FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 3 (2016), 717–740

WHAT IS THE LOCUS OF THE CENTERS OF MEUSNIER SPHERES IN THE MINKOWSKI 3-SPACE?

Beyhan Yilmaz, Zehra Özdemir, Ismail Gök and Yusuf Yayli

Abstract. Focal curves is the locus of the centers of the osculating spheres. In view of these basic facts, it is natural to ask the following geometric question:
"What is the locus of the centers of Meusnier spheres in 3-dimensional Minkowski space?"
In this study, we investigate the answer of the above question.
Keywords: Osculating sphere, Meusnier sphere, evolute curve, Minkowski 3−Space.

1. Introduction

The osculating sphere of a space curve α at a point P is the sphere having contact of order $n \geq 3$ with α at P and let us consider all the curves on the surface S each of them has the same the tangent line and the same normal curvatures at a common point $p \in S$. Then, their osculating circles are on a sphere which is called the Meusnier sphere. The locus of the centers of the osculating sphere called the focal curve [7]. In view of these basic facts, it is natural to ask the following geometric question: "What is the locus of the centers of Meusnier spheres ?" Just as focal curve is the locus of the centers of the osculating spheres, we investigate the geometrical interpretation on the locus of the centers of the Meusnier spheres. We proved that if the curve is a line of curvature, the locus of the centers of the Meusnier spheres of the curve is an evolute curve in Euclidean space [8].

In Euclidean space, the observation of the length contraction, time dilation, velocity addition, etc. and the relations of them are difficult. However, these relations are all very easy to observe in Minkowski space. Moreover to study in Minkowski space, sometimes give more realistic result to us than Euclidean space.

In the present paper, we investigate the locus of the centers of the Meusnier spheres given a curve in the Minkowski space. Moreover, we give some relations between locus of the centers of the Meusnier spheres and osculating spheres of the curve.

Received November 03, 2015; accepted May 01, 2016
2010 *Mathematics Subject Classification.* 14H45; 14H50; 53A04; 53A55

2. Preliminaries

Let $\mathbb{E}_1^3$ be the 3-dimensional Minkowski space which endowed with the standart flat metric given by

$$g(x,y) = x_1y_1 + x_2y_2 - x_3y_3$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are the usual coordinate system in $\mathbb{E}_1^3$. Due to semi-Riemannian metric there are three different kind of vectors, namely spacelike, timelike and lightlike (null) depending on the properties $g(x,x) > 0$, $g(x,x) < 0$ and $g(x,x) = 0$, respectively, for any vector x in $\mathbb{E}_1^3/\{0\}$. These can be generalized for curves depending on the casual character of their tangent vectors, that is, the curve α is called a spacelike (resp. timelike and lightlike) if its velocity vector $\alpha'(t)$ is spacelike (resp. timelike and lightlike) for any $t \in I$ [5].

In particular, the norm (length) of a vector x is given by $\|x\| = \sqrt{|g(x,x)|}$. Two vectors x and y are orthogonal, if $g(x,y) = 0$.

The Lorentzian vector product is defined by

$$x \times y = (x_2y_3 - y_2x_{3,}y_3x_1 - x_3y_{1,}y_1x_2 - x_1y_2)$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathbb{E}_1^3$ [5].

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the unit speed curve α in the Minkowski space $\mathbb{E}_1^3$, the following Frenet formulas are given,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.1}$$

where

$$g(T,T) = 1,\ g(N,N) = \varepsilon = \pm 1,\ g(B,B) = -\varepsilon,\ g(T,N) = g(T,B) = g(N,B) = 0$$

and κ and τ are curvature and torsion of the spacelike curve α, respectively. Here, ε determines the kind of spacelike curve. If $\varepsilon = 1$, then α is a spacelike curve with spacelike first principal normal N and timelike binormal B. If $\varepsilon = -1$, then α is a spacelike curve with timelike first principal normal N and spacelike binormal B. Furthermore, for a timelike curve α in the Minkowski space $\mathbb{E}_1^3$, the following Frenet formulas are given as follows,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.2}$$

where

$$g(T,T) = -1,\ g(N,N) = \varepsilon = 1,\ g(B,B) = 1,\ g(T,N) = g(T,B) = g(N,B) = 0$$

and κ and τ are curvature and torsion of the timelike curve α, respectively [5].

Let M be a oriented surface in the Minkowski space $\mathbb{E}_1^3$ and let consider a non-null curve α lying on M fully. Since the curve α is also in the space, there exists Frenet frame $\{T, N, B\}$ at each points of the curve where T is unit tangent vector, N is principal normal vector and B is binormal vector, respectively. Since the curve α lying on the surface M there exists another frame of the curve α which is called Darboux frame and denoted by $\{T, Y, Z\}$. In this frame T is the unit tangent of the curve, Z is the unit normal of the surface M and Y is a unit vector given by $Y = \pm Z \times T$. Since the unit tangent T is common in both Frenet frame and Darboux frame, the vectors N, B, Y and Z lie on the same plane [4].

If the surface M is an oriented timelike surface, then the curve α lying on M can be spacelike or timelike. If the curve α is a timelike curve then the relations between the Frenet frame and Darboux frame can be given as,

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.3}$$

and if the curve α is a spacelike curve, then the relations between these frames can be given as follows,

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.4}$$

If the surface M is an oriented spacelike surface, then the curve α lying on M can be spacelike curve. So, the relations between the frames can be given as follows

$$\begin{bmatrix} T \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.5}$$

In all cases, θ is the angle between the vectors Y and N. According to the Lorentzian causal characters of the surface M and the curve α lying on M, the derivative formulas of the Darboux frame can be changed as follows .

Case 1. If the surface M is a timelike surface, then the derivative formulas of the Darboux frame of α is given by

$$\begin{bmatrix} T' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & k_g & -\varepsilon k_n \\ k_g & 0 & \varepsilon t_r \\ k_n & t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix}. \tag{2.6}$$

$$g(T,T)=\varepsilon=\pm 1,\ g(Y,Y)=-\varepsilon,\ g(Z,Z)=1,\ g(T,Y)=g(T,Z)=g(Y,Z)=0$$

Case 2. If the surface M is a spacelike surface, then the derivative formulas of the Darboux frame of α is given by

$$\begin{bmatrix} T' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & t_r \\ k_n & t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix}. \tag{2.7}$$

$$g(T,T)=g(Y,Y)=1,\ g(Z,Z)=-1,\ g(T,Y)=g(T,Z)=g(Y,Z)=0$$

In these formulas k_g, k_n and t_r are called as geodesic curvature, normal curvature and geodesic torsion, respectively [5]. The relations between the $\{k_g,k_n,t_r\}$ and the $\{\kappa,\tau\}$ are given as follows.

if both the surface M and the curve α are timelike

$$k_g=\kappa\cos\theta,\ k_n=\kappa\sin\theta,\ t_r=\tau+\theta',\tag{2.8}$$

if both the surface M and the curve α are spacelike

$$k_g=\kappa\cosh\theta,\ k_n=-\kappa\sinh\theta,\ t_r=\tau+\theta',$$

if the surface M is timelike and the curve α is spacelike

$$k_g=\kappa\cosh\theta,\ k_n=-\kappa\sinh\theta,\ t_r=\tau+\theta',\tag{2.9}$$

where θ is the angle function between N and B. It is well known that, if $k_g=0$, then the curve α called as geodesic, if $k_n=0$, then the curve α called as asymptotic and if $t_r=0$, then the curve α called as line of curvature [4].

Theorem 2.1. *Let $\alpha: I\subset\mathbb{R}\longrightarrow\mathbb{E}_1^3$ be an arc lenghted parameter curve with non-zero curvatures κ and τ in $\mathbb{E}_1^3$ and $\{T(s),N(s),B(s)\}$ denote the Frenet frame of the curve α. Then the center of the osculating circle of the curve α is given by*

$$C=\alpha(s)+\varepsilon\frac{1}{\kappa(s)}N(s)\tag{2.10}$$

where $g(N,N)=\varepsilon=\pm1$ [3].

Theorem 2.2. *Let $\alpha: I\subset\mathbb{R}\longrightarrow\mathbb{E}_1^3$ be an arc-length parameter spacelike curve with non-zero curvatures κ and τ in $\mathbb{E}_1^3$ and $\{T(s),N(s),B(s)\}$ denote the Frenet frame of the curve α. The curve*

$$C_\alpha(s)=\alpha(s)+\varepsilon\frac{1}{\kappa(s)}N(s)-\left(\frac{1}{\kappa(s)}\right)'\frac{1}{\tau(s)}B(s),\tag{2.11}$$

where $g(N, N) = \varepsilon = \pm 1$ is consisting of the centers of the osculating spheres of the curve α. The locus of these centres is called the focal curve of α .

Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}_1^3$ be an arc-length parameter timelike curve with non-zero curvatures κ and τ in $\mathbb{E}_1^3$ and $\{T(s), N(s), B(s)\}$ denote the Frenet frame of the curve α. The curve

$$C_\alpha(s) = \alpha(s) - \frac{1}{\kappa(s)}N(s) + \left(\frac{1}{\kappa(s)}\right)' \frac{1}{\tau(s)}B(s) \tag{2.12}$$

is consisting of the centers of the osculating spheres of the curve α [6].

Theorem 2.3. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}_1^3$ be a unit speed spacelike curve with Serret-Frenet apparatus $\{\kappa, \tau, T, N, B\}$ and β be an evolute curve of α. Then the following equality*

$$\beta(s) = \alpha(s) + \varepsilon\frac{1}{\kappa(s)}(N(s) - \coth(\int \tau(s)ds + c)B(s)) \tag{2.13}$$

holds, where $g(N, N) = \varepsilon = \pm 1$.

Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}_1^3$ be a unit speed timelike curve with Serret-Frenet apparatus $\{\kappa, \tau, T, N, B\}$ and β be an evolute curve of α. Then the following equality

$$\beta(s) = \alpha(s) - \frac{1}{\kappa(s)}(N(s) + \cot(\int \tau(s)ds + c)B(s)) \tag{2.14}$$

holds [2].

Definition 2.1. A curve $\alpha : I \longrightarrow \mathbb{E}_1^3$, $\kappa(s) \neq 0$, is called a cylindrical helix if the tangent lines of α make a constant angle with a fixed direction. It has been known that the curve α is a cylindrical helix if and only if $(\frac{\tau}{\kappa})(s)$ =constant. If both $\kappa(s) \neq 0$ and $\tau(s)$ are constant, it is, of course, a cylindrical helix. We call such a curve a circular helix [5].

Definition 2.2. Let $\alpha : I \longrightarrow \mathbb{E}_1^3$ be a unit speed space curve with $\kappa(s) \neq 0$. A curve α is called a slant helix if there exists a non-zero constant vector field U in $\mathbb{E}_1^3$ such that the function $\langle N(s), U\rangle$ is constant [1].

Theorem 2.4. *Let M be a Lorentzian surface in $\mathbb{E}_1^3$ and $p \in M$, $X_p \in T_M(p)$. We assume that $X_p \in T_M(p)$ is not an asymptotic direction on M then*

i) The locus of the curvature centers of all the non-null section curves determined by X_p with spacelike second Frenet vectors is a pseudosphere

ii) The locus of the fourth vertex point of the parallelogram which constructed with one diagonal $[CC_i]$ and three vertices P, C, C_i is a pseudosphere where C_i and C are the curvature centers of any section curve and the normal section curve determined by X_p, respectively [3].

Theorem 2.5. *Let M be a Lorentzian surface in $\mathbb{E}_1^3$ and $p \in M$, $X_p \in T_M(p)$. We assume that $X_p \in T_M(p)$ is not an asymptotic direction on M. Let the points C and C_i denote the curvature centers of the normal section curve and a section curve determined by X_p. Then,*

i) All curvature circles of all the non-null section curves determined by X_p with spacelike second Frenet vectors lie on a pseudosphere centered at the point C.

ii) All the conjugate curvature circles of all non-null section curves determined by X_p with timelike second Frenet vectors lie on a pseudosphere or a pseudo-hyperbolic space and the center of the pseudosphere or the hyperbolic space is the fourth vertex point of the parallelogram which is determined by the vertex points, p, C and C_i and one diagonal the line segment $[CC_i]$ [3].

Lemma 2.1. *Let h be the second fundamental form of the Lorentzian surface M in $\mathbb{E}_1^3$. If X_p is a tangent vector to M and V and k_1 are the second Frenet vector and the first curvature function of the section curve determined by X_p, respectively. Then*

$$k_2(0)\langle V_p, N_p\rangle = -h(X_p, X_p),$$

where N_p is the unit normal to M at the point p [3].

Corollary 2.1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a curve on the Lorentzian manifold M and X_p is a non-asymptotic tangent vector to M. If g, g are the curvature radii of the normal section curve and a section curve determined N by X_p, respectively, then*

$$\begin{aligned}\langle V_2, N\rangle &= \frac{g}{g_N} = \frac{k_N}{k_1} \quad \textit{when} \quad \langle V_2, N\rangle > 0,\\ \langle V_2, N\rangle &= \frac{-g}{g_N} = \frac{-k_N}{k_1} \quad \textit{when} \quad \langle V_2, N\rangle < 0,\end{aligned}$$

where V is the second Frenet vector of α and N is the unit normal vector field to M and k_1, k_N denote the curvatures of α and the normal section curve determined by X_p [3].

Let k_1 and k_N denote the first curvature of the section curve α and the normal section curve determined by X_p, respectively. So, the locus of the centers of the Meusnier spheres in the Lorentz- Minkowski space $\mathbb{E}_1^3$ is given by as follows [3]. In the case of $\langle V_2, N\rangle > 0$,

$$\begin{aligned}C_i &= p + \frac{1}{k_1}V_2,\\ C &= p + \frac{1}{k_N}N_p,\end{aligned} \tag{2.15}$$

where N_p is the unit normal to M at the point p. It should be noticed that if $\langle V_2, N\rangle < 0$ then we have to take $N_p = -V_2$, i.e.

$$C = p - \frac{1}{k_N}N_p. \tag{2.16}$$

3. Evolutes-Involutes of Spatial Curves and Meusnier Spheres in the Minkowski 3-Space

In this section, we investigate the locus of the centers of the Meusnier spheres of the curve α on spacelike and timelike surface M in the Minkowski $3-$space $\mathbb{E}_1^3$. Moreover, we give some relations between the locus of the centers of the Meusnier spheres and some special curves.

3.1. Evolutes-Involutes of Spacelike Space Curves and Meusnier Spheres on Spacelike Surface

Theorem 3.1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed spacelike Frenet curve on spacelike surface M in the Minkowski $3-$space $\mathbb{E}_1^3$. Then the curve α is a geodesic if and only if the centers of the osculating circles coincide with the centers of the Meusnier spheres of the curve α and the osculating circles are geodesics on the Meusnier spheres.*

Proof. Let $Z(s)$ be the unit normal vector field of M and $\{T, N, B\}$ is the Frenet frame of the curve α. The locus of the center of the Meusnier sphere is given by (Eq. (2.16))

$$\begin{aligned}
\beta(s) &= \alpha(s) - \frac{1}{k_n} Z(s) \\
&= \alpha(s) - \frac{1}{k_n} (\sinh\theta(s) N(s) + \cosh\theta(s) B(s)) \\
&= \alpha(s) - \frac{1}{-\kappa \sinh\theta(s)} (\sinh\theta(s) N(s) + \cosh\theta(s) B(s)) \\
&= \alpha(s) + \frac{1}{\kappa} (N(s) + \frac{k_g}{k_n} B(s)) \qquad (3.1)
\end{aligned}$$

Since α is a geodesic curve, we get $\beta(s) = \alpha(s) + \frac{1}{\kappa} N(s)$. So, $\beta(s)$ is the locus of the center of the osculating circle.

Conversely, we assume that the center of the osculating circle coincides with the center of the Meusnier sphere of the curve α. We can easily see that $k_g = 0$, i.e. , the curve α is a geodesic curve. This completes the proof. $\square$

Theorem 3.2. *Let β be evolute curve of the unit speed spacelike curve α. Then the curve β can be written as*

$$\beta(s) = \alpha(s) + \frac{1}{\kappa} (N(s) - \coth(\int \tau(s)\, ds + b)) B(s). \qquad (3.2)$$

Proof. The tangent of the curve β at the point $\beta(s)$ is the line constructed by the vector $T^*(s)$. Since this line passes through the point $\alpha(s)$, the vector $\beta(s) - \alpha(s)$ is perpendicular to the vector $T(s)$. Then

$$\beta(s) = \alpha(s) + \lambda N(s) + \mu B(s). \qquad (3.3)$$

If we differentiate the Eq. (3.3) , then we have

$$T^{*}(s)=\beta^{'}(s)=(1-\lambda\kappa)T(s)+\left(\lambda^{'}+\mu\tau\right)N(s)+\left(\lambda\tau+\mu^{'}\right)B(s). \tag{3.4}$$

According to the definition of the evolute, since $\langle T^{*}(s),T(s)\rangle=0$, we get

$$\lambda=\frac{1}{\kappa} \tag{3.5}$$

and

$$\beta^{'}(s)=\left(\lambda^{'}+\mu\tau\right)N(s)+\left(\lambda\tau+\mu^{'}\right)B(s). \tag{3.6}$$

From the Eq. (3.3) and Eq. (3.6), the vector field $\beta^{'}$is parallel to the vector field $\beta-\alpha$. Then we have

$$\frac{\lambda^{'}+\mu\tau}{\lambda}=\frac{\lambda\tau+\mu^{'}}{\mu}.$$

After that, we have

$$\tau=\frac{-\left(\frac{\lambda}{\mu}\right)^{'}}{1-\left(\frac{\lambda}{\mu}\right)^{2}}.$$

If we integrate the last equation, we get

$$\int\tau(s)\,ds+c=-arc\tanh\left(\frac{\lambda}{\mu}\right).$$

Hence, we find

$$\mu=-\frac{1}{\kappa}\coth\left(\int\tau(s)\,ds+c\right). \tag{3.7}$$

If we substitute Eq. (3.5) and Eq. (3.7) into Eq. (3.3), we have

$$\beta(s)=\alpha(s)+\frac{1}{\kappa}(N(s)-\coth(\int\tau(s)\,ds+c))B(s).$$

Then we obtain an evolute curve for each $c\in\mathbb{R}$. □

Theorem 3.3. ***(Main Theorem)*** *Let $\alpha:I\subset\mathbb{R}\longrightarrow M$ be a unit speed spacelike Frenet curve on spacelike surface M in the Minkowski $3-$space E_1^3 with non zero curvatures κ and τ. Then the curve α is a line of curvature on M if and only if the locus of the centers of the Meusnier spheres of the curve α coincide with evolute curve of the curve α.*

Proof. Assume that the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be an arc-length parameter curve lying on the oriented spacelike surface M and α is a line of curvature. The locus of the centers of the Meusnier spheres of the curve α is

$$\beta(s) = \alpha(s) + \frac{1}{\kappa(s)}(N(s) + \coth\theta(s))B(s). \tag{3.8}$$

Since the curve α is the line of curvature on the spacelike surface M, then $t_r = 0$ and $\theta(s) = -\int \tau(s)\,ds + c$. If we consider the last equality with (3.8), then the curve β becomes as

$$\beta(s) = \alpha(s) + \frac{1}{\kappa(s)}(N(s) - \coth(\int \tau(s)\,ds + c)B(s))$$

So, from Theorem 3.2, the curve β is the evolute of the curve α.

Conversely, assume that the locus of the centers of the Meusnier spheres of the curve α coincides with evolute of the curve α. We can easily see that the curve α is a line of curvature on M. This completes the proof. □

Theorem 3.4. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed spacelike Frenet curve on spacelike surface M with non-zero curvatures κ, τ in the Minkowski $3-$space E_1^3. Let C_α be the focal curve of α and β be the locus of centers of the Meusnier spheres of α. Then the focal curve C_α coincides with the curve β if and only if $\kappa(s) = ce^{\int \tau \coth\theta(s)ds}$.*

Proof. Assume that the focal curve C_α coincides with β. By using the (Theorem (2.2)) and Eq.(3.8), we obtain

$$\begin{aligned}
\alpha(s) + \frac{1}{\kappa}N(s) - \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}B(s) &= \alpha(s) + \frac{1}{\kappa}N(s) + \frac{1}{\kappa}\coth\theta(s)\,B(s), \\
-\left(\frac{1}{\kappa}\right)' \frac{1}{\tau} &= \frac{1}{\kappa}\coth\theta(s), \\
\kappa(s) &= ce^{\int \tau \coth\theta(s)ds}.
\end{aligned}$$

Conversely, if we assume that the following equality $\kappa(s) = ce^{\int \tau \coth\theta(s)ds}$ holds, then it is obvious that the focal curve C_α coincides with β. This completes the proof. □

Corollary 3.1. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on M . The locus of the centers of the Meusnier spheres of α coincides with the locus of the centers of the osculating spheres of α if and only if $\theta(s) = \operatorname{arcsin} h(\frac{c}{\kappa(s)})$.*

Proof. Assume that the locus of the centers of the Meusnier spheres of α coincides with the locus of the centers of the osculating spheres of α. So, we have that

$$\begin{aligned}
\alpha(s) + \frac{1}{\kappa(s)}N(s) + \frac{1}{\kappa(s)}\coth\theta(s)\,B(s) &= \alpha(s) + \frac{1}{\kappa(s)}N(s) - (\frac{1}{\kappa(s)})' \frac{1}{\tau(s)}B(s), \\
\tau(s)\coth\theta(s) &= \frac{\kappa(s)'}{\kappa(s)}.
\end{aligned}$$

Since α is a line of curvature, $\theta(s) = -\int \tau(s)\,ds + c$, then we have

$$\begin{aligned} -\tau(s)\coth\Big(\int \tau(s)\,ds + c\Big) &= \frac{\kappa(s)'}{\kappa(s)}, \\ \ln(\sinh(\tau(s)\,ds)) + \ln\kappa(s) &= \ln c, \\ \int \tau(s)\,ds &= \arcsin h\Big(\frac{c}{\kappa(s)}\Big), \\ \theta(s) &= \arcsin h\Big(\frac{c}{\kappa(s)}\Big). \end{aligned}$$

Conversely, if we assume that $\theta(s) = \arcsin h(\frac{c}{\kappa(s)})$ holds, then it is obvious that the locus of the centers of the Meusnier spheres of α coincides with the locus of the centers of the osculating spheres of α. This completes the proof. □

Theorem 3.5. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed spacelike Frenet curve on spacelike surface M with non-zero curvatures κ, τ in the Minkowski $3-$space $\mathbb{E}^3_1$. If the centers of the osculating spheres coincide with the centers of the Meusnier spheres of α. Then the curve α is a geodesic if and only if κ is a constant function.*

Proof. Assume that the centers of the osculating spheres coincide with the centers of the Meusnier spheres of α, then we have that

$$-\Big(\frac{1}{\kappa(s)}\Big)'\frac{1}{\tau(s)} = \frac{1}{\kappa(s)}\frac{k_g}{k_n}.$$

Since the curve α is a geodesic, then we have $(\frac{1}{\kappa(s)})'\frac{1}{\tau(s)} = 0$ and then κ is a constant function.

Conversely, if we assume that $\kappa(s)$ is a constant function, then it is obvious that the centers of the osculating spheres coincide with the centers of the Meusnier spheres of α. This completes the proof. □

Theorem 3.6. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on M with Darboux frame $\{T,\ Y,\ Z\}$ and β with Frenet frame $\{T^*, N^*, B^*\}$ be the locus of the centers of the Meusnier spheres of α. Then there exist the following relations*

$$T^* = -Z,\ N^* = -T,\ B^* = Y$$

and

$$\kappa^* = \frac{k_n}{(\frac{1}{k_n})'},\ \tau^* = -\frac{k_g}{(\frac{1}{k_n})'}.$$

Proof. The locus of the centers of the Meusnier spheres of α is given by

$$\beta(s) = \alpha(s) - \frac{1}{k_n}Z(s). \tag{3.9}$$

If we differantiate Eq.(3.9), we get

$$\begin{aligned}\frac{d\beta}{ds^*}\frac{ds^*}{ds} &= T - (\frac{1}{k_n})' Z(s) - \frac{1}{k_n}(k_n T(s) + t_r Y(s)), \\ &= -(\frac{1}{k_n})' Z(s) - \frac{t_r}{k_n} Y(s).\end{aligned}$$

Using that α is a line of curvature, we obtain that the curve β is a timelike curve.

$$T^* \frac{ds^*}{ds} = -(\frac{1}{k_n})' Z \tag{3.10}$$

and calculating the norm of the last equation we get

$$\frac{ds^*}{ds} = (\frac{1}{k_n})'. \tag{3.11}$$

If we consider Eq. (3.11) in Eq. (3.10), then we obtain

$$T^* = -Z. \tag{3.12}$$

Differentiating Eq. (3.12) then we get

$$\frac{dT^*}{ds^*}\frac{ds^*}{ds} = -\frac{dZ}{ds}$$

or using Eq.(3.11) and the Frenet-Serret formulas, we have

$$\kappa^* N^* (\frac{1}{k_n})' = -(k_n T + t_r Y),$$

and since α is a line curvature, we get

$$\kappa^* N^* (\frac{1}{k_n})' = -k_n T$$

Calculating the norm of the last equation $\kappa^* = \frac{k_n}{(\frac{1}{k_n})'}$ and using the last equality, we get

$$N^* = -T. \tag{3.13}$$

Using Eq. (3.12) and Eq. (3.13), we obtain

$$B^* = Y. \tag{3.14}$$

Then if we differentiate Eq. (3.13) with respect to s, we get

$$\begin{aligned}\frac{dN^*}{ds^*}(\frac{1}{k_n})' &= -\frac{dT}{ds}\frac{ds}{ds^*}, \\ \frac{dN^*}{ds^*} &= (-k_g Y - k_n Z)\frac{1}{(\frac{1}{k_n})'}.\end{aligned}$$

If we consider Eq. (3.12) and Eq. (3.13) we have

$$\kappa^* T^* + \tau^* B^* = \frac{k_n}{(\frac{1}{k_n})'} T^* - \frac{k_g}{(\frac{1}{k_n})'} B^*.$$

So, using the equality $\kappa^* = \frac{k_n}{(\frac{1}{k_n})'}$ in the last equation we obtain $\tau^* = -\frac{k_g}{(\frac{1}{k_n})'}$ which completes the proof. □

Corollary 3.2. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be an arc-length parameter spacelike curve lying on an oriented spacelike surface M which is a constant angle surface. If the curve α is a line of curvature on M, then the curve β is a helix on $\mathbb{E}_1^3$.*

Proof. Since M is a constant angle surface (constant angle surface means that the unit normal of M makes a constant angle with a fixed direction U), $\langle Z, U \rangle$ is a constant function. Then using Eq. (3.12), $\langle T^*, U \rangle$ is a constant function, that is, β is a helix. This completes the proof. □

Corollary 3.3. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on the spacelike surface M and β be the locus of the centers of the Meusnier spheres on M. Then the following relation is given among curvatures of these curves.*

$$\frac{\tau^*}{\kappa^*} = -\frac{k_g}{k_n}.$$

Proof. It is obvious by Theorem (3.6). □

Corollary 3.4. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on the spacelike surface M and β be the locus of the centers of the Meusnier spheres on M. Then β is a general helix if and only if*

$$\frac{k_g}{k_n} = constant.$$

Proof. It is obvious by Corollary (3.3). □

Corollary 3.5. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on the spacelike surface M and β be the locus of the centers of the Meusnier spheres of α. The curve α is a helix in the Minkowski $3-$space E_1^3 if and only if β is a slant helix in the Minkowski $3-$space E_1^3.*

Proof. Assume that α is a helix in the Minkowski $3-$space E_1^3. So, $\langle T, U \rangle$ is a constant function. Using Eq. (3.13), we obtain that $\langle N^*, U \rangle$ is a constant function. So, β is a slant helix in the Minkowski $3-$space E_1^3.

Conversely, if we assume that β is a slant helix in Minkowski $3-$space E_1^3, then it is obvious that the curve α is a helix in the Minkowski $3-$space E_1^3. This completes the proof. □

Corollary 3.6. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on M and β be the locus of the centers of the Meusnier spheres of α. The curve α is a slant helix in the Minkowski $3-$space E_1^3 if and only if β is a helix in the Minkowski $3-$space $\mathbb{E}_1^3$.*

Proof. Assume that α is a slant helix in the Minkowski $3-$space $\mathbb{E}_1^3$. So, $\langle Z, U\rangle$ is a constant function. Then using the Eq. (3.12), we obtain that $\langle T^*, U\rangle$ is a constant function. So, β is a helix in the Minkowski $3-$space $\mathbb{E}_1^3$.

Conversely, assume that β is a helix in the Minkowski $3-$space $\mathbb{E}_1^3$. It is obvious that the curve α is a slant helix in the Minkowski $3-$space $\mathbb{E}_1^3$. This completes the proof. □

Corollary 3.7. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on the spacelike surface M and β be the locus of the centers of the Meusnier spheres of α. Then α is a geodesic if and only if β is a planar curve.*

Proof. Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be an arc-length parameter curve on M and β be the locus of the centers of the Meusnier spheres of α. If α is a geodesic, then the geodesic curvature k_g of α and the torsion τ^* of β are equal to zero. So, β is a planar curve.

Conversely, if we assume that β is a planar curve, then it is obvious that α is a geodesic. This completes the proof. □

Corollary 3.8. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on M and β is the locus of the centers of the Meusnier spheres on M. β is a helix in the Minkowski $3-$space $\mathbb{E}_1^3$ if and only if α is a planar curve.*

Proof. Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on M. If the β is a helix in the Minkowski $3-$space $\mathbb{E}_1^3$, then we obtain $\frac{\tau^*}{\kappa^*} = -\frac{k_g}{k_n}$ =constant. So, $\cot\theta\left(s\right)$ is a constant function and $\theta'\left(s\right) = 0$. Since α is a line of curvature on M, $\theta\left(s\right) = \tau\left(s\right) = 0$. So α is a planar curve.

Conversely, if we assume that the curve α is a planar curve, then it is obvious that β is a helix in the Minkowski $3-$space $\mathbb{E}_1^3$. This completes the proof. □

3.2. Evolutes-Involutes of Spacelike Space Curves and Meusnier Spheres on Timelike Surface

Theorem 3.7. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed spacelike Frenet curve on the timelike surface M in the Minkowski $3-$space $\mathbb{E}_1^3$. Then the curve α is a geodesic if and only if the centers of the osculating circles coincide with the centers of the Meusnier spheres of the curve α and the osculating circles are geodesics on the Meusnier spheres.*

Proof. Let $Z(s)$ be the unit normal vector field of M and $\{T, N, B\}$ be the Frenet frame of the curve α. Then the locus of the center of the Meusnier sphere is given by (Eq. (2.15))

$$\begin{aligned}
\beta(s) &= \alpha(s) + \frac{1}{k_n} Z(s) \\
&= \alpha(s) + \frac{1}{k_n}(\sinh\theta(s) N(s) + \cosh\theta(s) B(s)) \\
&= \alpha(s) + \frac{1}{-\kappa\sinh\theta(s)}(\sinh\theta(s) N(s) + \cosh\theta(s) B(s)) \\
&= \alpha(s) - \frac{1}{\kappa}(N(s) + \frac{k_g}{k_n} B(s)) \qquad (3.15)
\end{aligned}$$

Since α is a geodesic curve, then we get $\beta(s) = \alpha(s) - \frac{1}{\kappa}N(s)$. So, β is the locus of the center of the osculating circle.

Conversely, if we assume that the center of the osculating circle coincides with the center of the Meusnier sphere of the curve α, then we can easily see that $k_g = 0$, that is, α is a geodesic curve. This completes the proof. □

Theorem 3.8. *Let β be evolute curve of the unit speed spacelike curve α. Then the curve β can be written as*

$$\beta(s) = \alpha(s) - \frac{1}{\kappa}(N(s) - \coth(\int \tau(s)\,ds + b))B(s). \qquad (3.16)$$

Proof. The tangent of the curve β at the point $\beta(s)$ is the line constructed by the vector $T^*(s)$. Since this line passes through the point $\alpha(s)$, then the vector $\beta(s) - \alpha(s)$ is perpendicular to the vector $T(s)$. Then

$$\beta(s) = \alpha(s) + \lambda N(s) + \mu B(s). \qquad (3.17)$$

If we derivative the Eq. (3.17), then we have

$$T^*(s) = \beta'(s) = (1 + \lambda\kappa) T(s) + \left(\lambda' + \mu\tau\right) N(s) + \left(\lambda\tau + \mu'\right) B(s). \qquad (3.18)$$

According to the definition of the evolute, since $\langle T^*(s), T(s)\rangle = 0$, we get

$$\lambda = -\frac{1}{\kappa} \qquad (3.19)$$

and

$$\beta'(s) = \left(\lambda' + \mu\tau\right) N(s) + \left(\lambda\tau + \mu'\right) B(s). \qquad (3.20)$$

From the Eq. (3.17) and Eq. (3.20), the vector field β' is parallel to the vector field $\beta - \alpha$. Then we have

$$\frac{\lambda' + \mu\tau}{\lambda} = \frac{\lambda\tau + \mu'}{\mu}.$$

Therefore, we have

$$\tau=-\frac{\left(\frac{\lambda}{\mu}\right)'}{1-\left(\frac{\lambda}{\mu}\right)^2}.$$

If we take the integral the last equation, we get

$$\int\tau\left(s\right)ds+c=-arc\tanh\left(\frac{\lambda}{\mu}\right).$$

Hence, we find

$$\mu=\frac{1}{\kappa}\coth\left(\int\tau\left(s\right)ds+c\right). \tag{3.21}$$

If we substitute Eq. (3.19) and Eq. (3.21) into Eq. (3.17), we have

$$\beta\left(s\right)=\alpha\left(s\right)-\frac{1}{\kappa}(N\left(s\right)-\coth(\int\tau\left(s\right)ds+c))B\left(s\right). \tag{3.22}$$

Then we obtain an evolute curve for each $c\in\mathbb{R}$. □

Theorem 3.9. ***(Main Theorem)*** *Let $\alpha:I\subset\mathbb{R}\longrightarrow M$ be a unit speed spacelike Frenet curve on timelike surface M in the Minkowski $3-$space $\mathbb{E}_1^3$ with non zero curvatures κ and τ. Then the curve α is a line of curvature on M if and only if the locus of the centers of the Meusnier spheres of the curve α coincides with evolute curve of the curve α.*

Proof. Assume that the spacelike curve $\alpha:I\subset\mathbb{R}\longrightarrow M$ be an arc-length parameter curve lying on the oriented spacelike surface M and α is a line of curvature. The locus of the centers of the Meusnier spheres of the curve α is

$$\beta\left(s\right)=\alpha\left(s\right)-\frac{1}{\kappa\left(s\right)}(N\left(s\right)+\coth\theta\left(s\right))B\left(s\right). \tag{3.23}$$

Since the curve α is the line of curvature on the timelike surface M, then $t_r=0$ and $\theta\left(s\right)=-\int\tau\left(s\right)ds+c$. If this equalities replace in (3.23), then the curve β becomes as

$$\beta\left(s\right)=\alpha\left(s\right)+\frac{1}{\kappa\left(s\right)}(N\left(s\right)-\coth(\int\tau\left(s\right)ds+c)B\left(s\right))$$

So, from Theorem 3.8, the curve β is the evolute of the curve α.

Conversely, assume that the locus of the centers of the Meusnier spheres of the curve α coincides with evolute of the curve α. We can easily see that the curve α is a line of curvature on M. This completes the proof. □

Theorem 3.10. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed spacelike Frenet curve on timelike surface M with non-zero curvatures κ, τ in the Minkowski $3-$space $\mathbb{E}_1^3$. Let C_α be the focal curve of α and β be the locus of centers of the Meusnier spheres of α. Then the focal curve C_α coincides with the curve β if and only if $\kappa(s) = ce^{-\int \tau \coth\theta(s)ds}$.*

Proof. Assume that the focal curve C_α coincides with β. By using the (Theorem (2.2)) and Eq. (3.23), we obtain

$$\begin{aligned} \alpha(s) - \frac{1}{\kappa}N(s) - \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}B(s) &= \alpha(s) - \frac{1}{\kappa}N(s) - \frac{1}{\kappa}\coth\theta(s)B(s), \\ -\left(\frac{1}{\kappa}\right)' \frac{1}{\tau} &= -\frac{1}{\kappa}\coth\theta(s), \\ \kappa(s) &= ce^{-\int \tau \coth\theta(s)ds}. \end{aligned}$$

Conversely, if we assume that the following equality $\kappa(s) = ce^{-\int \tau \coth\theta(s)ds}$ holds, then it is obvious that the focal curve C_α coincides with β. This completes the proof. □

Corollary 3.9. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on M . The locus of the centers of the Meusnier spheres of α coincides with the locus of the centers of the osculating spheres of α if and only if $\theta(s) = \arcsin h(\frac{\kappa(s)}{c})$.*

Proof. Assume that the locus of the centers of the Meusnier spheres of α coincides with the locus of the centers of the osculating spheres of α. So, we have that

$$\begin{aligned} \alpha(s) - \frac{1}{\kappa(s)}N(s) - \frac{1}{\kappa(s)}\coth\theta(s)B(s) &= \alpha(s) - \frac{1}{\kappa(s)}N(s) - (\frac{1}{\kappa(s)})' \frac{1}{\tau(s)}B(s), \\ \tau(s)\coth\theta(s) &= -\frac{\kappa(s)'}{\kappa(s)}. \end{aligned}$$

Since α is a line of curvature, $\theta(s) = -\int \tau(s)\,ds + c$, then we have

$$\begin{aligned} -\tau(s)\coth(\int \tau(s)\,ds + c) &= -\frac{\kappa(s)'}{\kappa(s)}, \\ \ln(\sinh(\tau(s)\,ds)) - \ln\kappa(s) &= -\ln c, \\ \int \tau(s)\,ds &= \arcsin h(\frac{\kappa(s)}{c}), \\ \theta(s) &= \arcsin h(\frac{\kappa(s)}{c}). \end{aligned}$$

Conversely, if we assume that $\theta(s) = \arcsin h(\frac{\kappa(s)}{c})$ holds, then it is obvious that the locus of the centers of the Meusnier spheres of α coincides with the locus of the centers of the osculating spheres of α. This completes the proof. □

Theorem 3.11. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed spacelike Frenet curve on timelike surface M with non-zero curvatures κ, τ in the Minkowski $3-$space $\mathbb{E}_1^3$. Let the centers of the osculating spheres coincide with the centers of the Meusnier spheres of α. Then the curve α is a geodesic if and only if κ is a constant function.*

Proof. Assume that the centers of the osculating spheres coincide with the centers of the Meusnier spheres of α, than we have that

$$(\frac{1}{\kappa(s)})' \frac{1}{\tau(s)} = \frac{1}{\kappa(s)} \frac{k_g}{k_n}.$$

Since the curve α is a geodesic, then we have $(\frac{1}{\kappa(s)})' \frac{1}{\tau(s)} = 0$ and than κ is a constant function.

Conversely, if we assume that $\kappa(s)$ is a constant function, then it is obvious that the centers of the osculating spheres coincide with the centers of the Meusnier spheres of α. This completes the proof. □

Theorem 3.12. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on M with Darboux frame $\{T,\ Y,\ Z\}$ and β with Frenet frame $\{T^*, N^*, B^*\}$ be the locus of the centers of the Meusnier spheres of α. Then there exist the following relations*

$$T^* = Z,\ N^* = -T,\ B^* = -Y$$

and

$$\kappa^* = \frac{k_n}{(\frac{1}{k_n})'},\ \tau^* = -\frac{k_g}{(\frac{1}{k_n})'}.$$

Proof. The locus of the centers of the Meusnier spheres of α is given by

$$\beta(s) = \alpha(s) + \frac{1}{k_n} Z(s). \tag{3.24}$$

If we differentiate Eq. (3.24), we get

$$\begin{aligned} \frac{d\beta}{ds^*}\frac{ds^*}{ds} &= T + (\frac{1}{k_n})' Z(s) + \frac{1}{k_n}(-k_n T(s) + t_r Y(s)), \\ &= (\frac{1}{k_n})' Z(s) + \frac{t_r}{k_n} Y(s). \end{aligned}$$

Using that α is a line of curvature, we obtain that the curve β is a spacelike curve.

$$T^* \frac{ds^*}{ds} = (\frac{1}{k_n})' Z \tag{3.25}$$

and calculating the norm of the last equation we get

$$\frac{ds^*}{ds} = (\frac{1}{k_n})'. \tag{3.26}$$

If we consider Eq.(3.26) in Eq.(3.25), then we obtain

$$T^* = Z. \tag{3.27}$$

Differentiating Eq. (3.27) with respect to s, we get

$$\frac{dT^*}{ds^*}\frac{ds^*}{ds} = \frac{dZ}{ds}$$

or using Eq. (3.26) and Frenet-Serret formulas, we get

$$\kappa^* N^* (\frac{1}{k_n})' = (-k_n T + t_r Y)\,,$$

and since α is a line curvature, we have

$$\kappa^* N^* (\frac{1}{k_n})' = -k_n T$$

Calculating the norm of the last equation we get $\kappa^* = \frac{k_n}{(\frac{1}{k_n})'}$ and then we have

$$N^* = -T. \tag{3.28}$$

Using Eq. (3.27) and Eq. (3.28), we obtain

$$B^* = -Y. \tag{3.29}$$

Then if we differentiate Eq. (3.28) with respect to s, we get

$$\begin{aligned} \frac{dN^*}{ds^*}(\frac{1}{k_n})' &= -\frac{dT}{ds}\frac{ds}{ds^*}, \\ \frac{dN^*}{ds^*} &= -(-k_g Y + k_n Z)\,\frac{1}{(\frac{1}{k_n})'}. \end{aligned}$$

If we consider Eq. (3.27) and Eq. (3.28) we have

$$\kappa^* T^* + \tau^* B^* = \frac{k_n}{(\frac{1}{k_n})'} T^* - \frac{k_g}{(\frac{1}{k_n})'} B^*.$$

So, using the equality $\kappa^* = \frac{k_n}{(\frac{1}{k_n})'}$ in the last equality we obtain $\tau^* = -\frac{k_g}{(\frac{1}{k_n})'}$ which completes the proof. □

Corollary 3.10. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be an arc-length parameter spacelike curve lying on an oriented timelike surface M which is a constant angle surface. If the curve α is a line of curvature on M, then the curve β is a helix on $\mathbb{E}^3_1$.*

Proof. Since M is a constant angle surface, $\langle Z, U\rangle$ is a constant function. Then using Eq. (3.27), $\langle T^*, U\rangle$ is a constant function, that is, β is a helix. This completes the proof. □

Corollary 3.11. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on the timelike surface M and β be the locus of the centers of the Meusnier spheres on M. Then the following relation is given among curvatures of these curves.*

$$\frac{\tau^*}{\kappa^*} = -\frac{k_g}{k_n}.$$

Proof. It is obvious by Theorem (3.12). □

Corollary 3.12. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on the timelike surface M and β be the locus of the centers of the Meusnier spheres on M. Then β is a general helix if and only if*

$$\frac{k_g}{k_n} = constant.$$

Proof. It is obvious by Corollary (3.11). □

Corollary 3.13. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on the timelike surface M and β be the locus of the centers of the Meusnier spheres of α. The curve α is a helix in the Minkowski $3-$space $\mathbb{E}^3_1$ if and only if β is a slant helix in the Minkowski $3-$space $\mathbb{E}^3_1$.*

Proof. Assume that α is a helix in Minkowski $3-$space $\mathbb{E}^3_1$. So, $\langle T, U\rangle$ is a constant function. Using Eq. (3.28), we obtain that $\langle N^*, U\rangle$ is a constant function. So, β is a slant helix in the Minkowski $3-$space $\mathbb{E}^3_1$.

Conversely, if we assume that β is a slant helix in the Minkowski $3-$space $\mathbb{E}^3_1$, then it is obvious that the curve α is a helix in the Minkowski $3-$space $\mathbb{E}^3_1$. This completes the proof. □

Corollary 3.14. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on M and β be the locus of the centers of the Meusnier spheres of α. The curve α is a slant helix in the Minkowski $3-$space $\mathbb{E}^3_1$ if and only if β is a helix in the Minkowski $3-$space $\mathbb{E}^3_1$.*

Proof. Assume that α is a slant helix in the Minkowski $3-$space $\mathbb{E}^3_1$. So, $\langle Z, U\rangle$ is a constant function. Then using the Eq. (3.27), we obtain that $\langle T^*, U\rangle$ is a constant function. So, β is a helix in the Minkowski $3-$space $\mathbb{E}^3_1$.

Conversely, assume that β is a helix in the Minkowski $3-$space $\mathbb{E}^3_1$. It is obvious that the curve α is a slant helix in the Minkowski $3-$space $\mathbb{E}^3_1$. This completes the proof. □

Corollary 3.15. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on the timelike surface M and β be the locus of the centers of the Meusnier spheres of α. Then α is a geodesic if and only if β is a planar curve.*

Proof. Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be an arc-length parameter curve on M and β be the locus of the centers of the Meusnier spheres of α. If α is a geodesic, then the geodesic curvature k_g of α and the torsion τ^* of β are equal to zero. So, β is a planar curve.

Conversely, if we assume that β is a planar curve, then it is obvious that α is a geodesic. This completes the proof. □

Corollary 3.16. *Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on M and β is the locus of the centers of the Meusnier spheres on M. β is a helix in the Minkowski 3−space $\mathbb{E}_1^3$ if and only if α is a planar curve.*

Proof. Let the spacelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on M. If the β is a helix in the Minkowski 3−space $\mathbb{E}_1^3$, then we obtain $\frac{\tau^*}{\kappa^*} = -\frac{k_g}{k_n}$ =constant. So, $\cot\theta(s)$ is a constant function and $\theta'(s) = 0$. Since α is a line of curvature on M, $\theta(s) = \tau(s) = 0$. So α is a planar curve.

Conversely, if we assume that the curve α is a planar curve, then it is obvious that β is a helix in the Minkowski 3−space $\mathbb{E}_1^3$. This completes the proof. □

3.3. Evolutes-Involutes of Timelike Space Curves and Meusnier Spheres on Timelike Surface

Theorem 3.13. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed timelike Frenet curve on timelike surface M in the Minkowski 3−space $\mathbb{E}_1^3$. Then the curve α is a geodesic if and only if the centers of the osculating circles coincide with the centers of the Meusnier spheres of the curve α and the osculating circles are geodesics on the Meusnier spheres.*

Proof. Let $Z(s)$ be the unit normal vector field of M and $\{T, N, B\}$ is the Frenet frame of the curve α. The locus of the center of the Meusnier sphere is given by (Eq. (2.15))

$$\begin{aligned}
\beta(s) &= \alpha(s) + \frac{1}{k_n} Z(s) \\
&= \alpha(s) + \frac{1}{k_n}(-\sin\theta(s) N(s) + \cos\theta(s) B(s)) \\
&= \alpha(s) + \frac{1}{\kappa\sin\theta(s)}(-\sin\theta(s) N(s) + \cos\theta(s) B(s)) \\
&= \alpha(s) + \frac{1}{\kappa}(-N(s) + \frac{k_g}{k_n} B(s)) \qquad (3.30)
\end{aligned}$$

Since α is a geodesic curve, we get $\beta(s) = \alpha(s) - \frac{1}{\kappa}N(s)$. So, $\beta(s)$ is the locus of the center of the osculating circle.

Conversely, we assume that the center of the osculating circle coincides with the center of the Meusnier sphere of the curve α. We can easily see that $k_g = 0$, that is, α is a geodesic curve. This completes the proof. □

Theorem 3.14. *Let β be evolute curve of the unit speed timelike curve α. Then the curve β can be written as*

$$\beta\left(s\right)=\alpha\left(s\right)-\frac{1}{\kappa}(N\left(s\right)+\cot(\int\tau\left(s\right)ds+c))B\left(s\right). \tag{3.31}$$

Proof. The tangent of the curve β at the point $\beta\left(s\right)$ is the line constructed by the vector $T^{*}\left(s\right)$. Since this line passes through the point $\alpha\left(s\right)$, the vector $\beta\left(s\right)-\alpha\left(s\right)$ is perpendicular to the vector $T\left(s\right)$. Then

$$\beta\left(s\right)=\alpha\left(s\right)+\lambda N\left(s\right)+\mu B\left(s\right). \tag{3.32}$$

If we differentiate the Eq.(3.32) , then we have

$$T^{*}\left(s\right)=\beta^{'}\left(s\right)=\left(1+\lambda\kappa\right)T\left(s\right)+\left(\lambda^{'}-\mu\tau\right)N\left(s\right)+\left(\lambda\tau+\mu^{'}\right)B\left(s\right). \tag{3.33}$$

According to the definition of the evolute, since $\left\langle T^{*}\left(s\right),T\left(s\right)\right\rangle=0$, we get

$$\lambda=-\frac{1}{\kappa} \tag{3.34}$$

and

$$\beta^{'}\left(s\right)=\left(\lambda^{'}-\mu\tau\right)N\left(s\right)+\left(\lambda\tau+\mu^{'}\right)B\left(s\right). \tag{3.35}$$

From Eq.(3.32) and Eq.(3.35), the vector field $\beta^{'}$is parallel to the vector field $\beta-\alpha$. Then we have

$$\frac{\lambda^{'}-\mu\tau}{\lambda}=\frac{\lambda\tau+\mu^{'}}{\mu}.$$

After that, we have

$$\tau=\frac{\left(\frac{\lambda}{\mu}\right)^{'}}{1+\left(\frac{\lambda}{\mu}\right)^{2}}.$$

If we integrate the last equation, we get

$$\int\tau\left(s\right)ds+c=\arctan\left(\frac{\lambda}{\mu}\right).$$

Hence, we find

$$\mu=-\frac{1}{\kappa}\cot\left(\int\tau\left(s\right)ds+c\right). \tag{3.36}$$

If we substitute Eq. (3.34) and Eq. (3.36) into Eq. (3.32), we have

$$\beta\left(s\right)=\alpha\left(s\right)-\frac{1}{\kappa}(N\left(s\right)+\cot(\int\tau\left(s\right)ds+c))B\left(s\right).$$

Then we obtain an evolute curve for each $c\in\mathbb{R}$. □

Theorem 3.15. ***(Main Theorem)*** *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed timelike Frenet curve on timelike surface M in the Minkowski $3-$space $\mathbb{E}_1^3$ with non zero curvatures κ and τ. Then the curve α is a line of curvature on M if and only if the locus of the centers of the Meusnier spheres of the curve α coincides with evolute curve of the curve α.*

Proof. Assume that the timelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be an arc-length parameter curve lying on the oriented timelike surface M and α is a line of curvature. The locus of the centers of the Meusnier spheres of the curve α is

$$\beta(s) = \alpha(s) + \frac{1}{\kappa(s)}(-N(s) + \cot\theta(s))B(s). \tag{3.37}$$

Since the curve α is the line of curvature on the timelike surface M, then $t_r = 0$ and $\theta(s) = -\int \tau(s)\,ds + c$. If we consider the last equality with (3.37), then the curve β becomes as

$$\beta(s) = \alpha(s) + \frac{1}{\kappa(s)}(-N(s) - \cot(\int \tau(s)\,ds + c)B(s))$$

So, from Theorem 3.14, the curve β is the evolute of the curve α.

Conversely, assume that the locus of the centers of the Meusnier spheres of the curve α coincides with evolute of the curve α. We can easily see that the curve α is a line of curvature on M. This completes the proof. $\square$

Theorem 3.16. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed timelike Frenet curve on timelike surface M with non-zero curvatures κ, τ in the Minkowski $3-$space $\mathbb{E}_1^3$. Let C_α be the focal curve of α and β be the locus of centers of the Meusnier spheres of α. Then the focal curve C_α coincides with the curve β if and only if $\kappa(s) = ce^{-\int \tau \cot\theta(s) ds}$.*

Proof. Assume that the focal curve C_α coincides with β. By using the (Definition (2.2)) and Eq. (3.37), we obtain

$$\begin{aligned} \alpha(s) - \frac{1}{\kappa}N(s) + \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}B(s) &= \alpha(s) - \frac{1}{\kappa}N(s) + \frac{1}{\kappa}\cot\theta(s)\,B(s), \\ \left(\frac{1}{\kappa}\right)' \frac{1}{\tau} &= \frac{1}{\kappa}\coth\theta(s), \\ \kappa(s) &= ce^{-\int \tau \cot\theta(s) ds}. \end{aligned}$$

Conversely, if we assume that the following equality $\kappa(s) = ce^{-\int \tau \cot\theta(s) ds}$ holds, then it is obvious that the focal curve C_α coincides with β. This completes the proof. $\square$

Corollary 3.17. *Let the timelike curve $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on M . The locus of the centers of the Meusnier spheres of α coincides with the locus of the centers of the osculating spheres of α if and only if $\theta(s) = \arcsin(\frac{c}{\kappa(s)})$.*

Proof. Assume that the locus of the centers of the Meusnier spheres of α coincides with the locus of the centers of the osculating spheres of α. So, we have that

$$\begin{aligned} \alpha(s) - \frac{1}{\kappa(s)}N(s) + \frac{1}{\kappa(s)}\cot\theta(s)\,B(s) &= \alpha(s) - \frac{1}{\kappa(s)}N(s) + \left(\frac{1}{\kappa(s)}\right)'\frac{1}{\tau(s)}B(s)\,, \\ \tau(s)\cot\theta(s) &= -\frac{\kappa(s)'}{\kappa(s)}. \end{aligned}$$

Since α is a line of curvature, $\theta(s) = -\int \tau(s)\,ds + c$, then we have

$$\begin{aligned} -\tau(s)\cot\left(\int \tau(s)\,ds + c\right) &= -\frac{\kappa(s)'}{\kappa(s)}, \\ \ln(\sin(\tau(s)\,ds)) - \ln\kappa(s) &= -\ln c, \\ \int \tau(s)\,ds &= \arcsin\left(\frac{c}{\kappa(s)}\right), \\ \theta(s) &= \arcsin\left(\frac{c}{\kappa(s)}\right). \end{aligned}$$

Conversely, if we assume that $\theta(s) = \arcsin(\frac{c}{\kappa(s)})$ holds, then it is obvious that the locus of the centers of the Meusnier spheres of α coincides with the locus of the centers of the osculating spheres of α. This completes the proof. □

Theorem 3.17. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed timelike Frenet curve on timelike surface M with non-zero curvatures κ, τ in the Minkowski $3-$space $\mathbb{E}_1^3$. Let the centers of the osculating spheres coincides with the centers of the Meusnier spheres of α. Then the curve α is a geodesic if and only if κ is a constant function.*

Proof. Assume that the centers of the osculating spheres coincide with the centers of the Meusnier spheres of α, than we have that

$$\left(\frac{1}{\kappa(s)}\right)'\frac{1}{\tau(s)} = \frac{1}{\kappa(s)}\frac{k_g}{k_n}.$$

Since the curve α is geodesic, then we have $(\frac{1}{\kappa(s)})'\frac{1}{\tau(s)} = 0$ and than κ is a constant function.

Conversely, if we assume that $\kappa(s)$ is a constant function, then it is obvious that the centers of the osculating spheres coincide with the centers of the Meusnier spheres of α. This completes the proof. □

REFERENCES

1. T. A. Ali, R. Lopez: *Slant helices in Minkowski space* $\mathbb{E}_1^3$. J. Korean. Math. Soc.doi: 10.4134/JKMS.2011.48.1.159.

2. B.Bukcu and M. K. Karaca: *On incomplete polynomials IIOn the Involute and Evolute Curves of the Spacelike Curve with a Spacelike Binormal in Minkowski* $3-$*Space*. Int J Contemp Math. Sci. **2** (2007), 221-232.
3. E. Iyigün, and E. Özdamar: *On the Meusnier's theorem for Lorentzian Surfaces*. Commun Fac Sci Univ Ank Series (1994) A_1. doi: 10.1501/Commua1_0000000468.
4. S. Kiziltuğ, and Y. Yayli: *Timelike Curves on Timelike Parallel Surfaces in Minkowski* $3-$*space* $\mathbb{E}_1^3$.. Math Aeterna **2** (2007), 689-700.
5. R. Lopez, and Y. Yayli: *Differential Geometry of curves and surfaces in Lorentz-Minkowski space*. (2008), arXiv:0810.3351v1[math.DG].
6. E. Soytürk, K. Ilarslan, and D. Sağlam: *Osculating Spheres and Osculating Circles of a Curve in Semi-Riemannian Space*. Commun Fac Sci Univ Ank Series A_1. doi: 10.1501/Commua1_0000000330.
7. V. R. Uribe: *On Vertices focal curvatures and differential geometry of space curves*. Bull Braz Math Soc. doi: 10.1007/s00574-005-0040-4.
8. B. Uzunoglu, İ Gök,Y. Yayli: *Locus of the centers of Meusnier spheres in Euclidean 3-space*.(2013) Univer J Math Sci **7** (2015), 51-66.

B. YILMAZ
Faculty of Science
Department of Mathematics
Ankara University
Ankara/ TURKEY
buzunoglu@ankara.edu.tr

Z. ÖZDEMİR
Faculty of Science
Department of Mathematics
Ankara University
Ankara/ TURKEY
zbozkurt@ankara.edu.tr

I. GÖK
Faculty of Science
Department of Mathematics
Ankara University
Ankara/ TURKEY
igok@science.ankara.edu.tr

Y. YAYLI
Faculty of Science
Department of Mathematics
Ankara University
Ankara/ TURKEY
yayli@science.ankara.edu.tr