{uu}^2 + K_0 f_u^2 \geq 0$, ie $\frac{1}{4}\frac{f_{uu}^2}{f_u^2} \geq K_0$
If $K_0 > 0$ then always true and

$$\beta(v) = \frac{-\frac{1}{2}f_{uu} \pm \sqrt{\frac{1}{4}f_{uu}^2 + K_0 f_u^2}}{2K_0}.$$

Then

$$b'(v) = \frac{1}{2}\frac{-\frac{1}{2}f_{uu} \pm \sqrt{\frac{1}{4}f_{uu}^2 + K_0 f_u^2}}{2\epsilon K_0} - \epsilon r^2 - f$$

Hense

$$b(v) = \frac{1}{2}\int(\frac{-\frac{1}{2}f_{uu} \pm \sqrt{\frac{1}{4}f_{uu}^2 + K_0 f_u^2}}{2\epsilon K_0} - \epsilon r^2 - f)dv$$

If $K_0 < 0$ ie $f_u \geq A(v)e^{2\sqrt{-K_0}}$ then

$$b(v) = \frac{1}{2}\int(\frac{-\frac{1}{2}f_{uu} \pm \sqrt{\frac{1}{4}f_{uu}^2 + K_0 f_u^2}}{2\epsilon K_0} - \epsilon r^2 - f)dv$$

□

Case 2: Constant Mean Curvature H_0.
From the mean curvature expression in (3.8), the condition $H = H_0$ gives the equation:

$$H_0 = \frac{1}{2}\frac{\epsilon(b'' + \frac{1}{2}f_v)}{\epsilon(2b' + \epsilon r^2 + f)^{3/2}}, \quad (3.12)$$

Theorem 3.2. *The family $\Sigma(r, b))$ of time-like ruled surfaces (3.2.) in a strict Walker 3-manifold is minimal if and only if the following holds:*

1. *$f_v = 0$ and $b(v) = av + c_0$ is a smooth affine function, where $(a, c_0) \in \mathbb{R}$*

2. $f_v \neq 0$ *and* $b(v) = \frac{-1}{2} \int f_v dv + c$

Proof. First, we will distinguish the following cases
From 3.8 the condition reads

$$\frac{1}{2}(\epsilon(b'' + \frac{1}{2} f_v)) = 0$$

if $f_v = 0$ then $b'' = 0$ i.e. $b(v) = av + c_0$ smooth affine function Where $(a, c_0) \in \mathbb{R}$
if $f_v \neq 0$ then the equation reads to

$$b''(v) = -\frac{1}{2} f_v$$

and

$$b'(v) = -\frac{1}{2} f$$

Then

$$b(v) = -\int \frac{1}{2} f dv.$$

□

Theorem 3.3. *The family* $\Sigma(r, b))$ *of timelike ruled surfaces (3.2.) in a strict Walker 3-manifold has constant non-zero mean curvature if and only if* $b(v) = \frac{\epsilon}{2} \int \lambda(v) - \epsilon r^2 - f)dv$ *where* $\lambda(v) = \frac{1}{-2H_0 v + C_1}$.

Proof.

We suppose that $H = H_0 \neq 0$. From 3.8 we obtain

$$H_0 = \frac{1}{4} \frac{\epsilon(2b'' + f_v)}{[\epsilon(2b' + \epsilon r^2 + f)]^{3/2}}.$$

We put

$$\lambda(v) = \epsilon(2b' + \epsilon r^2 + f)$$

and the derivative with respect to z gives

$$\lambda'(v) = \epsilon(2b'' + f_v).$$

Then we obtain

$$4H_0 = \frac{\lambda'(v)}{\lambda^{\frac{3}{2}}}(v)$$

We integrate

$$4 \int H_0 dv = \int \frac{\lambda'(v)}{\lambda^{\frac{3}{2}}}(v) dv$$

Hense

$$\lambda(v) = \frac{1}{(-2H_0 v - \frac{c}{2})^2}$$

Then we have

$$b(v) = \int \left(\frac{1}{(-2H_0 v - \frac{c}{2})^2} - \frac{r^2 + \epsilon f}{2} \right) dv.$$

□

3.3. Spacelike ruled Surfaces with Constant Curvature in a strict Walker 3-manifold

In this part, we consider the case of spacelike ruled surfaces given by (3.2.) with $\varphi(u,v) = (-\epsilon ru + b(v), u + rv, v)$, and r $\in \mathbb{R}$. We will discuss both the cases where K_0 or H_0 is zero or not. Let us first consider the constant Gaussian curvature case $K = K_0$, which yields the same equation as in the timelike case:

$$K_0 = \frac{-\frac{1}{2}f_{uu}(\epsilon(2b' + \epsilon r^2 + f)) - \frac{1}{4}f_u^2}{(\epsilon(2b' + \epsilon r^2 + f)^2} \tag{3.13}$$

which gives the same thing as in the case where le ruled surface is timelike. Thus, we obtain the same conclusions as in Theorem 3.1.

Now, we consider the constant mean curvature $H = H_0$. Hence we get from (3.6),that

$$H_0 = -\frac{1}{2}\frac{\epsilon(b'' + \frac{1}{2}f_v)}{\epsilon(2b' + \epsilon r^2 + f)^{3/2}}, \tag{3.14}$$

Theorem 3.4. *The family $\Sigma(r,b))$ of spacelike ruled surfaces (3.2.) in a strict Walker 3-manifold has constant mean curvature H_0 if and only if the following hold:*

1. $H_0 = 0$
   (a) *If $f_v = 0$ then $b(v) = av+c_0$ is a smooth affine function. Where $(a, c_0) \in \mathbb{R}$*
   (b) *If $f_v \neq 0$ then $b(v) = \frac{\epsilon}{2}\int f_v dv + c$*
2. *If $H_0 \neq 0$ then $b(v) = -\frac{\epsilon}{2}\int(\lambda(v) - \epsilon r^2 - f)dv$ where $\lambda(v) = \frac{1}{-2H_0 v + C_1}$*

3.4. Lightlike Ruled Surfaces with Constant Curvature in a strict Walker 3-manifold

In this section, we will examine the case of lightlike ruled surfaces with constant null sectional or null mean curvatures in a strict Walker 3-manifold. If $\Sigma(r,b)$ is degenerate and spanned by (η,ν) where η is a null vector of $T\Sigma(r,b)$ and $\nu \in T\Sigma(r,b)$ such that $g_f^\epsilon(\eta,\nu) = 0$ and $g_f^\epsilon(\nu,\nu) \neq 0$, then the null mean curvature H and the null sectional curvature K_η (see [7]) are defined respectively by:

$$H = trace(B) \tag{3.15}$$

and

$$K_\eta\left(\Sigma(r,b)\right) = \frac{g_f^\epsilon(R(\nu,\eta)\eta,\nu)}{g_\epsilon^f(\nu,\nu)}, \tag{3.16}$$

where B is the second fundamental form of the lightlike surface $\Sigma(r,b)$ and R is the Riemann curvature tensor. The ruled surface $\Sigma(r,b)$ given by (3.2.) where $(r) \in \mathbb{R}$, is a lightlike ruled surface if and only if b is a solution of the partial differential equation:

$$b' = \epsilon(-\epsilon r^2 - f). \tag{3.17}$$

The tangent plane of $\Sigma(r,b)$ is spanned by (φ_z, η), where $\eta = (-\epsilon r^2 - b') - f, r, 1)$. Thus, the radical distribution is given by:

$$Rad(T\Sigma(r,b)) = span\{\eta\}.$$

The vector field $\tau = \partial_x$ is a parallel vector field such that $g_f^\epsilon(X,\eta) = 1$, and thus $\tau = \partial_x$ is a global rigging on $T\Sigma(r,b)$. The transversal vector τ of the lightlike ruled surface $\Sigma(r,b)$ is defined by:

$$N := \tau - \frac{1}{2}g_f^\epsilon(\tau,\tau)\eta = \tau = \partial_x.$$

The ruled distribution $ltr(T\Sigma(r,b)$ and the screen distribution $S(T\Sigma(r,b))$ are given respectively by:

$$ltr(T\Sigma(r,b)) = span\{N = \partial_x\}$$
$$S(T\Sigma(r,b)) = span\{\varphi_v = b'\partial_x + r\partial_u + \partial_v\}.$$

The Gauss formula is given by:

$$\nabla^\circ_X Y = \nabla_X Y + B(X,Y)N \tag{3.18}$$

for all $X, Y \in \Gamma(T\Sigma(r,b))$, $N \in \Gamma(ltr(T\Sigma(r,b))$, where $\nabla_X Y \in \Gamma(T\Sigma(r,b))$ and B is the second fundamental form of $\Sigma(r,b)$. Let us now compute $\nabla^\circ_\xi \xi$, $\nabla^\circ_{\varphi_u}\varphi_u$, $\nabla^\circ_\xi \varphi_u$ et $\nabla^\circ_{\varphi_u}\xi$. By direct calculations, we obtain :

$$\nabla^\circ_\eta \eta = -(2b'' + f_v)\partial_x, \qquad \nabla^\circ_{\varphi_u}\varphi_u = (2rf_u + f_v)\partial_x$$

$$\nabla^\circ_\eta \varphi_v = (2rf_u + f_v)\partial_x, \qquad \nabla^\circ_{\varphi_u}\eta = -(2b'' + f_v)\partial_x$$

and from Levi-Civita connection and from $\nabla^\circ_X Y = \nabla_X Y + B(X,Y)N$ we find

$$\begin{aligned}
\nabla_\eta \nabla_{\varphi_v}\eta &= -(b''' + \frac{1}{2}f_{vv})\partial_x \\
\nabla_{\varphi_v}\nabla_\eta \eta &= -(b''' + \frac{1}{2}f_{vv})\partial_x \\
\nabla_{[\varphi_v,\eta]}\eta &= -(2b'' + f_v)\partial_x
\end{aligned} \tag{3.19}$$

Let us consider the case where the lightlike ruled surface $\Sigma(r,b)$ has a constant null sectional curvature $K_\eta(\Sigma(r,b)) = K_0$. In this case the null sectional curvature K_η of the lightlike ruled surface $\Sigma(r,b)$ with respect to η and using (3.16) is given by

$$K_\eta(\Sigma(r,b)) = \frac{g_f^\epsilon(R(\varphi_v,\eta)\eta,\varphi_v)}{g_f^\epsilon(\varphi_v,\varphi_v)} = \frac{-2b''-f_v}{2b'+\epsilon r^2+f} \tag{3.20}$$

By degenerate lightlike ruled surface with a constant null sectional curvature K_η we have:

Theorem 3.5. *The family $\Sigma(r,b))$ of lightlike ruled surfaces (3.2.) in a strict Walker 3-manifold has constant null sectional curvature K_0 if and only if the following hold:*

1. *If $K_0 = 0$*
   (a) *$f_v = 0 \Rightarrow b(v)$ is any smooth affine function.*
   (b) *$f_v \neq 0 \Rightarrow b(v) = \int -\frac{1}{2}f(v)dv + c_1 v + c_0, \quad c_1, c_0 \in \mathbb{R}$*
2. *If $K_0 \neq 0$ then $b(v) = \frac{1}{2}\int \left(e^{K_0+c_0} - \epsilon r^2 - f\right)dv$, where $c_0 \in \mathbb{R}$.*

Proof.

Case 1: If we suppose $K_0 = 0$, then we have $-2b'' - f_v = 0$. The case where $_v = 0$ mean that $b''(v) = 0$, that is $b(v)$ is any smooth affine function.
Now suppose that $f_v \neq 0$

$$2b''(v) = -f_v$$
$$b'(v) = -\frac{1}{2}f(v)dv + c_0.$$

Hence by integration we get

$$b(v) = \int -\frac{1}{2}f(v)dv + c_1 v + c_0, \quad c_1, c_0 \in \mathbb{R}$$

case 2: If we suppose that $K_0 \neq 0$, then we have

$$-2b'' - f_v = K_0\left(2b' + \epsilon r^2 + f\right).$$

We put $\omega(v) = 2b' + \epsilon r^2 + f$

$$\omega'(v) = 2b''(v) + f_v$$
$$-\omega'(v) = K_0\omega(v)$$
$$K_0 = \frac{\omega'(v)}{\omega(v)}$$
$$\omega(v) = e^{K_0+c_0}.$$

Then by integrating we get

$$b(v) = \frac{1}{2}\int \left(e^{K_0+c_0} - \epsilon r^2 - f\right)dv$$

where $c_0 \in \mathbb{R}$. □

Remark 3.1. Physical interpretation of $K_0 = \frac{\omega'(v)}{\omega(v)}$

1. Radioactive decay: If $y(t)$ represents the amount of radioactive material, k_0 is related to the decay constant, and the equation describes how the amount of material decreases over time.
2. Electrical circuit (capacitor discharge): In an RC circuit, $y(t)$ could represent the charge of the capacitor, and k_0 would be related to the resistance and capacitance of the circuit. The equation describes the exponential discharge of the capacitor over time.
3. Cooling: If $y(t)$ represents the temperature difference between an object and its environment, and k_0 is proportional to the thermal conductance, this equation models the cooling of a body according to Newton's law of cooling.

Now, we consider the constant null mean curvature $H = H_0$ situation on lightlike ruled surfaces in a strict Walker 3-manifold.
Thus, from the Gauss formula and (3.26) we get

$$B(X,Y) = g_f^\epsilon(\nabla_X Y, \eta), \forall X, Y \in T\Sigma(r,b),$$

and

$$B(\varphi_v, \varphi_v) = \frac{1}{2} r f_u + \frac{1}{2} f_v, \qquad B(\eta, \eta) = b'' - \frac{1}{2} r f_u - \frac{1}{2} f_v$$

Then, we can state the following theorem.

Theorem 3.6. *The family $\Sigma(r,b))$ of lightlike ruled surfaces (3.2.) in a strict Walker 3-manifold has constant null mean curvature H_0 if and only if the following hold:*

$$b(v) = H_0 v^2 + av + b, \tag{3.21}$$

and

$$f(v) = \epsilon\left(2H_0 v + r^2 + \epsilon a\right), \tag{3.22}$$

where $(a,b) \in \mathbb{R}^2$.

Proof.

By the definition of H, one get $H = B(\varphi_v, \varphi_v) + B(\eta, \eta)$. Hense we have

$$H = \frac{1}{2} r f_u + \frac{1}{2} f_v + b'' - \frac{1}{2} r f_u - \frac{1}{2} f_v = b''$$

Hense

$$b(v) = H_0 v^2 + av + b$$

From $b' = \epsilon(-\epsilon r^2 - f)$

$$f(v) = \epsilon\left(2H_0 v + r^2 + \epsilon a\right).$$

□

4. Conclusion

In this work, we have studied the differential geometry of a special family of ruled surfaces in a strict Walker 3-manifold. For each surface we distinguish the case where the surface is space-like, time-like and light-like and we have the classifications of this family of ruled surfaces with constant mean curvature and constant Gauss curvature.
The important question in the rest of this work is to solve the equation of geodesics and to define the concept of ruled surfaces in a 3-dimensional Walker manifold and to study the properties of their differential geometry.

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