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A NOTE ON THE PERTURBATIONS OF PSEUDOSPECTRA OF 2×2 UPPER-TRIANGULAR OPERATOR MATRICES

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Abstract. In this paper we completely describe the sets $\bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma_\varepsilon(M_C)$ and $\bigcup_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma_\varepsilon(M_C)$. We also analyze the similarities and differences between these results in the case when the spectrum is perturbed.
Keywords: Hilbert space, operator matrix, 2×2 operator matrix, spectrum, pseudospectrum, perturbations of pseudospectrum.

1. Introduction and motivation

For subspaces $\mathcal{M}$ and $\mathcal{N}$ of a Hilbert space $\mathcal{H}$ with $\mathcal{M} \subseteq \mathcal{N}$, we set $codim_{\mathcal{N}}\,\mathcal{M} = dim\,\mathcal{N}/\mathcal{M}$ and, if $\mathcal{M}$ is closed, use the symbol $P_{\mathcal{M}}$ to denote the orthogonal projection onto $\mathcal{M}$. If $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, then by $P_{\mathcal{M},\mathcal{N}}$ we denote the projection onto $\mathcal{M}$ parallel to $\mathcal{N}$. For a given operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively. We use the standard notations $n(A) = \dim \mathcal{N}(A)$, $\beta(A) = \mathrm{codim}\mathcal{R}(A)$ and $d(A) = \dim \mathcal{R}(A)^\perp$. Also, by $\mathcal{P} \oplus \mathcal{Q}$ we denote the direct sum of the closed subspaces $\mathcal{P}$ and $\mathcal{Q}$.

If $A \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $\mathcal{M}$ is a subspace of $\mathcal{K}$ then the restriction of the operator A to the subspace $\mathcal{M}$ will be denoted by $A|_{\mathcal{M}}$. Also, for $\mathcal{S} \subseteq \mathcal{H}$ by $A^{-1}(\mathcal{S})$ we denote the inverse image of $\mathcal{S}$ i.e. $A^{-1}(\mathcal{S}) = \{x \in \mathcal{K} : Ax \in \mathcal{S}\}$. For $A \in \mathcal{B}(\mathcal{H})$ the spectrum (left-spectrum, right-spectrum) is denoted by $\sigma(A)$ ($\sigma_l(A)$, $\sigma_r(A)$), while $\rho(A) = \mathbb{C} \setminus \sigma(A)$ ($\rho_l(A) = \mathbb{C} \setminus \sigma_l(A)$, $\rho_r(A) = \mathbb{C} \setminus \sigma_r(A)$) denotes the resolvent (left

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resolvent, right resolvent) set. Throughout the paper, we will abbreviate $A - \lambda I$ as $A - \lambda$ for a bounded linear operator A.

The definition of the ε-pseudospectrum that will be used throughout this paper is:

Definition 1.1. Let $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$ be arbitrary. The ε-pseudospectrum $\sigma_\varepsilon(A)$ is the set

$$\sigma_\varepsilon(A) = \left\{ \lambda \in \mathbb{C} : \lambda \in \sigma(A) \text{ or } \left(\lambda \in \rho(A) \text{ and } \|(A-\lambda)^{-1}\| > \frac{1}{\varepsilon} \right) \right\}.$$

More on the topic and basic definition can be found in the seminal monograph [20] by L. N. Trefethen and M. Embree on this topic. The notion of the pseudospectrum has found many applications in both mathematics and other sciences. In [10] Driscoll and Trefethen applied the psedospectrum to the study of the wave equation, while Davies in [9] discussed pseudospectra in quantum mechanics and differential operators. In [19], Reddy, Schmid, and Henningson applied an analysis based on the pseudospectrum to fluid dynamics and stability of shear flows. In [1] Burke, Lewis Overton investigated the use of pseudospectra in optimization problems, particularly focusing on robust stability analysis in control systems. Jaramillo, Macedo, and Al Sheikh applied pseudospectral analysis to black hole physics in [14].

The study of operator matrices, primarily upper-triangular operator matrices has been extensive over the years with important results and applications. The motivation for this paper is to extend this branch in a new direction.

The study of 2×2 upper triangular operator matrices naturally arises from the fact that if a bounded linear operator A acting on a Banach space $\mathcal{X}$ has a complemented invariant subspace $\mathcal{S}$, we can represent A as

$$A = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} : \mathcal{S} \oplus \mathcal{P} \to \mathcal{S} \oplus \mathcal{P},$$

where $\mathcal{X} = \mathcal{S} \oplus \mathcal{P}$.

In this paper, we will study the ε-pseudospectra of an upper-triangular matrix

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K},$$

where $\mathcal{H}$ and $\mathcal{K}$ are infinite dimensional separable Hilbert spaces. To be more precise, for an arbitrary $\varepsilon > 0$ we will completely describe the sets

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_\varepsilon(M_C) \text{ and } \bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_\varepsilon(M_C).$$

In the case of the spectrum the set $\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C)$ and $\bigcup_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C)$ were for the first time described by Du and Pan in [11] with the following result for the Hilbert space case:

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C) = \sigma_l(A) \cup \sigma_r(B) \cup \{\lambda \in \mathbb{C} : n(B - \lambda I_\mathcal{K}) \neq d(A - \lambda I_\mathcal{H})\}.$$

Han, Lee, and Lee in [12] obtained equivalent results in the case of Banach spaces,

$$\bigcap_{C\in\mathcal{B}(\mathcal{X},\mathcal{Y})} \sigma(M_C) = \sigma_l(A)\cup\sigma_r(B)\cup\{\lambda\in\mathbb{C} : \mathcal{N}(B-\lambda I_{\mathcal{K}}) \not\cong \mathcal{X}/\mathcal{R}(A-\lambda I_{\mathcal{H}})\}.$$

Du and Pan raised the following 3 questions:

Question 1. For fixed $A\in\mathcal{B}(\mathcal{H})$ and $B\in\mathcal{B}(\mathcal{K})$ does there exists an operator $C\in\mathcal{B}(\mathcal{K},\mathcal{H})$ such that $\sigma(M_C)\subset\sigma(M_0)$,

Question 2. Completely describe $\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma(M_C)$ for fixed $A\in\mathcal{B}(\mathcal{H})$ and $B\in\mathcal{B}(\mathcal{K})$,

Question 3. Does an operator $C_0\in\mathcal{B}(\mathcal{K},\mathcal{H})$ exists such that $\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma(M_C)=\sigma(M_{C_0})$,

and for the first two questions they provided answers. Question 3. is still only partially answered. This paper will primarily focus on the equivalent of Question 2. for the pseudospectrum, and partially on Questions 1. and 3.

A systematic study of the spectrum (and other types of spectra) of upper-triangular matrices was done in [3], which also included partial answers to Question 3. Various results regarding operator matrices can be found in [4, 5, 6, 13, 18, 21], which together with the mentioned papers can serve as a good introduction into the topic.

One of the reasons for the extensive study of operator matrices is the application of their results to specific problems in operator theory. Results concerning upper-triangular operator matrices were applied to the study of linear combinations of operators on Hilbert spaces in [7, 15, 8]. Completions of upper-triangular operator matrices were applied in [17] to the study of the reverse-order law for {1}-inverses on Hilbert spaces, while in [16] new representations for the generalized Bott-Duffin inverse were achieved.

2. Preliminary results

Before we proceed with the results we should briefly comment on the operator norm of $M_C\in\mathcal{B}(\mathcal{X}\oplus\mathcal{Y})$ and how it relates to the norms of $A\in\mathcal{B}(\mathcal{X})$, $B\in\mathcal{B}(\mathcal{Y})$ and $C\in\mathcal{B}(\mathcal{Y},\mathcal{X})$, where $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, and present some auxiliary results.

For a vector $\begin{bmatrix} x\\ y\end{bmatrix}\in\mathcal{X}\oplus\mathcal{Y}$ the norm on $\mathcal{X}\oplus\mathcal{Y}$ is naturally defined by

$$\left\|\begin{bmatrix} x\\ y\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}} = \sqrt{\|x\|_{\mathcal{X}}^2+\|y\|_{\mathcal{Y}}^2},$$

which allows us to write the definition of the standard norm of $\begin{bmatrix} A & C\\ D & B\end{bmatrix}\in\mathcal{B}(\mathcal{X}\oplus\mathcal{Y})$

as

$$\left\| \begin{bmatrix} A & C \\ D & B \end{bmatrix} \right\|_{\mathcal{X}\oplus\mathcal{Y}} = \sup_{\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = 1} \sqrt{\|Ax+Cy\|_{\mathcal{X}}^2 + \|Dx+By\|_{\mathcal{Y}}^2}$$
$$= \sup_{\|x\|_{\mathcal{X}}^2 + \|y\|_{\mathcal{Y}}^2 = 1} \sqrt{\|Ax+Cy\|_{\mathcal{X}}^2 + \|Dx+By\|_{\mathcal{Y}}^2}$$

In the case of Hilbert spaces, $\mathcal{H}$ and $\mathcal{K}$, this norm coincides with the norm generated by the inner product defined on their sum. This norm is generalized naturally to direct sums of 3 or more Banach spaces in the same manner.

Remark: Notice that it is easy to establish that

$$\min\{\|A\|_{\mathcal{X}}, \|B\|_{\mathcal{Y}}\} \leq \left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{\mathcal{X}\oplus\mathcal{Y}} \leq \max\{\|A\|_{\mathcal{X}}, \|B\|_{\mathcal{Y}}\}. \tag{2.1}$$

The previous inequality can be extended to diagonal and off-diagonal cases for 3×3 operator matrices as well. The following lemma covers one case:

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{X}_1, \mathcal{Y}_1)$, $B \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_3)$, $C \in \mathcal{B}(\mathcal{X}_3, \mathcal{Y}_3)$, where $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$ are Banach spaces. Then the norm of the operator matrix*

$$\begin{bmatrix} A & 0 & 0 \\ 0 & 0 & C \\ 0 & B & 0 \end{bmatrix} : \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus X_3 \to \mathcal{Y}_1 \oplus \mathcal{Y}_2 \oplus \mathcal{Y}_3$$

satisfies the following inequality

$$\left\| \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & C \\ 0 & B & 0 \end{bmatrix} \right\| \leq \max\{\|A\|, \|B\|, \|C\|\} \tag{2.2}$$

where $\|\cdot\|$ denotes the operator norm on the appropriate space.

Proof. Let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an arbitrary unit vector in the Banach space $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$, that is $\|x_1\|_{\mathcal{X}_1}^2 + \|x_2\|_{\mathcal{X}_2}^2 + \|x_3\|_{\mathcal{X}_3}^2 = 1$. Notice that

$$\begin{aligned}
\left\| \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & C \\ 0 & B & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\|_{\mathcal{Y}_1\oplus\mathcal{Y}_2\oplus\mathcal{Y}_3}^2 &= \left\| \begin{bmatrix} Ax_1 \\ Cx_3 \\ Bx_2 \end{bmatrix} \right\|_{\mathcal{Y}_1\oplus\mathcal{Y}_2\oplus\mathcal{Y}_3}^2 \\
&= \|Ax_1\|_{\mathcal{Y}_1}^2 + \|Cx_3\|_{\mathcal{Y}_2}^2 + \|Bx_2\|_{\mathcal{Y}_3}^2 \\
&\leq \|A\|^2\|x_1\|_{\mathcal{X}_1}^2 + \|B\|^2\|x_2\|_{\mathcal{X}_2}^2 + \|C\|^2\|x_3\|_{\mathcal{X}_3}^2 \\
&\leq (\max\{\|A\|, \|B\|, \|C\|\})^2 \left(\|x_1\|_{\mathcal{X}_1}^2 + \|x_2\|_{\mathcal{X}_2}^2 + \|x_3\|_{\mathcal{X}_3}^2\right) \\
&= (\max\{\|A\|, \|B\|, \|C\|\})^2
\end{aligned}$$

Since the choice of the unit vector in $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ was arbitrary, it follows that

$$\left\| \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & C \\ 0 & B & 0 \end{bmatrix} \right\| \leq \max\{\|A\|, \|B\|, \|C\|\}$$

which is what we wanted to show. □

The result we will primarily rely on is the following Theorem from [3]:

Theorem 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ given operators. The operator matrix M_C is invertible for some $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ if and only it:*

(i) A is left invertible;

(ii) B is right invertible;

(iii) $\mathcal{N}(B) \cong \mathcal{X}/\mathcal{R}(A)$.

If conditions $(i)-(iii)$ are satisfied, the set of all $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_C is invertible is given by

$$S(A,B) = \Big\{ C \in \mathcal{B}(\mathcal{Y}, \mathcal{X}) : C = \begin{bmatrix} C_1 & 0 \\ 0 & C_4 \end{bmatrix} : \mathcal{P} \oplus \mathcal{N}(B) \to \mathcal{R}(A) \oplus \mathcal{S},$$
$$C_4 \text{ is invertible}, \mathcal{X} = \mathcal{R}(A) \oplus \mathcal{S},\, \mathcal{Y} = \mathcal{P} \oplus \mathcal{N}(B) \Big\}$$

This Theorem can be rephrased in the case of Hilbert spaces as:

Remark: If the conditions of Theorem 2.1 are satisfied, then the operator matrix M_C can be represented as

$$M_C = \begin{bmatrix} A_1 & C_1 & 0 \\ 0 & 0 & C_4 \\ 0 & B_1 & 0 \end{bmatrix} : \mathcal{X} \oplus \mathcal{P} \oplus \mathcal{N}(B) \to \mathcal{R}(A) \oplus \mathcal{S} \oplus \mathcal{K} \tag{2.3}$$

where A_1 and B_1 are invertible operators. Using this representation it is easy to verify that

$$M_C^{-1} = \begin{bmatrix} A_1^{-1} & 0 & -A_1^{-1} C_1 B_1^{-1} \\ 0 & 0 & B_1^{-1} \\ 0 & C_4^{-1} & 0 \end{bmatrix} : \mathcal{R}(A) \oplus \mathcal{S} \oplus \mathcal{K} \to \mathcal{X} \oplus \mathcal{P} \oplus \mathcal{N}(B). \tag{2.4}$$

In the case when either A or B is invertible, M_C will be invertible if and only if both A and B are invertible, and it will be invertible for all $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. The inverse of M_C will in this case be given by

$$M_C^{-1} = \begin{bmatrix} A^{-1} & -A^{-1} C B^{-1} \\ 0 & B^{-1} \end{bmatrix}.$$

From the previous remark, we see that $\sigma(M_0) = \sigma(A) \cup \sigma(B)$, and in the result that follows, we establish an analogous result for the ε-pseudospectrum:

Lemma 2.2. *Let $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y})$ and $\varepsilon > 0$. We have that $\sigma_\varepsilon(M_0) = \sigma_\varepsilon(A) \cup \sigma_\varepsilon(B)$.*

Proof. Since $\sigma(M_0) = \sigma(A) \cup \sigma(B)$ and

$$(M_0 - \lambda)^{-1} = \begin{bmatrix} (A-\lambda)^{-1} & 0 \\ 0 & (B-\lambda)^{-1} \end{bmatrix}, \lambda \in \rho(A) \cap \rho(B)$$

it is sufficient to show that for any $\lambda \in \rho(M_0)$ we have that

$$\|(M_0 - \lambda)^{-1}\|_{\mathcal{X}\oplus\mathcal{Y}} > \frac{1}{\varepsilon} \Leftrightarrow \left(\|(A-\lambda)^{-1}\|_{\mathcal{X}} > \frac{1}{\varepsilon} \text{ or } \|(B-\lambda)^{-1}\|_{\mathcal{Y}} > \frac{1}{\varepsilon} \right).$$

Since

$$\|(M_0 - \lambda)^{-1}\|_{\mathcal{X}\oplus\mathcal{Y}} \leq \max\{\|(A-\lambda)^{-1}\|_{\mathcal{X}}, \|(B-\lambda)^{-1}\|_{\mathcal{Y}}\}$$

it immediately follows that if $\|(M_0 - \lambda)^{-1}\|_{\mathcal{X}\oplus\mathcal{Y}} > \frac{1}{\varepsilon}$, then $\|(A-\lambda)^{-1}\|_{\mathcal{X}} > \frac{1}{\varepsilon}$ or $\|(B-\lambda)^{-1}\|_{\mathcal{Y}} > \frac{1}{\varepsilon}$. Conversely, let us suppose that $\|(A-\lambda)^{-1}\|_{\mathcal{X}} > \frac{1}{\varepsilon}$ or $\|(B-\lambda)^{-1}\|_{\mathcal{Y}} > \frac{1}{\varepsilon}$. Without loss of generality we can suppose that $\|(A-\lambda)^{-1}\|_{\mathcal{X}} > \frac{1}{\varepsilon}$ so there exists a vector $x \in \mathcal{X}$ such that $\|x\|_{\mathcal{X}} = 1$ and $\|(A-\lambda)^{-1}x\|_{\mathcal{X}} > \frac{1}{\varepsilon}$. Then $\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{X} \oplus \mathcal{Y}$, $\left\| \begin{bmatrix} x \\ 0 \end{bmatrix} \right\|_{\mathcal{X}\oplus\mathcal{Y}} = 1$ and

$$\left\| (M_0 - \lambda)^{-1} \begin{bmatrix} x \\ 0 \end{bmatrix} \right\|_{\mathcal{X}\oplus\mathcal{Y}} = \left\| \begin{bmatrix} (A-\lambda)^{-1}x \\ 0 \end{bmatrix} \right\|_{\mathcal{X}\oplus\mathcal{Y}} = \|(A-\lambda)^{-1}x\|_{\mathcal{X}} > \frac{1}{\varepsilon}$$

completes the proof. □

3. Main Results

With the preliminary results now in place we proceed with the main results of this paper:

Theorem 3.1. *Let $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y})$ and $\varepsilon > 0$. Then*

$$\bigcap_{C \in \mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma_\varepsilon(M_C) = \sigma_0 \cup ((\sigma_\varepsilon(A) \cup \sigma_\varepsilon(B)) \setminus (\sigma(A) \cup \sigma(B))) \cup \Sigma_\varepsilon. \tag{3.1}$$

where

$$\sigma_0 = \bigcap_{C \in \mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma(M_C),$$

$$\Sigma_\varepsilon = \left\{ \lambda \in (\sigma(A) \setminus \sigma_l(A)) \cup (\sigma(B) \setminus \sigma_r(B)) \, : \, ||(A-\lambda)_l^{-1}||_{\mathcal{X}} > \frac{1}{\varepsilon} \right.$$
$$\left. \text{or } ||(B-\lambda)_r^{-1}||_{\mathcal{Y}} > \frac{1}{\varepsilon} \right\}$$

Proof. ($\supseteq$): Let us first show that $\sigma_0 \cup ((\sigma_\varepsilon(A) \cup \sigma_\varepsilon(B)) \setminus (\sigma(A) \cup \sigma(B))) \cup \Sigma_\varepsilon \subseteq \bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma_\varepsilon(M_C)$. From the definition of the ε-pseudospectrum it is clear that

$$\sigma_0 \subseteq \bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma_\varepsilon(M_C).$$

Now, let $\lambda \in (\sigma_\varepsilon(A)\cup\sigma_\varepsilon(B))\setminus(\sigma(A)\cup\sigma(B))$ be arbitrary. Since $\lambda \notin \sigma(A)\cup\sigma(B)$ we have that $M_C - \lambda$ is invertible for every $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$. If $\lambda \in \sigma_\varepsilon(A)$ then there exists a unit vector $x \in \mathcal{X}$ such that $\|(A-\lambda)^{-1}x\|_{\mathcal{X}} > \frac{1}{\varepsilon}$, so for an arbitrary $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ we have that

$$\left\|(M_C-\lambda)^{-1}\begin{bmatrix}x\\0\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}} = \left\|\begin{bmatrix}(A-\lambda)^{-1}x\\0\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}} = \|(A-\lambda)^{-1}x\|_{\mathcal{X}} > \frac{1}{\varepsilon}.$$

Similarly, if $\lambda \in \sigma_\varepsilon(B)$ then there exists a unit vector $y \in \mathcal{Y}$ such that $\|(B-\lambda)^{-1}y\|_{\mathcal{Y}} > \frac{1}{\varepsilon}$, so for an arbitrary $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ we have that

$$\begin{aligned}\left\|(M_C-\lambda)^{-1}\begin{bmatrix}0\\y\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}} &= \left\|\begin{bmatrix}-(A-\lambda)^{-1}C(B-\lambda)^{-1}y\\(B-\lambda)^{-1}y\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}}\\ &= \sqrt{\|(A-\lambda)^{-1}C(B-\lambda)^{-1}y\|_{\mathcal{X}}^2 + \|(B-\lambda)^{-1}y\|_{\mathcal{Y}}^2}\\ &\geq \|(B-\lambda)^{-1}y\|_{\mathcal{Y}} > \frac{1}{\varepsilon}.\end{aligned}$$

Since $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ was arbitrary in these considerations we can conclude that $(\sigma_\varepsilon(A)\cup\sigma_\varepsilon(B))\setminus(\sigma(A)\cup\sigma(B)) \subseteq \bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})}\sigma_\varepsilon(M_C)$.

Next, let $\lambda \in \Sigma_\varepsilon$ be arbitrary. From the definition of the set Σ_ε we know that $A-\lambda$ and $B-\lambda$ are not invertible, but $A-\lambda$ is left invertible and $B-\lambda$ is right invertible, and $\|(A-\lambda)_l^{-1}\|_{\mathcal{X}} > \frac{1}{\varepsilon}$ or $\|(B-\lambda)_r^{-1}\|_{\mathcal{Y}} > \frac{1}{\varepsilon}$. Let $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ be arbitrary. We have two possibilities, either $M_C-\lambda$ is invertible, or it is not. If it is not invertible then $\lambda \in \sigma(M_C) \subseteq \sigma_\varepsilon(M_C)$. If $M_C-\lambda$ is invertible we can represent it as

$$M_C-\lambda = \begin{bmatrix}(A-\lambda)_1 & C_1 & 0\\0 & 0 & C_4\\0 & (B-\lambda)_1 & 0\end{bmatrix} : \mathcal{X}\oplus\mathcal{P}\oplus\mathcal{N}(B-\lambda) \to \mathcal{R}(A-\lambda)\oplus\mathcal{S}\oplus\mathcal{K}$$

$\mathcal{X} = \mathcal{R}(A-\lambda)\oplus\mathcal{S}$, $\mathcal{Y} = \mathcal{P}\oplus\mathcal{N}(B-\lambda)$, and where $(A-\lambda)_1$ and $(B-\lambda)_1$ are invertible operators such that

$$\begin{aligned}(A-\lambda)_l^{-1} &= \begin{bmatrix}(A-\lambda)_1^{-1} & 0\end{bmatrix} : \mathcal{R}(A-\lambda)\oplus\mathcal{S} \to \mathcal{X}\\ (B-\lambda)_r^{-1} &= \begin{bmatrix}(B-\lambda)_1^{-1}\\0\end{bmatrix} : \mathcal{Y} \to \mathcal{P}\oplus\mathcal{N}(B-\lambda).\end{aligned}$$

By (2.4) we know that $(M_C-\lambda)^{-1}$ can be represented as

$$\begin{bmatrix}(A-\lambda)_1^{-1} & 0 & -(A-\lambda)_1^{-1}C_1(B-\lambda)_1^{-1}\\0 & 0 & (B-\lambda)_1^{-1}\\0 & C_4^{-1} & 0\end{bmatrix} : \mathcal{R}(A-\lambda)\oplus\mathcal{S}\oplus\mathcal{K} \to \mathcal{X}\oplus\mathcal{P}\oplus\mathcal{N}(B-\lambda).$$

If $\|(A-\lambda)_l^{-1}\|_{\mathcal{X}} > \frac{1}{\varepsilon}$ then there exists a unit vector $x_1 \in \mathcal{R}(A-\lambda)$ such that $\|(A-\lambda)_l^{-1}x_1\|_{\mathcal{X}} > \frac{1}{\varepsilon}$ from which we get

$$\begin{aligned}
\left\|(M_C-\lambda)^{-1}\begin{bmatrix}x_1\\0\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}} &= \left\|\begin{bmatrix}A_1^{-1} & 0 & -A_1^{-1}C_1B_1^{-1}\\ 0 & 0 & B_1^{-1}\\ 0 & C_4^{-1} & 0\end{bmatrix}\begin{bmatrix}x_1\\0\\0\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}}\\
&= \left\|\begin{bmatrix}(A-\lambda)_1^{-1}x_1\\0\\0\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}}\\
&= \left\|\begin{bmatrix}(A-\lambda)_l^{-1}x_1\\0\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}}\\
&= \|(A-\lambda)_l^{-1}x_1\|_{\mathcal{X}} > \frac{1}{\varepsilon}.
\end{aligned}$$

This implies that $\|(M_C-\lambda)^{-1}\|_{\mathcal{X}\oplus\mathcal{Y}} > \frac{1}{\varepsilon}$. Similarly, if $\|(B-\lambda)_r^{-1}\| > \frac{1}{\varepsilon}$ then there exists a unit vector $y \in \mathcal{Y}$ such that $\|(B-\lambda)_r^{-1}y\|_{\mathcal{Y}} > \frac{1}{\varepsilon}$, and in this case we have that

$$\begin{aligned}
\left\|(M_C-\lambda)^{-1}\begin{bmatrix}0\\y\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}} &= \left\|\begin{bmatrix}(A-\lambda)_1^{-1} & 0 & -(A-\lambda)_1^{-1}C_1(B-\lambda)_1^{-1}\\ 0 & 0 & (B-\lambda)_1^{-1}\\ 0 & C_4^{-1} & 0\end{bmatrix}\begin{bmatrix}0\\0\\y\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}}\\
&= \left\|\begin{bmatrix}-(A-\lambda)_1^{-1}C_1(B-\lambda)_1^{-1}y\\0\\(B-\lambda)_1^{-1}y\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}}\\
&= \left\|\begin{bmatrix}-(A-\lambda)_l^{-1}C(B-\lambda)_r^{-1}y\\(B-\lambda)_r^{-1}y\end{bmatrix}\right\|_{\mathcal{X}\oplus\mathcal{Y}}\\
&= \sqrt{\|(A-\lambda)_l^{-1}C(B-\lambda)_r^{-1}y\|_{\mathcal{X}}^2 + \|(B-\lambda)_r^{-1}y\|_{\mathcal{Y}}^2}\\
&\geq \|(B-\lambda)_r^{-1}y\|_{\mathcal{Y}} > \frac{1}{\varepsilon}
\end{aligned}$$

which again shows that $\|(M_C-\lambda)^{-1}\|_{\mathcal{X}\oplus\mathcal{Y}} > \frac{1}{\varepsilon}$. We can now conclude that $\Sigma_\varepsilon \subseteq \bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})}\sigma_\varepsilon(M_C)$ which completes the proof of the right inclusion.

($\subseteq$): We will now show that

$$\bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})}\sigma_\varepsilon(M_C) \subseteq \sigma_0 \cup ((\sigma_\varepsilon(A)\cup\sigma_\varepsilon(B)) \setminus (\sigma(A)\cup\sigma(B))) \cup \Sigma_\varepsilon.$$

Let $\lambda \in \bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})}\sigma_\varepsilon(M_C)$ be arbitrary, and we will consider 3 cases.

Case 1: $\lambda \in \sigma(M_C)$ for all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$. This case is trivial since $\lambda \in \sigma_0$ in this case.

Case 2: $\lambda \in \bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})}\sigma_\varepsilon(M_C)$ and $\lambda \in \rho(A)\cap\rho(B)$. In this case $M_C-\lambda$ is invertible for all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ so the assumption that $\lambda \in \bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})}\sigma_\varepsilon(M_C)$ implies that

$\|(M_C - \lambda)^{-1}\|_{\mathcal{X}\oplus\mathcal{Y}} > \frac{1}{\varepsilon}$ for all $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. This will hold for $M_0 - \lambda$ as well, which implies that

$$\max\{\|(A-\lambda)^{-1}\|_{\mathcal{X}}, \|(B-\lambda)^{-1}\|_{\mathcal{Y}}\} \geq \|(M_0 - \lambda)^{-1}\|_{\mathcal{X}\oplus\mathcal{Y}} > \frac{1}{\varepsilon}.$$

This allows us to conclude that in this case $\lambda \in (\sigma_\varepsilon(A) \cup \sigma_\varepsilon(B)) \setminus (\sigma(A) \cup \sigma(B))$.
Case 3: $\lambda \in \bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma_\varepsilon(M_C)$, $\lambda \notin \sigma_0$, $\lambda \notin \rho(A) \cap \rho(B)$. So, $(A-\lambda)_l^{-1}$ and $(B-\lambda)_r^{-1}$ exist. Futhermore, $\mathcal{N}(B-\lambda) \cong \mathcal{S}$, where $\mathcal{S}$ is a complement of $\mathcal{R}(A-\lambda)$ in $\mathcal{X}$. This complement exists and is non-trivial since $A-\lambda$ is left-invertible, but not invertible. Let J be an isomorphism from $\mathcal{N}(B-\lambda)$ to $\mathcal{S}$.

There exists an operator $C' \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that his representation with respect to the decompositions $\mathcal{X} = \mathcal{R}(A-\lambda) \oplus \mathcal{S}$, $\mathcal{Y} = \mathcal{P} \oplus \mathcal{N}(B-\lambda)$ is of the form

$$C' = \begin{bmatrix} 0 & 0 \\ 0 & C_4 \end{bmatrix} : \mathcal{P} \oplus \mathcal{N}(B-\lambda) \to \mathcal{R}(A-\lambda) \oplus \mathcal{S}$$

where C_4 is invertible and $\|C_4^{-1}\| < \min\{\|(A-\lambda)_l^{-1}\|_{\mathcal{X}}, \|(B-\lambda)_r^{-1}\|_{\mathcal{Y}}\}$. Notice that in this case $\|C_4^{-1}\|$ refers to the norm on $\mathcal{B}(\mathcal{N}(B-\lambda), \mathcal{S})$ induced by the norms on $\mathcal{Y}$ and $\mathcal{X}$, respectively.

Indeed, let $m = \min\{\|(A-\lambda)_l^{-1}\|_{\mathcal{X}}, \|(B-\lambda)_r^{-1}\|_{\mathcal{Y}}\}$. We will define $C' \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ via

$$C' = \begin{bmatrix} 0 & 0 \\ 0 & C_4 \end{bmatrix} : \mathcal{P} \oplus \mathcal{N}(B-\lambda) \to \mathcal{R}(A-\lambda) \oplus \mathcal{S},$$
$$C_4 = \frac{2\|J^{-1}\|}{m} J.$$

For this choice of C', we have that $C_4^{-1} = \frac{m}{2\|J^{-1}\|} J^{-1}$ and

$$\|C_4^{-1}\| = \frac{m}{2\|J^{-1}\|}\|J^{-1}\| = \frac{m}{2} < m.$$

Since $\lambda \in \sigma_\varepsilon(M_{C'})$ and $M_{C'} - \lambda$ is invertible we have that

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(M_{C'} - \lambda)^{-1}\|_{\mathcal{X}\oplus\mathcal{Y}} \\ &\leq \max\{\|C_4^{-1}\|, \|(A-\lambda)_l^{-1}\|_{\mathcal{X}}, \|(B-\lambda)_r^{-1}\|_{\mathcal{Y}}\} \\ &\leq \max\{\|(A-\lambda)_l^{-1}\|_{\mathcal{X}}, \|(B-\lambda)_r^{-1}\|_{\mathcal{Y}}\}, \end{aligned}$$

where the third inequality follows from $\|C_4^{-1}\| < \min\{\|(A-\lambda)_l^{-1}\|_{\mathcal{X}}, \|(B-\lambda)_r^{-1}\|_{\mathcal{Y}}\}$, and the second inequality follows from Lemma 2.1. Indeed, from our choice of C' we see that $M_{C'} - \lambda$ can be represented as

$$M_{C'} - \lambda = \begin{bmatrix} (A-\lambda)_1 & 0 & 0 \\ 0 & 0 & C_4 \\ 0 & (B-\lambda)_1 & 0 \end{bmatrix} : \mathcal{X} \oplus \mathcal{P} \oplus \mathcal{N}(B-\lambda) \to \mathcal{R}(A-\lambda) \oplus \mathcal{S} \oplus \mathcal{K}$$

$\mathcal{X} = \mathcal{R}(A-\lambda) \oplus \mathcal{S}$, $\mathcal{Y} = \mathcal{P} \oplus \mathcal{N}(B-\lambda)$, and where $(A-\lambda)_1$ and $(B-\lambda)_1$ are invertible operators such that

$$(A-\lambda)_l^{-1} = \begin{bmatrix}(A-\lambda)_1^{-1} & 0\end{bmatrix} : \mathcal{R}(A-\lambda) \oplus \mathcal{S} \to \mathcal{X}$$
$$(B-\lambda)_r^{-1} = \begin{bmatrix}(B-\lambda)_1^{-1} \\ 0\end{bmatrix} : \mathcal{Y} \to \mathcal{P} \oplus \mathcal{N}(B-\lambda).$$

By (2.4) we know that

$$(M_{C'}-\lambda)^{-1} = \begin{bmatrix}(A-\lambda)_1^{-1} & 0 & 0 \\ 0 & 0 & (B-\lambda)_1^{-1} \\ 0 & C_4^{-1} & 0\end{bmatrix} : \mathcal{R}(A-\lambda)\oplus\mathcal{S}\oplus\mathcal{K} \to \mathcal{X}\oplus\mathcal{P}\oplus\mathcal{N}(B-\lambda),$$

which allows us to apply Lemma 2.1 and establish

$$\|(M_{C'}-\lambda)^{-1}\|_{\mathcal{X}\oplus\mathcal{Y}} \le \max\{\|C_4^{-1}\|, \|(A-\lambda)_l^{-1}\|_{\mathcal{X}}, \|(B-\lambda)_r^{-1}\|_{\mathcal{Y}}\}$$

since $\|(A-\lambda)_1^{-1}\| = \|(A-\lambda)_l^{-1}\|$ and $\|(B-\lambda)_1^{-1}\| = \|(B-\lambda)_r^{-1}\|$

This inequality implies that

$$\|(A-\lambda)_l^{-1}\|_{\mathcal{X}} > \frac{1}{\varepsilon} \text{ or } \|(B-\lambda)_r^{-1}\|_{\mathcal{Y}} > \frac{1}{\varepsilon},$$

which proves that $\lambda \in \Sigma_\varepsilon$ in this case, and completes the proof. □

The result we obtained regarding the perturbations of the ε-pseudospectrum of 2×2 upper-triangular operator matrices is completely analogous to the results converting the spectrum, and it would be natural to expect that a similar result holds for the sets $\cup_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})}\sigma(M_C)$ and $\cup_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})}\sigma_\varepsilon(M_C)$, but we will show that this is not the case.

We have already made use of the fact that if $\lambda \in \rho(A)\cap\rho(B)$ then $M_C - \lambda$ is invertible for all C which in turn implies that

$$\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B),$$

which (together with $\sigma(M_0) = \sigma(A)\cup\sigma(B)$) in turn allows us to conclude that

$$\bigcup_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma(M_C) = \sigma(A)\cup\sigma(B).$$

In the following Theorem, we show that in the case of the ε-pseudospectrum we cannot produce a similar result:

Theorem 3.2. *Let $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y})$ and $\varepsilon > 0$. Then*

$$\bigcup_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_\varepsilon(M_C) = \mathbb{C}. \tag{3.2}$$

Proof. As $\sigma_\varepsilon(M_0) = \sigma_\varepsilon(A) \cup \sigma_\varepsilon(B)$ it is sufficient to show that for each $\lambda \in \mathbb{C} \setminus (\sigma_\varepsilon(A) \cup \sigma_\varepsilon(B))$ there exists an operator $C_\lambda \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that $\lambda \in \sigma_\varepsilon(M_{C_\lambda})$. We can refine this further since $\sigma(A) \cup \sigma(B) \subseteq \sigma_\varepsilon(A) \cup \sigma_\varepsilon(B)$, so if $\lambda \in \mathbb{C} \setminus (\sigma_\varepsilon(A) \cup \sigma_\varepsilon(B))$ then $A - \lambda$, $B - \lambda$ will be invertible, and $M_C - \lambda$ will be invertible for every $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$.

Let $\lambda \in \mathbb{C} \setminus (\sigma_\varepsilon(A) \cup \sigma_\varepsilon(B))$ be arbitrary, and let $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$ be arbitrary, but fixed vectors such that $\|y_0\|_{\mathcal{Y}} = 1$.

We will define C_λ in the following way. Let $x_1 = (A - \lambda)x_0$, $y_1 = (B - \lambda)^{-1} y_0$, and

$$C_\lambda y = \begin{cases} Lx_1, & y = y_1 \\ 0, & \text{otherwise} \end{cases} \text{ where } L = \frac{2}{\varepsilon \|x_0\|_{\mathcal{X}}}.$$

The operator C_λ is well-defined as an operator of rank 1, and

$$\begin{aligned} \|(A - \lambda)^{-1} C_\lambda (B - \lambda)^{-1} y_0\|_{\mathcal{X}} &= \|(A - \lambda)^{-1} C_\lambda y_1\|_{\mathcal{X}} \\ &= \|L(A - \lambda)^{-1} x_1\|_{\mathcal{X}} \\ &= L\|x_0\|_{\mathcal{X}} = \frac{2}{\varepsilon}. \end{aligned}$$

We now have an unit vector in $\mathcal{X} \oplus \mathcal{Y}$, $\begin{bmatrix} 0 \\ y_0 \end{bmatrix}$ such that

$$\begin{aligned} \left\| (M_{C_\lambda} - \lambda)^{-1} \begin{bmatrix} 0 \\ y_0 \end{bmatrix} \right\|_{\mathcal{X} \oplus \mathcal{Y}} &= \left\| \begin{bmatrix} -(A - \lambda)^{-1} C_\lambda (B - \lambda)^{-1} y_0 \\ (B - \lambda)^{-1} y_0 \end{bmatrix} \right\|_{\mathcal{X} \oplus \mathcal{Y}} \\ &= \sqrt{\|(A - \lambda)^{-1} C_\lambda (B - \lambda)^{-1} y_0\|_{\mathcal{X}}^2 + \|(B - \lambda)^{-1} y_0\|_{\mathcal{Y}}^2} \\ &\geq \|(A - \lambda)^{-1} C_\lambda (B - \lambda)^{-1} y_0\|_{\mathcal{Y}} = \frac{2}{\varepsilon} > \frac{1}{\varepsilon}. \end{aligned}$$

This allows us to conclude that $\|(M_{C_\lambda} - \lambda)^{-1}\|_{\mathcal{X} \oplus \mathcal{Y}} > \frac{1}{\varepsilon}$, so $\lambda \in \sigma_\varepsilon(M_{C_\lambda})$, which completes the proof. □

We can now briefly comment on the problem of finding an operator $C' \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that

$$\sigma_\varepsilon(M_{C'}) = \bigcap_{C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})} \sigma_\varepsilon(M_C).$$

Remark: We have already mentioned that this question in the case of the spectrum has been partially answered in several papers, such as [2, 3, 11, 21], and it is interesting to note that in some cases the solution to this question is the same for the ε-pseudospectrum and spectrum.

In the case when $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ are Riesz or polynomially Riesz, it is easy to verify that $\sigma_0 = \sigma(A) \cup \sigma(B)$ and that $\Sigma_\varepsilon = \emptyset$ so

$$\bigcap_{C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})} \sigma_\varepsilon(M_C) = \sigma_\varepsilon(A) \cup \sigma_\varepsilon(B),$$

and therefore any $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ has the desired property.

Remark: There is room for further research on this topic in several directions. One direction would be to better describe the ε-pseudospectrum of an upper-triangular operator matrix M_C for fixed $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y})$ and $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. A second direction would be to investigate the perturbations of ε-pseudospectra for other types of operator matrices, such as 2×2 operator matrices of the form

$$M_{T,S} = \begin{bmatrix} A & B \\ T & S \end{bmatrix}$$

where $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ are given operators, and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, $S \in \mathcal{B}(\mathcal{Y})$ and

$$M_X = \begin{bmatrix} A & C \\ X & B \end{bmatrix},$$

where $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y})$ and $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ are given operators and $X \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$.

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