ON THE EXISTENCE OF BESICOVITCH ALMOST PERIODIC SOLUTIONS FOR A CLASS OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. We study the existence of a Besicovitch almost periodic solution for a class of second order neutral delay differential equations

$$u''(t-r) + D_1F(u(t-r), u(t-2r), t-r) + D_2F(u(t), u(t-r), t) = 0,$$

in a Hilbert space, under some hyptoheses on the function $F(\cdot, \cdot, t)$. Here, $F: H \times H \times \mathbb{R} \to \mathbb{R}$ denotes a differentiable function, D_j, $j = 1, 2$, denotes the partial differential with respect to the jth vector variable and $r \in (0, \infty)$ is a fixed real number. The approach we use is based on a variational method applied on a Hilbert space of Besicovitch almost periodic functions.

1. Introduction

The variational method with retarded argument is a subject tracing its origins in the early papers of Elsgolc and Sabbagh. Later on, some new improvements have been established by Hughes and Sabbagh. For further details, we refer the reader to [16, 17, 23].

Using variational methods, Shu and Xu [24] and Yung Li [20] study the periodic solutions of the second order neutral differential equations.

Moreover, a new variational formalism has been established later by J. Blot (See [9], [10], [11]) in order to study the almost periodic solutions of Lagrangian system. This formalism, also called calculus of variations in mean, can be extended in the frame of the retarded functional differential equations, [3, 4, 5, 6]. More recently, inspired by the works of Blot, Parasyuk and Rustamova [22] have used a variational approach to study the almost periodic solutions of Lagragian system in the framework of Riemannian manifolds.

In this article, we aim to study the *weak* almost periodic solutions of a class of second order neutral delay differential equations having the following form:

(1.1) $u''(t-r) + D_1F(u(t-r), u(t-2r), t-r) + D_2F(u(t), u(t-r), t) = 0.$

Here, $F : H \times H \times \mathbb{R} \to \mathbb{R}$ denotes a differentiable function, D_j, $j = 1, 2$, denotes the partial differential with respect to the jth vector variable and $r \in (0, \infty)$ is a fixed real number. H stands for a Hilbert space.

By a *weak* almost periodic solution of 1.1 we mean a function $u : \mathbb{R} \to H$ which is a.p. in the sense of Besicovitch [8, 21] and which possesses a first-order and a second-order generalized derivative and such that the equality in 1.1 is equivalent to the fact that the difference between the two members has a quadratic mean value equal to zero.

In order to study the almost periodic solutions of 1.1, we use here a variational formalism which we apply on a Hilbert space of almost periodic functions via the following functional:

$$(1.2) \qquad J(u) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \Big(\frac{1}{2} \|\nabla u(t)\|_H^2 - F(u(t), u(t - r), t) \Big) dt$$

where ∇u is a generalized derivative of u.

The present paper is organized as follows. In Section 2, we review some known results on almost periodic functions, we provide our notations and we give our fundamental hypotheses which will be the main ingredients to prove the main result. In Section 3, we state our main theorem. In Section 4, we prove a list of basic lemmas which will be used. The main result will be proved in the last section.

2. Definitions, Notations and Hypotheses

Throughout this paper, H stands for a Hilbert space endowed with a norm $\| \cdot \|_H := \langle \cdot, \cdot \rangle_H^{1/2}$ associated to its inner product $\langle \cdot, \cdot \rangle_H$. The set of nonnegative integer numbers is denoted by $\mathbb{N}^*$, that is $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

Our first assumption is that we can find a nonempty subset S in H such that:

$$(2.1) \qquad \qquad S \text{ is convex, closed, and bounded.}$$

For $R := \sup \big\{ \|(x_1, x_2)\|_{H \times H}; \ (x_1, x_2) \in S \times S \big\}$, we suppose that the following hypothesis holds true:

$$(2.2) \qquad \begin{cases} \text{There exists } \tilde{R} \in (R, \infty) \text{ s.t. } \forall t \in \mathbb{R}, \forall (x_1, x_2) \in S \times S \\ \forall (y_1, y_2) \in B_{H \times H}(0, \tilde{R}) \setminus (S \times S), \ F(x_1, x_2, t) \geq F(y_1, y_2, t). \end{cases}$$

Here, we recall that for a fixed nonnegative number $\rho > 0$,

$$B_{H \times H}(0, \rho) := \{(x_1, x_2) \in H \times H; \|(x_1, x_2)\|_{H \times H} < \rho\}$$

and

$$B(0, \rho) := \{(x \in H; \|(x_1, x_2)\|_H < \rho\}.$$

The idea is to find critical points of the functional J defined as 1.2. Indeed, we shall obtain the existence of a minimizer on a subset of almost periodic functions taking their values into S and using the hypothesis 2.2 we show that this minimizer is a critical point of the functional J on a space of functions with values into the whole space H. We conclude finally that the minimizer that we found is a Besicovitch almost periodic solution to the equation 1.1.

Now, let us recall some useful results about almost periodic functions.

Definition 2.1. *A continuous function $u : \mathbb{R} \to H$ (respectively $u : \mathbb{R} \to \mathbb{R}$) is said to be a Bohr-almost periodic function if for any $\epsilon > 0$, there is a constant $l_\epsilon > 0$ such that any interval of length l_ϵ contains at least a number τ satisfying*

$$\sup_{t \in \mathbb{R}} \|u(t + \tau) - u(t)\|_H \leq \epsilon$$

(respectively $\sup_{t \in \mathbb{R}} |u(t + \tau) - u(t)| \leq \epsilon$).

$AP^0(H)$ (respectively $AP^0(\mathbb{R})$) denotes the space of the Bohr almost periodic functions from $\mathbb{R}$ into H (respectively $\mathbb{R}$). When equipped with the norm $\|u\|_\infty := \sup_{t \in \mathbb{R}} \|u(t)\|_H$ (respectively $\|u\|_\infty := \sup_{t \in \mathbb{R}} |u(t)|$), $AP^0(H)$ (respectively $AP^0(\mathbb{R})$) is a Banach space.

Definition 2.2. *When $u \in AP^0(H)$ (respectively $AP^0(\mathbb{R})$), its mean value*

$$\mathcal{M}\{u\} = \mathcal{M}_t\{u(t)\} := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(t)\,dt$$

exists in H (respectively $\mathbb{R}$).

For more details about these notions, we refer the reader to [8, 13].

Definition 2.3. *For a given integer $k \in \mathbb{N}^*$, $C^k(\mathbb{R}, H)$ denotes the space of the k-times continuously differentiable functions from $\mathbb{R}$ into H and*

$$AP^k(H) := \{u \in C^k(\mathbb{R}, H) : \forall j = 0, \cdots, k, u^{(j)} \in AP^0(H)\}$$

where $u^{(j)}(t) := \frac{d^j u(t)}{dt^j}$ if $j \geq 1$ and $u^{(0)} = u$.

Endowed with the norm $\|u\|_{C^1} := \|u\|_\infty + \|u'\|_\infty$, $AP^1(H)$ is a Banach space [14, Corollary 2.12].

If A is a subset of H, we denote by $AP^0(A) := \{u \in AP^0(H) : u(\mathbb{R}) \subset A\}$ and by $AP^k(A) := AP^k(H) \cap AP^0(A)$.

Definition 2.4. *[12, page 45] Let X and Y be two Banach spaces, a function $f : X \times \mathbb{R} \to Y$ belongs to $APU(X \times \mathbb{R}, Y)$ when f is continuous and it satisfies the following condition: for all $\epsilon > 0$, for all compact subset K in X, there exists $\ell = \ell(\epsilon, K)$ such that, for all $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + \ell]$ satisfying $\|f(x, t + \tau) - f(x, t)\|_Y \le \epsilon$ for all $t \in \mathbb{R}$ and for all $x \in K$.*

Definition 2.5. *When $p = 1, 2$, $\mathcal{B}^p(H)$ denotes the closure of $AP^0(H)$ into the Lebesgue space $L^p_{loc}(\mathbb{R}, H)$ for the semi-norm*

$$\mathcal{M}\{\|u\|^p\}^{1/p} := \left(\lim_{T \to \infty} \frac{1}{2T} \int_T^T \|u(t)\|_H^p dt \right)^{1/p}$$

and $B^p(H)$ is the quotient space $\mathcal{B}^p(H)/ \sim_p$ where $u \sim_p v$ means $\mathcal{M}\{\|u - v\|^p\} = 0$.

Endowed with the inner product $(u \mid v)_{B^2} := \mathcal{M}_t\{\langle u(t), v(t) \rangle_H\}$, $B^2(H)$ is a Hilbert space; its norm is denoted by $\|u\|_{B^2} := ((u \mid u)_{B^2})^{1/2}$. Endowed with the norm $\|u\|_{B^1} := \mathcal{M}\{\|u\|\}$, $B^1(H)$ is a Banach space. The (classes of) functions of $B^2(H)$ and $B^1(H)$ are Besicovitch almost periodic functions, [8], [21, pages 11-13]. Thanks to the Cauchy-Schwarz-Buniakovski inequality [8, page 69] we have: $B^2(H) \subset B^1(H)$ and $\|u\|_{B^1} \le \|u\|_{B^2}$ for all $u \in B^2(H)$.

Definition 2.6. *$B^{1,2}(H)$ is the space of the $u \in B^2(H)$ such that*

$$\nabla u := \lim_{\tau \to 0} \frac{1}{\tau}(u(\cdot + \tau) - u)$$

exists in $B^2(H)$.

Equipped with the inner product $(u \mid v)_{B^{1,2}} := (u \mid v)_{B^2} + (\nabla u \mid \nabla v)_{B^2}$, $B^{1,2}(H)$ becomes an Hilbert space, (See [10]). In [3], [4], [5], [9], [10], [11], this latter space was used in the study of the *weak* almost periodic solutions of various classes of differential equations.

Similarly to the above notations, giving a subset A of H, $B^2(A)$ stands for the closure of $AP^0(A)$ into $B^2(H)$, and $B^{1,2}(A)$ denotes the closure of $AP^1(A)$ into $B^{1,2}(H)$.

Now, we assume the following assumptions

$$(2.3) \qquad\qquad F \in APU((H \times H) \times \mathbb{R}, \mathbb{R}).$$

$$(2.4) \quad \begin{cases} \text{For all } (x_1, x_2, t) \in H \times H \times \mathbb{R} \text{ and } k = 1, 2, \\[2mm] \text{the partial differentials } D_k F(x_1, x_2, t) \text{ exist} \\[2mm] \text{and } D_k F \in APU((H \times H) \times \mathbb{R}, \mathcal{L}(H, \mathbb{R})). \end{cases}$$

$$(2.5) \qquad \begin{cases} \text{There exists } a \in (0, +\infty) \text{ s.t. } \forall X, Y \in H \times H, \forall t \in \mathbb{R}, \\ \\ |F(X, t) - F(Y, t)| \leq a.\|X - Y\|_{H \times H}. \end{cases}$$

$$(2.6) \qquad \begin{cases} \text{There exists } b \in (0, +\infty) \text{ s.t. } \forall X, Y \in H \times H, \forall t \in \mathbb{R}, \\ \\ \|D_X F(X, t) - D_X F(Y, t)\|_{\mathcal{L}(H \times H, \mathbb{R})} \leq b.\|X - Y\|_{H \times H}, \end{cases}$$

where D_X is the partial differential with respect to $X \in H \times H$.

$$(2.7) \qquad\qquad\qquad F \text{ is bounded } on \ S \times S \times \mathbb{R}.$$

$$(2.8) \qquad\qquad\qquad \text{For all } t \in \mathbb{R}, \ F(\cdot, \cdot, t) \text{ is concave } on \ S \times S.$$

The above assumptions are fundamental to use the variational formalism on the functional J.

Now, we state our main result.

Theorem 2.7. *Assume that 2.1- 2.8 hold true. Then, the equation 1.1 possesses a weak almost periodic solution, i.e. there exists $\underline{u} \in B^{1,2}(S)$ satisfying*

$$\nabla^2 \underline{u}(t - r) + D_1 F(\underline{u}(t - r), \underline{u}(t - 2r), t - r) + D_2 F(\underline{u}(t), \underline{u}(t - r), t) = 0$$

The above equality holds in B^2.

3. Proof of the Main result

In this section, we assume that all the assumptions 2.1-2.8 are satisfied. We begin by establishing the following elementary lemmas which will be used in the proof of our main result.

Lemma 3.1. *The function $J : B^{1,2}(H) \to \mathbb{R}$ defined by*

$$J(u) := \mathcal{M}_t \left\{ \frac{1}{2} \|\nabla u(t)\|_H^2 - F(u(t), u(t - r), t) \right\}$$

is continuously differentiable on $B^{1,2}(H)$ and its differential is given, for all $u, \ h \in B^{1,2}(H)$, by
$DJ(u).h =$
$\mathcal{M}_t \Big\{ \langle \nabla u(t), \nabla h(t) \rangle_H - [D_1 F(u(t), u(t - r), t) + D_2 F(u(t + r), u(t), t + r)].h(t) \Big\}.$

Proof. Let us consider the functional $J_1 : B^{1,2}(H) \to \mathbb{R}$ defined by

$$J_1(u) := \mathcal{M}_t\left\{\frac{1}{2}\|\nabla u(t)\|_H^2\right\}.$$

Since the derivation operator $\nabla : B^{1,2}(H) \to B^2(H)$ is linear continuous, it is of class C^1. Besides, the mapping $L : B^2(H) \to B^2(H) \times B^2(H)$ defined by $L(u) := (u, u)$ is also of class C^1 as a linear continuous one.

By virtue of the Cauchy-Schwarz-Buniakovski inequality (See [8, page 69]), the inner product $(\cdot|\cdot)_{B^2} : B^2(H) \times B^2(H) \to \mathbb{R}$ defined as $(u|v)_{B^2} := \mathcal{M}_t\{\langle u(t), v(t)\rangle_H\}$ is a bilinear continuous function and so it is of class C^1.

Taking into account the following formula $J_1 = \frac{1}{2}(\cdot|\cdot)_{B^2} \circ L \circ \nabla$, we deduce that J_1 is a continuously differentiable function on $B^{1,2}(H)$ as a composition of C^1-mappings. Furthermore, using the chain rule and the formulas of the differentials of linear and bilinear mappings, we obtain for all $u, h \in B^{1,2}(H)$

$$DJ_1(u).h = \mathcal{M}_t\{\langle \nabla u(t), \nabla h(t)\rangle_H\}.$$

On the other hand, let us define the functional $J_2 : B^{1,2}(H) \to \mathbb{R}$ by

$$J_2(u) := \mathcal{M}_t\{F(u(t), u(t-r), t)\}.$$

Note that the canonical injection $in : B^{1,2}(H) \to B^2(H)$ defined as $in(u) := u$ is of class C^1 since it is a linear continuous mapping.

As the mean value is invariant under translation transform, the linear mapping $T : B^2(H) \to B^2(H) \times B^2(H) \equiv B^2(H \times H)$ defined for all $t \in \mathbb{R}$ by $T(u)(t) := (u(t), u(t-r))$ is clearly a continuous mapping and consequently of class C^1 on $B^2(H)$.

Now, let us suppose that the above conditions 2.3, 2.4 and 2.6 are fulfilled. By using [4, Lemma 4.2], these conditions imply that the well-defined *Nemytskii* operator $\mathcal{N}_F : B^2(H \times H) \to B^1(\mathbb{R})$, built on the function F, and given for all $t \in \mathbb{R}$ by $\mathcal{N}_F(U)(t) := F(U(t), t)$ is continuously differentiable and we have for each $U \in B^2(H \times H)$, $D\mathcal{N}_F(U) = \mathcal{N}_{D_X F}(U)$.

Since the mean value $\mathcal{M} : B^1(\mathbb{R}) \to \mathbb{R}$ is a linear continuous mapping, it follows immediately that it is of class C^1. Hence, by noticing that $J_2 = \mathcal{M} \circ \mathcal{N}_F \circ T \circ in$, the functional J_2, being a composition of C^1-mappings, is continuously differentiable. Moreover, thanks to the chain rule formula and the invariance of the mean value under translation transform, we get, for all $u, h \in B^{1,2}(H)$,

$$DJ_2(u).h = \mathcal{M}_t\{D_1 F(u(t), u(t-r), t).h(t) + D_2 F(u(t), u(t-r), t).h(t-r)\}$$

$$= \mathcal{M}_t\{[D_1 F(u(t), u(t-r), t) + D_2 F(u(t+r), u(t), t+r)].h(t)\}.$$

Finally, we have proven that our function $J := J_1 - J_2$ is of class $C^1(B^{1,2}(H), \mathbb{R})$ and we have $DJ(u).h = DJ_1(u).h - DJ_2(u).h = \mathcal{M}_t\{\langle \nabla u(t), \nabla h(t)\rangle_H - [D_1 F(u(t), u(t-r), t) + D_2 F(u(t+r), u(t), t+r)].h(t)\}$ for all $u, h \in B^{1,2}(H)$. $\square$

Lemma 3.2. *The following assertions hold.*

(1) $B^{1,2}(S)$ *is a convex set which is closed into* $B^{1,2}(H)$.

(2) *J is convex on* $B^{1,2}(S)$.

(3) *J is weakly lower semi-continuous on* $B^{1,2}(S)$.

Proof. (1) This statement has been shown in [11, Lemma 4.1].

(2) We write $J := J_1 - J_2$, where the functions J_1 and J_2 are those introduced in the previous lemma.

It is obvious that $\|.\|_{B^2}$ is a convex mapping on $B^2(H)$ and that the differential operator $\nabla : B^{1,2}(H) \to B^2(H)$ is linear. Then, since $J_1 = \frac{1}{2}\|.\|_{B^2} \circ \nabla$, it follows immediately that J_1 is a convex functional on $B^{1,2}(S)$.

Let $u, v \in B^{1,2}(S)$ and $\lambda \in (0,1)$. Then there exist two sequences $(u_m)_m$ and $(v_m)_m$ in $AP^1(S)$ such that $\lim\limits_{m\to\infty} \|u - u_m\|_{B^{1,2}} = 0$ and $\lim\limits_{m\to\infty} \|v - v_m\|_{B^{1,2}} = 0$, so that, by using the fact that $B^{1,2}$ is continuously embedded into B^2, we have $\lim\limits_{m\to\infty} \|u - u_m\|_{B^2} = \lim\limits_{m\to\infty} \|v - v_m\|_{B^2} = 0$. Therefore, due to the invariance of the mean value under translation transform, we obtain that $\lim\limits_{m\to\infty} \|\tau_r u - \tau_r u_m\|_{B^2} = \lim\limits_{m\to\infty} \|\tau_r v - \tau_r v_m\|_{B^2} = 0$ where $\tau_r u := u(\cdot + r)$ denotes the operator of translation by r.

Since the function $F(\cdot, \cdot, t)$ is concave on the convex set $S \times S$ 2.8, we have for all $t \in \mathbb{R}$

$$F((1-\lambda)u_m(t) + \lambda v_m(t), (1-\lambda)u_m(t-r) + \lambda v_m(t-r), t)$$
$$\geq (1-\lambda)F(u_m(t), u_m(t-r), t) + \lambda F(v_m(t), v_m(t-r), t)$$

or equivalently

(3.1) $\quad \mathcal{N}_F((1-\lambda)(u_m, \tau_r u_m) + \lambda(v_m, \tau_r v_m)) \geq (1-\lambda)\mathcal{N}_F(u_m, \tau_r u_m) + \lambda\mathcal{N}_F(v_m, \tau_r v_m)$

On the other hand, using [4, Lemma 4.1], one has $\mathcal{N}_F \in C^0(B^2(H \times H), B^2(\mathbb{R}))$. Then, taking the limit when $m \to \infty$ in the inequality 3.1, we get

$$\mathcal{N}_F((1-\lambda)(u, \tau_r u) + \lambda(v, \tau_r v)) \geq (1-\lambda)\mathcal{N}_F(u, \tau_r u) + \lambda\mathcal{N}_F(v, \tau_r v).$$

This implies that $\mathcal{N}_F \circ T$ is concave on $B^2(S)$.

By linearity of the mean value, we deduce that the function $J_2 = \mathcal{M} \circ \mathcal{N}_F \circ T \circ in$ is concave on $B^{1,2}(S)$ or, similarly, $(-J_2)$ is convex.

As a consequence, observing that $J = J_1 - J_2$, we infer that the function J is convex as the sum of two convex functionals.

(3) Since J is a convex function on $B^{1,2}(S)$, for each $\alpha \in \mathbb{R}$, the level set

$$[J \leq \alpha] := \{u \in B^{1,2}(S); J(u) \leq \alpha\}$$

is a convex. But, by continuity of J, the set $[J \leq \alpha]$ is closed for the strong topology. Then, the fact that the level set $[J \leq \alpha]$ is convex and closed allows us to deduce by arbitrariness of α, that $[J \leq \alpha]$ is weakly closed, and consequently, the function J is weakly lower semi-continuous. $\quad\square$

Lemma 3.3. *There exists $\underline{u} \in B^{1,2}(S)$ such that $J(\underline{u}) = \inf\{J(u); \ u \in B^{1,2}(S)\}$.*

Proof. Since the function F is bounded on $S \times S \times \mathbb{R}$ 2.7, we set the finite real number $M := \sup\{F(x_1, x_2, t); \ (x_1, x_2, t) \in S \times S \times \mathbb{R}\}$.

Then, we have for all $u \in AP^1(S)$,

$$J(u) \geq \mathcal{M}_t\left\{\frac{1}{2}\|u'\|_H^2 - F(u(t), u(t-r), t)\right\} \geq \frac{1}{2}\|u'\|_{B^2} - M \geq -M > -\infty.$$

Therefore, the function J is bounded on $AP^1(S)$. By the continuity of J on $B^{1,2}(H)$ and the density of $AP^1(S)$ in $B^{1,2}(S)$, we deduce that $J(u) \geq -M$ for all the functions $u \in B^{1,2}(S)$ and $\inf\{J(u); \ u \in B^{1,2}(S)\} = \inf\{J(u); \ u \in AP^1(S)\}$.

Now, by definition of the infimum, let us consider $(u_k)_k$ a minimizing sequence of J on $AP^1(S)$ such that $J(u_k) \leq \inf\{J(u); \ u \in B^{1,2}(S)\} + \frac{1}{k}$ for all $k \in \mathbb{N}^*$. So, we get

$$\frac{1}{2}\|u_k'\|_{B^2}^2 \leq \mathcal{M}_t\{F(u_k(t), u_k(t-r), t)\} + \inf\{J(u); \ u \in B^{1,2}(S)\} + \frac{1}{k}$$

$$\leq M + \inf\{J(u); \ u \in B^{1,2}(S)\} + \frac{1}{k}$$

$$\leq M + \inf\{J(u); \ u \in B^{1,2}(S)\} + 1$$

We deduce then that the (u_k') is a bounded sequence in $B^2(H)$ and due to the fact that $u_k(\mathbb{R}) \subset S \subset B(0, \tilde{R})$ we have also $\|u_k(t)\|_H \leq \tilde{R}$ for each $t \in \mathbb{R}$, and so $\|u_k\|_{B^2} \leq \tilde{R}$ for all $k \in \mathbb{N}$. As a consequence, we have shown that the sequence $(u_k)_k$ is bounded on $B^{1,2}(H)$.

As the space $B^{1,2}(H)$ is a Hilbert space and consequently a reflexive one, we can extract a sub-sequence $(u_{k'})$ of (u_k) which converges weakly to $\underline{u} \in B^{1,2}(H)$ and by convexity and closedness of $B^{1,2}(S)$, we have necessarily $\underline{u} \in B^{1,2}(S)$.

Finally, since the function J is weakly lower semi-continuous, we deduce that $(u_{k'})$ is a minimizing sequence and consequently that

$$J(\underline{u}) = \inf\{J(u); \ u \in B^{1,2}(S)\}.$$

$\square$

Lemma 3.4. *The following statement holds true.*

$$\inf\{J(u); \ u \in AP^1(S)\} = \inf\{J(u); \ u \in AP^1(B(0, R))\}.$$

Proof. • As $AP^1(S) \subset B^{1,2}(S) \subset B^{1,2}(B(0, \tilde{R}))$, we have certainly that

$$\inf\Big\{J(u); \ u \in AP^1(S)\Big\} \geq \inf\Big\{J(u); \ u \in B^{1,2}(B(0, \tilde{R}))\Big\}.$$

But, $\inf\Big\{J(u); \ u \in B^{1,2}(B(0, \tilde{R}))\Big\} = \inf\Big\{J(u); \ u \in AP^1(B(0, \tilde{R}))\Big\}$. So we get

$$\inf\Big\{J(u); \ u \in AP^1(S)\Big\} \geq \inf\Big\{J(u); \ u \in AP^1(B(0, \tilde{R}))\Big\}.$$

• Let us, now, prove that $\inf\Big\{J(u); \ u \in AP^1(S)\Big\} \leq \inf\Big\{J(u); \ u \in AP^1(B(0, \tilde{R}))\Big\}$.

Consider $u \in AP^1(B(0, \tilde{R}))$ and define the projection operator on the convex closed set S, $P : H \to S$. We set, then, the function defined for all real t by $v(t) := P(u(t))$. As known, the projection operator is 1-Lipschitzean function [2, page 16] and consequently continuous. This implies that $v \in AP^0(S)$, [12, Lemma 3.2]. Besides, the function $u' \in AP^0(H)$ since $u \in AP^1(H)$. Then, u' is a bounded function on $\mathbb{R}$. As a consequence, the function u is a Lipschitzean one on $\mathbb{R}$. We deduce then that the function v is Lipschitzean on $\mathbb{R}$ as it is a composition of Lipschitzean mappings. Therefore, the function v is locally absolutely continuous on $\mathbb{R}$ which infers that v is Lebesgue-almost everywhere differentiable on $\mathbb{R}$, [7] (Corollaire A.2 page 145).

Let $t \in \mathbb{R}$ be a point where v is differentiable and consider $\delta \in \mathbb{R} \setminus \{0\}$. One has then

$$\|\frac{1}{\delta}(v(t+\delta) - v(t))\|_H = \|\frac{1}{\delta}(P(u(t+\delta)) - P(u(t)))\|_H$$

$$= \frac{1}{|\delta|}\|P(u(t+\delta)) - P(u(t))\|_H$$

$$\leq \frac{1}{|\delta|}\|u(t+\delta) - u(t)\|_H.$$

Taking the limit when δ tends to 0, we obtain

(3.2) $$\|v'(t)\|_H \leq \|u'(t)\|_H \quad \text{a.e. } t \in \mathbb{R}$$

Now, we consider the functions of Bochner-Féjer defined as $K_m(t) := \sum_{\xi=-m}^{m}(1-\frac{|\xi|}{m})e^{-i\xi t}$ and $\varphi_m(t) := \prod_{j=1}^{m} K_{(m!)^2}(\frac{\beta_j t}{m})$ for all $m \in \mathbb{N}^*$ and for all $t \in \mathbb{R}$, where $(\beta_j)_j$ is a $\mathbb{Z}$-basis of the set of the Fourier-Bohr exponents of v. According to [13, pages 86-88] and [15, page 115], we have that $K_m(t) \geq 0$, $\mathcal{M}\{K_m\} = 1$, $\varphi_m \geq 0$ and $\mathcal{M}\{\varphi_m\} = 1$, for all $m \in \mathbb{N}^*$ and $t \in \mathbb{R}$.
Let us set for all $m \in \mathbb{N}^*$ and all $t \in \mathbb{R}$,

$$\sigma_m(t) := \mathcal{M}_s\{\varphi_m(s).v(s+t)\}.$$

Using [15, page 116], and changing $\mathbb{R}$ by H if necessary, we obtain that $\sigma_m \in AP^1(H)$ and

$$\lim_{m\to\infty} \|\sigma_m - v\|_{B^2} = 0.$$

0 By continuity of the functional J_2 on $B^2(H)$, we have $\lim_{m\to\infty} J_2(\sigma_m) = J_2(v)$.

Thanks to the condition 2.2, we have $F(v(t), v(t - r), t) \geq F(u(t), u(t - r), t)$, for all $t \in \mathbb{R}$. Then, $J_2(v) \geq J_2(u)$ and consequently, by definition of the limit,

$$(3.3) \qquad \forall \epsilon > 0, \exists m_\epsilon \in \mathbb{N}^*, \forall m \geq m_\epsilon, J_2(\sigma_m) \geq J_2(u) - \epsilon.$$

On the other hand, using the theorem of Mazur [19, page 88], [1, Corollaire 5.62 page 194], we can write that $\overline{co}(v(\mathbb{R})) = \cap_{(p,\alpha)\in\Pi}[p \geq \alpha]$ where $\Pi := \{(p, \alpha) \in \mathcal{L}(H, \mathbb{R}) \times \mathbb{R} : \forall y \in v(\mathbb{R}), \langle p, y \rangle_H \geq \alpha\}$ and $[p \geq \alpha] := p^{-1}([\alpha, \infty))$. But since $\varphi_m \geq 0$, we get for all $t \in \mathbb{R}$ and all $(p, \alpha) \in \Pi$,

$$\langle p, \sigma_m(t) \rangle_H = \mathcal{M}_s\{\langle p, \varphi_m(s).v(s + t) \rangle_H\}$$
$$= \mathcal{M}_s\{\varphi_m(s)\langle p, v(s + t) \rangle_H\}$$
$$\geq \mathcal{M}_s\{\varphi_m(s).\alpha\} = \mathcal{M}_s\{\varphi_m(s)\}.\alpha = \alpha.$$

This shows that $\sigma_m(t) \in \overline{co}(v(\mathbb{R})) \subset S$ and thus, $\sigma_m \in AP^1(S)$ for all $m \in \mathbb{N}^*$

Using [11, Lemma 4.3 iii.], we have for all $m \in \mathbb{N}^*$ and all $t \in \mathbb{R}$,

$$\|\sigma'_m(t)\|_H = \|\mathcal{M}_s\{\varphi_m(s)v'(t + s)\}\|_H$$
$$\leq \mathcal{M}_s\{\varphi_m(s)\|v'(t + s)\|_H\}$$
$$= \mathcal{M}_s\{(\varphi_m(s))^{1/2}(\varphi_m(s))^{1/2}\|v'(t + s)\|_H\}.$$

And according to the Cauchy-Schwarz-Buniakovski equality and 3.2, we deduce that

$$\|\sigma'_m(t)\|_H \leq \left(\mathcal{M}_s\{\varphi_m(s)\}\right)^{1/2}\left(\mathcal{M}_s\{\varphi_m(s)\|v'(t + s)\|^2_H\}\right)^{1/2}$$
$$\leq \mathcal{M}_s\{\varphi_m(s)\|u'(t + s)\|^2_H\}^{1/2}.$$

Then,
$$\mathcal{M}_t\{\|\sigma'_m(t)\|^2_H\} \leq \mathcal{M}_t\{\mathcal{M}_s\{\varphi_m(s)\|u'(t + s)\|^2_H\}\}.$$

By using [11, Lemma 4.3 i.], we obtain that

$$\mathcal{M}_t\{\mathcal{M}_s\{\varphi_m(s)\|u'(t + s)\|^2_H\}\} = \mathcal{M}\{\varphi_m\}.\mathcal{M}\{\|u'\|^2\}.$$

Therefore, we get for each $m \in \mathbb{N}^*$,

$$(3.4) \qquad \mathcal{M}_t\|\sigma'_m(t)\|^2_H \leq \mathcal{M}\{\|u'\|^2\}.$$

Let us fix arbitrarily $\epsilon > 0$ and consider $m_\epsilon \in \mathbb{N}^*$ defined as (3.3). Using (3.2) and (3.4) we can write

$$J(\sigma_m) = \frac{1}{2}\mathcal{M}\{\|\sigma'_m\|^2\} - J_2(\sigma_m) \leq \frac{1}{2}\mathcal{M}\{\|u'\|^2\} - J_2(u) + \epsilon = J(u) + \epsilon.$$

This last inequality implies that $\inf\{J(u);\ u \in AP^1(S)\} \leq J(u) + \epsilon$. By passing to the limit when $\epsilon \to 0$, we obtain $\inf\{J(u);\ u \in AP^1(S)\} \leq J(u)$. Hence, we have shown that $\inf\{J(u);\ AP^1(S)\} \leq \inf\{J(u);\ u \in AP^1(B(0, \tilde{R}))\}$ which completes the proof of the statement. $\square$

Lemma 3.5. *The following assertion holds true.*

$$\inf\{J(u);\ u \in B^{1,2}(S)\} = \inf\{J(u);\ u \in B^{1,2}(B(0, \tilde{R}))\}.$$

Proof. From the fact that $AP(S) \subset B^{1,2}(S)$, it follows immediately that

(3.5) $$\inf\{J(u);\ u \in B^{1,2}(S)\} \leq \inf\{J(u);\ u \in B^{1,2}(S)\}$$

Let $u \in B^{1,2}(S)$. Then, we can find a sequence $(u_m)_m \in (AP^1(S))^{\mathbb{N}}$ such that $\lim_{m\to\infty} \|u - u_m\|_{B^{1,2}} = 0$. But since $J \in C^0(B^{1,2}(H), \mathbb{R})$, we have $\inf\{J(u);\ u \in AP^1(S)\} \leq \lim_{m\to\infty} J(u_m) = J(u)$. Hence,

(3.6) $$\inf\{J(u);\ u \in AP^1(S)\} \leq \inf\{J(u);\ u \in B^{1,2}(S)\}.$$

Taking into account the two inequalities 3.5 and 3.6, we get $\inf\{J(u);\ u \in AP^1(S)\} = \inf\{J(u);\ u \in B^{1,2}(S)\}$.

We reason similarly to obtain that $\inf\{J(u);\ u \in AP^1(B(0, \tilde{R}))\} = \inf\{J(u);\ u \in B^{1,2}(B(0, \tilde{R}))\}$.

Finally, thanks to Lemma 3.4, we deduce the desired result. $\square$

Now, we shall prove the main result stated in the Theorem 2.7.

By virtue of Lemma 3.3, there exists $\underline{u} \in B^{1,2}(S)$ which satisfies that $J(\underline{u}) = \inf\{J(u);\ u \in B^{1,2}(S)\}$.

Let $(u_m)_m$ be a sequence in $AP^1(S)$ such that $\lim_{m\to\infty} \|u_m - u\|_{B^{1,2}} = 0$.

Consider an element $h \in AP^1(H)$ with $h \neq 0$ and define the scalar λ as $\lambda := \frac{\tilde{R} - R}{\|h\|_\infty} \in (0, \infty)$. For each $m \in \mathbb{N}$ and each $t \in \mathbb{R}$, we have for any $\theta \in (-\lambda, \lambda)$,

$$\|u_m(t) + \theta h(t)\|_H \leq \|u_m(t)\|_H + |\theta|.\|h(t)\|_H \leq R + (\tilde{R} - R) = \tilde{R}.$$

Then, it follows that $u_m + \theta.h \in AP^1(B(0, \tilde{R}))$. But taking into account that the sequence $(u_m + \theta.h)_m$ converges in B^2 to $u + \theta.h$, we obtain that $u + \theta.h \in B^{1,2}(B(0, \tilde{R}))$ for all $\theta \in (-\lambda, \lambda)$.

Lemma 3.3 and Lemma 3.5 imply that

$$J(\underline{u}) \leq J(\underline{u} + \theta.h),$$

for all $\theta \in (-\lambda, \lambda)$.

On the other hand, J is of class C^1, by virtue of Lemma 3.1. Therefore, the mapping $\theta \mapsto J(\underline{u} + \theta h)$ is differentiable. We infer that
$DJ(\underline{u}).h = \frac{d}{d\theta}_{|\theta=0} J(\underline{u} + \theta.h) = 0$, or equivalently $DJ(\underline{u}).h = 0$, for all $h \in AP^1(H)$.
The linearity of $DJ(\underline{u})$ and its continuity gives, by density of $AP^1(H)$ into $B^{1,2}(H)$,
that $DJ(\underline{u}).h = 0$ holds true for each function $h \in B^{1,2}(H)$.

Besides, using the Lemma 3.1, we have that

$$DJ(\underline{u}).h = \mathcal{M}_t\{\langle \nabla \underline{u}(t), \nabla h(t)\rangle_H -$$
$$\left[D_1F(\underline{u}(t), \underline{u}(t-r), t) + D_2F(\underline{u}(t+r), \underline{u}(t), t+r)\right].h(t)\}$$
$$= 0.$$

Thanks to [10, proposition 10], we deduce that the function $\underline{u}$ belongs to $B^{2,2}(H)$ and that $\nabla^2\underline{u}(t-r) + D_1F(\underline{u}(t), u(t-r), t) + D_2F(\underline{u}(t-r), \underline{u}(t-2r), t-r) = 0$. Our principal result has, finally, been proved.

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