FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 1 (2016), 79–95

BERNSTEIN DURRMEYER OPERATORS BASED ON TWO PARAMETERS

Vijay Gupta and Ali Aral

Abstract. In the present paper, we study the applications of the extension of quantum calculus based on two parameters. We define beta function and establish an identity with gamma function, for two parameters (p, q), i e. the post-quantum calculus. We also propose the (p, q)-Durrmeyer operators, estimate moments and establish some direct results. Depending on the selection of p and q, the rate of convergence of the our new operators can provide better approximation than those of the Bernstein-Durrmeyer operators and its q-analogue. In the end, we provide some graphs using the software Mathematica.
Keywords: Bernstein Durrmeyer operator, quantum calculus, approximation theory, Bernstein operator.

1. Introduction

In the last eighteen years the applications of quantum calculus (q-calculus) in the field of approximation theory has been an active area of research. Several new operators have been generalized to their q variants and their approximation behavior have been discussed. We mention here the recent books [1] and [6] on this topic. Further generalization of q calculus is the post quantum calculus, denoted by (p, q)-calculus. Some papers have appeared recently in this area related to approximation theory. This area is in the developing stage. For (p, q)-calculus, we mention here some basic definitions and theorems, some of them may be found in the recent papers [7], [10], [8], [11] etc.

The (p, q)-number is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q},$$

Obviously, it can be seen that $[n]_{p,q} = p^{n-1} [n]_{q/p}$. It is obvious from the definition, that q-integers and (p, q)-integers are different, that is we cannot obtain (p, q) integers

Received October 30, 2015; Accepted November 12, 2015
2010 *Mathematics Subject Classification.* Primary 41A25; Secondary 41A30

just by replacing q by q/p in the definition of q-integers. But if we put $p = 1$ in definition of (p,q) integers then q-integers becomes a particular case of (p,q) integers. Thus we can say that (p,q)-integers can be taken as a generalization of q-integers. The (p,q)-factorial is defined by

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, n \geq 1, \quad [0]_{p,q}! = 1.$$

The (p,q)-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}, \quad 0 \leq k \leq n.$$

Definition 1.1. The (p,q)-power basis is defined below

$$(x \ominus a)^n_{p,q} = (x-a)(px-qa)(p^2x-q^2a)\cdots(p^{n-1}x-q^{n-1}a).$$

Definition 1.2. The (p,q)-derivative of the function f is defined as

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x}, \quad x \neq 0$$

and $D_{p,q}f(0) = f'(0)$, provided that f is differentiable at 0. It is obvious that $D_{p,q}x^n = [n]_{p,q}x^{n-1}$. Note also that for $p = 1$, the (p,q)-derivative reduces to the $q-$derivative given by

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad x \neq 0$$

Definition 1.3. The (p,q)-derivative fulfils the following product rules

$$\begin{aligned} D_{p,q}(f(x)g(x)) &= f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x) \\ D_{p,q}(f(x)g(x)) &= g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x). \end{aligned}$$

The following assertion is valid:

$$\begin{aligned} D_{p,q}(x \ominus a)^n_{p,q} &= [n]_{p,q}(px \ominus a)^{n-1}_{p,q}, n \geq 1 \\ D_{p,q}(a \ominus x)^n_{p,q} &= -[n]_{p,q}(a \ominus qx)^{n-1}_{p,q}, n \geq 1, \end{aligned}$$

and $D_{p,q}(x \ominus a)^0_{p,q} = 0$.

Definition 1.4. Let f be an arbitrary function and a be a real number. The (p,q)-integral of $f(x)$ on $[0,a]$ is defined as

$$\int_0^a f(x)\,d_{p,q}x = (q-p)\,a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}a\right) \quad \text{if} \quad \left|\frac{p}{q}\right| < 1$$

and

$$\int_0^a f(x)\, d_{p,q}x = (p-q)\, a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}} a\right) \quad \text{if} \quad \left|\frac{p}{q}\right| > 1$$

Theorem 1.1. *(Fundamental theorem of (p,q)-calculus) If $F(x)$ is an antiderivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, we have*

$$\int_a^b f(x)\, d_{p,q}x = F(b) - F(a),$$

where $0 \leq a < b \leqslant \infty$.

Proposition 1.1. *The formula of (p,q)-integration by part is given by*

$$\int_a^b f(px) D_{p,q} g(x) d_{p,q}x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) D_{p,q} f(x) d_{p,q}x. \tag{1.1}$$

Definition 1.5. ([9]) Let n is a nonnegative integer, we define the (p,q)-Gamma function as

$$\Gamma_{p,q}(n+1) = \frac{(p \ominus q)_{p,q}^n}{(p-q)^n} = [n]_{p,q}!, \quad 0 < q < p.$$

In the present paper, we define (p,q)-Beta function and establish a relation between (p,q)-Beta and (p,q)-Gamma functions. As the (p,q)-Beta function is generalization of Beta function of first kind, we propose the (p,q)-Bernstein-Durrmeyer operators. Using some identities of (p,q)-calculus, we estimate moments and establish some direct results for (p,q)-Bernstein-Durrmeyer operators.

2. (p,q)-Beta Function

In this section, we propose the (p,q) Beta function and then find a relation between (p,q)-Beta and (p,q)-Gamma functions.

Let $m, n \in \mathbb{N}$, we define (p,q)-Beta function of first kind as

$$B_{p,q}(m,n) = \int_0^1 (px)^{m-1} (p \ominus pqx)_{p,q}^{n-1}\, d_{p,q}x. \tag{2.1}$$

In [9], the author has given the relation between (p,q)-Beta and (p,q)-Gamma functions, without mentioning the integral representation of the (p,q)-Beta functions. For the above form of (p,q)-Beta function, we have the following relation, which is not commutative.

Theorem 2.1. *The (p,q)-Gamma and (p,q)-Beta functions fulfill the following fundamental relation*

$$B_{p,q}(m,n) = p^{[n(2m+n-2)+n-2]/2}\frac{\Gamma_{p,q}(m)\,\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}, \tag{2.2}$$

where $m,n\in\mathbb{N}$.

Proof. For any $m,n\in\mathbb{N}$ since

$$B_{p,q}(m,n) = \int_0^1 (px)^{m-1}(p\ominus pqx)_{p,q}^{n-1}\,d_{p,q}x,$$

using (1.1) for $f(x)=x^{m-1}$ and $g(x) = -\frac{(p\ominus px)_{p,q}^n}{p[n]_{p,q}}$ with the equality

$$D_{p,q}(p\ominus px)^n = -[n]_{p,q}\,p\,(p\ominus pqx)^{n-1}$$

we have

$$\begin{aligned} B_{p,q}(m,n) &= \frac{[m-1]_{p,q}}{p^{m-1}[n]_{p,q}}\int_0^1 (px)^{m-2}(p\ominus pqx)_{p,q}^n\,d_{p,q}x \\ &= \frac{[m-1]_{p,q}}{p^{m-1}[n]_{p,q}}B_{p,q}(m-1,n+1). \end{aligned} \tag{2.3}$$

Also we can write for positive integer n

$$\begin{aligned} B_{p,q}(m,n+1) &= \int_0^1 (px)^{m-1}(p\ominus pqx)_{p,q}^n\,d_{p,q}x \\ &= \int_0^1 (px)^{m-1}(p\ominus pqx)_{p,q}^{n-1}(p^n-pq^nx)\,d_{p,q}x \\ &= p^n\int_0^1 (px)^{m-1}(p\ominus pqx)_{p,q}^{n-1}d_{p,q}x \\ &\qquad -q^n\int_0^1 (px)^m(p\ominus pqx)_{p,q}^{n-1}\,d_{p,q}x \\ &= p^nB_{p,q}(m,n) - q^nB_{p,q}(m+1,n). \end{aligned}$$

Using (2.3), we have

$$B_{p,q}(m,n+1) = p^nB_{p,q}(m,n) - q^n\frac{[m]_{p,q}}{p^m[n]_{p,q}}B_{p,q}(m,n+1),$$

which implies that

$$B_{p,q}(m,n+1) = p^{n+m}\frac{p^n-q^n}{p^{n+m}-q^{n+m}}B_{p,q}(m,n).$$

Further, by definition of (p, q) integration

$$B_{p,q}(m,1) = \int_0^1 (px)^{m-1}\, d_{p,q}x = \frac{p^{m-1}}{[m]_{p,q}}.$$

We immediately have

$$\begin{aligned} B_{p,q}(m,n) &= p^{n+m-1}\frac{p^{n-1}-q^{n-1}}{p^{n+m-1}-q^{n+m-1}}B_{p,q}(m,n-1) \\ &= p^{n+m-1}\frac{p^{n-1}-q^{n-1}}{p^{n+m-1}-q^{n+m-1}}p^{n+m-2}\frac{p^{n-2}-q^{n-2}}{p^{n+m-2}-q^{n+m-2}}B_{p,q}(m,n-2) \\ &= p^{n+m-1}\frac{p^{n-1}-q^{n-1}}{p^{n+m-1}-q^{n+m-1}}p^{n+m-2}\frac{p^{n-2}-q^{n-2}}{p^{n+m-2}-q^{n+m-2}}\cdots \\ &\quad \cdots p^{m+1}\frac{p-q}{p^{m+1}-q^{m+1}}B_{p,q}(m,1) \\ &= p^{n+m-1}\frac{p^{n-1}-q^{n-1}}{p^{n+m-1}-q^{n+m-1}}p^{n+m-2}\frac{p^{n-2}-q^{n-2}}{p^{n+m-2}-q^{n+m-2}}\cdots \\ &\quad \cdots p^{m+1}\frac{p-q}{p^{m+1}-q^{m+1}}\frac{p^{m-1}}{[m]_{p,q}} \\ &= \frac{p^{(m-1)+m+(m+1)+\cdots+(m+n-1)}}{p^m}\frac{(p\ominus q)^{n-1}_{p,q}}{(p^m\ominus q^m)^n_{p,q}}(p-q), \end{aligned}$$

i.e.,

$$B_{p,q}(m,n) = p^s\frac{(p\ominus q)^{n-1}_{p,q}}{(p^m\ominus q^m)^n_{p,q}}(p-q), \tag{2.4}$$

where $s = [n(2m+n-2)+n-2]/2$.

Following [9], we have $(a\ominus b)^{n+m}_{p,q} = (a\ominus b)^n_{p,q}(ap^n\ominus bq^n)^m_{p,q}$ thus (2.4) leads to

$$\begin{aligned} B_{p,q}(m,n) &= p^s\frac{(p\ominus q)^{n-1}_{p,q}}{(p^m\ominus q^m)^n_{p,q}}(p-q) \\ &= p^s\frac{(p\ominus q)^{n-1}_{p,q}}{(p-q)^{n-1}}.\frac{(p\ominus q)^{m-1}_{p,q}}{(p-q)^{m-1}}.\frac{(p-q)^{m-1}(p-q)^{n-1}}{(p\ominus q)^{m-1}_{p,q}(p^m\ominus q^m)^n_{p,q}}(p-q) \\ &= p^s\frac{(p\ominus q)^{n-1}_{p,q}}{(p-q)^{n-1}}.\frac{(p\ominus q)^{m-1}_{p,q}}{(p-q)^{m-1}}.\frac{(p-q)^{m+n-1}}{(p\ominus q)^{m+n-1}_{p,q}} = p^s\frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}. \end{aligned}$$

This completes the proof of the theorem. □

3. (p,q) Bernstein-Durrmeyer-Operators and Moments

The (p,q)-analogue of Bernstein operators for $x \in [0,1]$ and $0 < q < p \leqslant 1$ can be defined as

$$B_{n,p,q}(f;x) = \sum_{k=0}^{n} b_{n,k}^{p,q}(1,x) f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right),$$

where

$$b_{n,k}^{p,q}(1,x) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{[k(k-1)-n(n-1)]/2} x^k (1 \ominus x)_{p,q}^{n-k}.$$

Although the different forms of the (p,q)-Bernstein polynomials have been considered in [7] and [8], but due to technical problems, the forms considered in these two papers do not preserve even the constant functions.

Remark 3.1. The link between q-binomial coefficient and (p,q)-binomial coefficients can be described as:

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_{q/p} &= \frac{[n]_{q/p}!}{[k]_{q/p}![n-k]_{q/p}!} \\
&= \frac{[n]_{q/p}[n-1]_{q/p}\cdots[2]_{q/p}.1}{([k]_{q/p}[k-1]_{q/p}\cdots[2]_{q/p}.1)([n-k]_{q/p}[n-k-1]_{q/p}\cdots[2]_{q/p}.1)} \\
&= \frac{p^{k(k-1)/2}p^{(n-k)(n-k-1)/2}[n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}.1}{p^{n(n-1)/2}([k]_{p,q}[k-1]_{p,q}\cdots[2]_{p,q}.1)([n-k]_{p,q}[n-k-1]_{p,q}\cdots[2]_{p,q}.1)} \\
&= \frac{p^{k(k-1)/2-n(n-1)/2}p^{(n-k)(n-k-1)/2}[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!} = p^{k(k-n)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}.
\end{aligned}$$

Also, we have (p,q)-power basis as

$$\begin{aligned}
(x \ominus a)_{p,q}^n &= (x-a)(px-qa)(p^2x-q^2a)\cdots(p^{n-1}x-q^{n-1}a) \\
&= 1\cdot p \cdot p^2 \cdots p^{n-1}(x-a)\left(x-a\frac{q}{p}\right)\left(x-a\frac{q^2}{p^2}\right)\cdots\left(x-a\frac{q^{n-1}}{p^{n-1}}\right) \\
&= p^{n(n-1)/2}(x-a)_{q/p}^n.
\end{aligned}$$

Using the above identities and using the moments of q-Bernstein polynomial, it can easily be verified by simple computation that

$$B_{n,p,q}(1;x) = 1, \quad B_{n,p,q}(t;x) = x, \quad B_{n,p,q}\left(t^2;x\right) = x^2 + \frac{p^{n-1}x(1-x)}{[n]_{p,q}}.$$

The q-Durrmeyer operators were proposed in [5] and also studied in [4]. We introduce below the (p,q) variant of the well known Durrmeyer operators.

Definition 3.1. The (p,q)-analogue of Bernstein-Durrmeyer operator for $x \in [0,1]$ and $0 < q < p \leqslant 1$ is defined as

$$D_n^{p,q}(f;x) = [n+1]_{p,q} \sum_{k=0}^{n} p^{-[n^2+3n-k^2-k]/2} b_{n,k}^{p,q}(1,x) \int_0^1 b_{n,k}^{p,q}(p,pqt) f(t)\, d_{p,q}t$$

where $b_{n,k}^{p,q}(p,pqt) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (pt)^k \, (p \ominus pqt)_{p,q}^{n-k}$.

It may be remarked here that for $p = 1$ these operators will not reduce to the q-Durrmeyer operators, but for $p = q = 1$ these will reduce to the Durrmeyer operators.

Lemma 3.1. *For $x \in [0,1]$, $0 < q < p \leq 1$, we have*

1° $D_n^{p,q}(1;x) = 1$,

2° $D_n^{p,q}(t;x) = \frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}}$,

3° $D_n^{p,q}(t^2;x) = \frac{p^{2n}[2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{(2q^2+qp)p^n[n]_{p,q}x}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{q^3[n]_{p,q}[x^2[n]_{p,q}+p^{n-1}x(1-x)]}{[n+2]_{p,q}[n+3]_{p,q}}$.

Proof. Using (2.2) and (2.1) and Remark 3.1, we have

$$\begin{aligned}
D_n^{p,q}(1;x) &= [n+1]_{p,q} \sum_{k=0}^{n} p^{-[n^2+3n-k^2-k]/2} b_{n,k}^{p,q}(1,x) \int_0^1 \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (pt)^k (p \ominus pqt)_{p,q}^{n-k} d_{p,q}t \\
&= [n+1]_{p,q} \sum_{k=0}^{n} p^{-[n^2+3n-k^2-k]/2} b_{n,k}^{p,q}(1,x) \begin{bmatrix} n \\ k \end{bmatrix} B_{p,q}(k+1, n-k+1) \\
&= [n+1]_{p,q} \sum_{k=0}^{n} p^{-[n^2+3n-k^2-k]/2} b_{n,k}^{p,q}(1,x) \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{[n^2+3n-k^2-k]/2} \frac{[k]_{p,q}![n-k]_{p,q}!}{[n+1]_{p,q}!} \\
&= B_{n,p,q}(1;x) = 1.
\end{aligned}$$

Next using the identity $[k+1]_{p,q} = p^k + q[k]_{p,q}$ and applying Remark 3.1, we have

$$\begin{aligned}
D_n^{p,q}(t;x) &= [n+1]_{p,q} \sum_{k=0}^{n} p^{-[n^2+3n-k^2-k]/2} b_{n,k}^{p,q}(1,x) p^{-1} \int_0^1 \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (pt)^{k+1} (p \ominus pqt)_{p,q}^{n-k} d_{p,q}t \\
&= [n+1]_{p,q} \sum_{k=0}^{n} p^{-[n^2+3n-k^2-k]/2} b_{n,k}^{p,q}(1,x) \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{-1} B_{p,q}(k+2, n-k+1) \\
&= [n+1]_{p,q} \sum_{k=0}^{n} p^{-[n^2+3n-k^2-k]/2} b_{n,k}^{p,q}(1,x) \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n^2+5n-k^2-3k)/2} \frac{[k+1]_{p,q}![n-k]_{p,q}!}{[n+2]_{p,q}!} \\
&= \sum_{k=0}^{n} p^{n-k} b_{n,k}^{p,q}(1,x) \frac{[k+1]_{p,q}}{[n+2]_{p,q}} \\
&= \frac{1}{[n+2]_{p,q}} \sum_{k=0}^{n} p^{n-k} b_{n,k}^{p,q}(1,x)(p^k + q[k]_{p,q})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{p^n}{[n+2]_{p,q}} \sum_{k=0}^{n} b_{n,k}^{p,q}(1,x) + \frac{q[n]_{p,q}}{[n+2]_{p,q}} \sum_{k=0}^{n} b_{n,k}^{p,q}(1,x) p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} \\
&= \frac{1}{[n+2]_{p,q}} p^n B_{n,p,q}(1;x) + \frac{q[n]_{p,q}}{[n+2]_{p,q}} B_{n,p,q}(t;x) \\
&= \frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}}.
\end{aligned}
$$

Further, using the identity $[k+2]_{p,q} = p^{k+1} + qp^k + q^2[k]_{p,q}$ and by Remark 3.1, we get

$$
\begin{aligned}
D_n^{p,q}(t^2;x) &= [n+1]_{p,q} \sum_{k=0}^{n} p^{-[n^2+3n-k^2-k]/2} b_{n,k}^{p,q}(1,x) p^{-2} \int_0^1 \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (pt)^{k+2} (p \ominus pqt)_{p,q}^{n-k} d_{p,q}t \\
&= [n+1]_{p,q} \sum_{k=0}^{n} p^{-[n^2+3n-k^2-k]/2} b_{n,k}^{p,q}(1,x) \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{-2} B_{p,q}(k+3, n-k+1) \\
&= [n+1]_{p,q} \sum_{k=0}^{n} p^{-[n^2+3n-k^2-k]/2} b_{n,k}^{p,q}(1,x) \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \\
&\qquad \times p^{[n^2+7n-k^2-5k]/2} \frac{[k+2]_{p,q}! [n-k]_{p,q}!}{[n+3]_{p,q}!} \\
&= \sum_{k=0}^{n} p^{2(n-k)} b_{n,k}^{p,q}(1,x) \frac{[k+1]_{p,q}[k+2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \\
&= \frac{1}{[n+2]_{p,q}[n+3]_{p,q}} \sum_{k=0}^{n} p^{2n-2k} b_{n,k}^{p,q}(1,x) (p^k + q[k]_{p,q}).(p^{k+1} + qp^k + q^2[k]_{p,q}) \\
&= \frac{1}{[n+2]_{p,q}[n+3]_{p,q}} \sum_{k=0}^{n} p^{2n-2k} b_{n,k}^{p,q}(1,x) \\
&\qquad \times [p^{2k}(p+q) + p^k[k]_{p,q}(2q^2+qp) + q^3[k]_{p,q}^2] \\
&= \frac{p^{2n}[2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \sum_{k=0}^{n} b_{n,k}^{p,q}(1,x) \\
&\qquad + \frac{(2q^2+qp)p^n[n]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \sum_{k=0}^{n} b_{n,k}^{p,q}(1,x) \left(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} \right) \\
&\qquad + \frac{q^3[n]_{p,q}^2}{[n+2]_{p,q}[n+3]_{p,q}} \sum_{k=0}^{n} b_{n,k}^{p,q}(1,x) \left(p^{2n-2k} \frac{[k]_{p,q}^2}{[n]_{p,q}^2} \right) \\
&= \frac{p^{2n}[2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} B_{n,p,q}(1;x) + \frac{(2q^2+qp)p^n[n]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} B_{n,p,q}(t;x) \\
&\qquad + \frac{q^3[n]_{p,q}^2}{[n+2]_{p,q}[n+3]_{p,q}} B_{n,p,q}\left(t^2;x\right),
\end{aligned}
$$

i.e.,

$$
D_n^{p,q}(t^2;x) = \frac{p^{2n}[2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{(2q^2+qp)p^n[n]_{p,q}x}{[n+2]_{p,q}[n+3]_{p,q}}
$$

$$+\frac{q^3[n]_{p,q}[x^2[n]_{p,q}+p^{n-1}x(1-x)]}{[n+2]_{p,q}[n+3]_{p,q}}. \qquad \square$$

Lemma 3.2. *Let $n > 3$ be a given natural number and let $0 < q < p \le 1$, $q_0 = q_0(n) \in (0, p)$ be the least number such that*

$$p^{2n+1}q - p^{n+1}q^{n+1} + p^{2n-1}q^3 - p^{n-1}q^{n+3} + p^{2n}q^2 - p^nq^{n+2} - 2p^{2n+3} + 2p^nq^{n+3} > 0$$

for every $q \in (q_0, 1)$. Then

$$D_n^{p,q}((t-x)^2, x) \le \frac{2}{[n+2]_{p,q}}\left(\varphi^2(x) + \frac{1}{[n+3]_{p,q}}\right),$$

where $\varphi^2(x) = x(1-x)$, $x \in [0,1]$.

Proof. In view of Lemma 3.1, we obtain

$$D_n^{p,q}((t-x)^2, x) = x^2\frac{q^3[n]_{p,q}([n]_{p,q}-p^{n-1}) - 2q[n]_{p,q}[n+3]_{p,q} + [n+2]_{p,q}[n+3]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}}$$
$$+ x\frac{p^{n-1}q(p+q)^2[n]_{p,q} - 2p^n[n+3]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{p^{2n}(p+q)}{[n+2]_{p,q}[n+3]_{p,q}}$$

By direct computations, using the definition of the (p,q)-numbers, we get

$$\begin{aligned} p^{n-1}q(p+q)^2[n]_{p,q} - 2p^n[n+3]_{p,q} &= p^{n-1}q(p+q)^2\frac{p^n-q^n}{p-q} - 2p^n\frac{p^{n+3}-q^{n+3}}{p-q} \\ &= \frac{1}{p-q}[p^{2n+1}q - p^{n+1}q^{n+1} + p^{2n-1}q^3 - p^{n-1}q^{n+3} \\ &\quad + p^{2n}q^2 - p^nq^{n+2} - 2p^{2n+3} + 2p^nq^{n+3}] > 0, \end{aligned}$$

for every $q \in (q_0, 1)$. Furthermore, $p^{n-1}q(p+q)^2[n]_{p,q} - 2p^n[n+3]_{p,q} \le 2[n+3]_{p,q}$ and following [5], we have

$$\begin{aligned} &p^{n-1}q(p+q)^2[n]_{p,q} - 2p^n[n+3]_{p,q} + q^3[n]_{p,q}([n]_{p,q} - p^{n-1}) \\ &\qquad -2q[n]_{p,q}[n+3]_{p,q} + [n+2]_{p,q}[n+3]_{p,q} \\ &= p^{n-1}q(p+q)^2[n]_{p,q} - 2p^n(p^{n+2} + qp^{n+1} + q^2p^n + q^3[n]_{p,q}) \\ &\quad +q^3[n]_{p,q}^2 - q^3p^{n-1}[n]_{p,q} - 2q[n]_{p,q}(p^{n+2} + qp^{n+1} + q^2p^n + q^3[n]_{p,q}) \\ &\quad +(p^{n+1} + qp^n + q^2[n]_{p,q})(p^{n+2} + qp^{n+1} + q^2p^n + q^3[n]_{p,q}) \leqslant 0. \end{aligned}$$

In conclusion, for $x \in [0,1]$, we have

$$D_n^{p,q}((t-x)^2, x) = \frac{p^{2n}(p+q)}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{p^{n-1}q(p+q)^2[n]_{p,q} - 2p^n[n+3]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}}x(1-x)$$
$$+\Bigg(\frac{p^{n-1}q(p+q)^2[n]_{p,q} - 2p^n[n+3]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}}$$

$$+\frac{q^3[n]_{p,q}([n]_{p,q}-p^{n-1})-2q[n]_{p,q}[n+3]_{p,q}+[n+2]_{p,q}[n+3]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}}\Bigg)x^2$$
$$\leq \frac{2[n+3]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}}\varphi^2(x)+\frac{2}{[n+2]_{p,q}[n+3]_{p,q}}$$
$$\leq \frac{2}{[n+2]_{p,q}}\left(\varphi^2(x)+\frac{1}{[n+3]_{p,q}}\right),$$

which was to be proved. □

4. Direct Estimates

We denote $W^2=\left\{g\in C[0,1]:g'',g''\in C[0,1]\right\}$ for $\delta>0$, $K-$functional is defined as

$$K_2(f,\delta)=\inf\left\{\|f-g\|+\eta\|g''\|:g\in W^2\right\},$$

where norm-$\|.\|$ denotes the uniform norm on $C[0,1]$. Following the well-known inequality due to DeVore and Lorentz [2], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\leq C\omega_2(f,\sqrt{\delta}), \tag{4.1}$$

where the second order modulus of smoothness for $f\in C[0,1]$ is defined as

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\leq\sqrt{\delta}}\ \sup_{x,x+h\in[0,1]}|f(x+h)-f(x)|.$$

The usual modulus of continuity for $f\in C[0,1]$ is defined as

$$\omega(f,\delta)=\sup_{0<h\leq\delta}\ \sup_{x,x+h\in[0,1]}|f(x+h)-f(x)|.$$

Our first main result is the following local theorem:

Theorem 4.1. *Let $n>3$ be a natural number and let $0<q<p\leq 1$, $q_0=q_0(n)\in(0,p)$ be defined as in Lemma 3.2. Then there exists an absolute constant $C>0$ such that*

$$|D_n^{p,q}(f,x)-f(x)|\ \leq\ C\omega_2\left(f,[n+2]_{p,q}^{-1/2}\delta_n(x)\right)+\omega\left(f,\frac{1-x}{[n+2]_{p,q}}\right),$$

where $f\in C[0,1]$, $\delta_n^2(x)=\varphi^2(x)+\dfrac{1}{[n+3]_{p,q}}$, $x\in[0,1]$ and $q\in(q_0,1)$.

Proof. For $f\in C[0,1]$ we define

$$\widetilde{D}_n^{p,q}(f,x)\ =\ D_n^{p,q}(f,x)+f(x)-f\left(\frac{p^n+q[n]_{p,q}x}{[n+2]_{p,q}}\right).$$

Then, by Lemma 3.1, we immediately get

$$\widetilde{D}_n^{p,q}(1,x) = D_n^{p,q}(1,x) = 1 \tag{4.2}$$

and

$$\widetilde{D}_n^{p,q}(t,x) = D_n^{p,q}(t,x) + x - \frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}} = x. \tag{4.3}$$

By Taylor's formula

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)\,du,$$

we get

$$\begin{aligned}\widetilde{D}_n^{p,q}(g,x) &= g(x) + \widetilde{D}_n^{p,q}\left(\int_x^t (t-u)\,g''(u)\,du, x\right)\\ &= g(x) + D_n^{p,q}\left(\int_x^t (t-u)g''(u)\,du, x\right)\\ &\quad - \int_x^{\frac{p^n+q[n]_{p,q}x}{[n+2]_{p,q}}} \left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}} - u\right) g''(u)\,du.\end{aligned}$$

Thus

$$\begin{aligned}|\widetilde{D}_n^{p,q}(g,x) - g(x)| &\le D_n^{p,q}\left(\left|\int_x^t |t-u||g''(u)|\,du\right|, x\right)\\ &\quad + \left|\int_x^{\frac{p^n+q[n]_{p,q}x}{[n+2]_{p,q}}} \left|\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}} - u\right| |g''(u)|\,du\right|\\ &\le D_n^{p,q}((t-x)^2,x)\|g''\| + \left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}} - x\right)^2 \|g''\| \qquad (4.4)\end{aligned}$$

Also, we have

$$\begin{aligned}&D_n^{p,q}((t-x)^2,x) + \left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}} - x\right)^2 \qquad (4.5)\\ &\le \frac{2}{[n+2]_{p,q}}\left(\varphi^2(x) + \frac{1}{[n+3]_{p,q}}\right) + \left(\frac{p^n - ([n+2]_{p,q} - q[n]_{p,q})x}{[n+2]_{p,q}}\right)^2.\end{aligned}$$

Obviously

$$1 \le [n+2]_{p,q} - q[n]_{p,q} \le 2. \tag{4.6}$$

Then, using (4.6), we get

$$\left(\frac{p^n - ([n+2]_{p,q} - q[n]_{p,q})x}{[n+2]_{p,q}}\right)^2 \delta_n^{-2}(x)$$
$$= \frac{p^{2n} - 2p^n([n+2]_{p,q} - q[n]_{p,q})x + ([n+2]_{p,q} - q[n]_{p,q})^2x^2}{[n+2]_{p,q}^2}$$
$$\times \frac{[n]_{p,q}}{[n]_{p,q}x(1-x)+1}$$
$$\leq \frac{p^{2n} - 2p^n x + 4x^2}{[n+2]_{p,q}} \cdot \frac{[n]_{p,q}}{[n+2]_{p,q}} \cdot \frac{1}{[n]_{p,q}x(1-x)+1},$$

i.e.,

$$\left(\frac{p^n - ([n+2]_{p,q} - q[n]_{p,q})x}{[n+2]_{p,q}}\right)^2 \delta_n^{-2}(x) \leq \frac{3}{[n+2]_{p,q}}, \tag{4.7}$$

for $n \in \mathbb{N}$ and $0 < q < p \leqslant 1$. In conclusion, by (4.5) and (4.7), for $x \in [0,1]$, we obtain

$$D_n^{p,q}((t-x)^2, x) + \left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}} - x\right)^2 \leq \frac{5}{[n+2]_{p,q}}\delta_n^2(x). \tag{4.8}$$

Hence, by (4.4) and with the conditions $n > 3$ and $x \in [0,1]$, we have

$$|\widetilde{D}_n^{p,q}(g,x) - g(x)| \leq \frac{5}{[n+2]_{p,q}}\, \delta_n^2(x)\, \|g''\|. \tag{4.9}$$

Furthermore, for $f \in C[0,1]$ we have $\|D_n^{p,q}(f,x)\| \leqslant \|f\|$, thus

$$|\widetilde{D}_n^{p,q}(f,x)| \leq |D_n^{p,q}(f,x)| + |f(x)| + \left| f\left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}}\right)\right| \leq 3\|f\|. \tag{4.10}$$

for all $f \in C[0,1]$.

Now, for $f \in C[0,1]$ and $g \in W^2$, we obtain

$$|D_n^{p,q}(f,x) - f(x)|$$
$$= \left| \widetilde{D}_n^{p,q}(f,x) - f(x) + f\left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}}\right) - f(x) \right|$$
$$\leq |\widetilde{D}_n^{p,q}(f-g,x)| + |\widetilde{D}_n^{p,q}(g,x) - g(x)| + |g(x) - f(x)|$$
$$+ \left| f\left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}}\right) - f(x) \right|$$

$$\leq 4\,\|f-g\| + \frac{5}{[n+2]_{p,q}}\cdot\delta_n^2(x)\cdot\|g''\| + \omega\left(f, \left|\frac{p^n-([n+2]_{p,q}-q[n]_{p,q})x}{[n+2]_{p,q}}\right|\right)$$

$$\leq 5\left(\|f-g\| + \frac{1}{[n+2]_{p,q}}\cdot\delta_n^2(x)\cdot\|g''\|\right) + \omega\left(f, \frac{1-x}{[n+2]_{p,q}}\right),$$

where we have used (4.9) and (4.10). Taking the infimum on the right hand side over all $g \in W^2$, we obtain at once

$$|D_n^{p,q}(f,x)-f(x)| \leq 5\,K_2\left(f, \frac{1}{[n+2]_{p,q}}\delta_n^2(x)\right) + \omega\left(f, \frac{1-x}{[n+2]_{p,q}}\right).$$

Finally, in view of (4.1), we find

$$|D_n^{p,q}(f,x)-f(x)| \;\leq\; C\,\omega_2\left(f, [n+2]_{p,q}^{-1/2}\delta_n(x)\right) + \omega\left(f, \frac{1-x}{[n+2]_{p,q}}\right).$$

This completes the proof of the theorem. □

The weighted modulus of continuity of second order for $f \in C[0,1]$ and $\varphi(x) = \sqrt{x(1-x)}$ is defined as:

$$\omega_2^{\varphi}(f, \sqrt{\delta}) \;=\; \sup_{0<h\leq\sqrt{\delta}}\;\sup_{x,x\pm h\varphi\in[0,1]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|$$

the corresponding K functional is defined by

$$\overline{K}_{2,\varphi}(f,\delta) \;=\; \inf\{\|f-g\| + \delta\|\varphi^2 g''\| + \delta^2\|g''\| : g \in W^2(\varphi)\},$$

where

$$W^2(\varphi) = \left\{g \in C[0,1] : g' \in AC_{loc}[0,1], \varphi^2 g'' \in C[0,1]\right\}$$

and $g' \in AC_{loc}[0,1]$ means that g is differentiable and g' is absolutely continuous on every closed interval $[a,b] \subset [0,1]$. By the property due to Ditzian-Totik (see [3, p. 24, Theorem 1.3.1]), we have

$$\overline{K}_{2,\varphi}(f,\delta) \;\leq\; C\,\omega_2^{\varphi}(f, \sqrt{\delta}) \tag{4.11}$$

for some absolute constant $C > 0$. Moreover, with ψ the admissible step-weight function on $[0,1]$, the Ditzian-Totik moduli of first order is given by

$$\vec{\omega}_{\psi}(f,\delta) \;=\; \sup_{0<h\leq\delta}\;\sup_{x,x\pm h\psi(x)\in[0,1]} |f(x+h\psi(x)) - f(x)|.$$

Now we state and prove the following global direct result:

Theorem 4.2. *Let $n > 3$ be a natural number and let $0 < q < p \leq 1$, $q_0 = q_0(n) \in (0, p$ be defined as in Lemma 3.2. Then there exists an absolute constant $C > 0$ such that$0 < q < p \leq 1$, $q_0 = q_0(n) \in (0, p)$*

$$\|D_n^{p,q} f - f\| \;\leq\; C\, \omega_2^{\varphi}(f, [n+2]_q^{-1/2}) \;+\; \vec{\omega}_{\psi}(f, [n+2]_q^{-1}),$$

where $f \in C[0,1]$, $q \in (q_0, 1)$ and $\psi(x) = 1 - x$, $x \in [0,1]$.

Proof. Let us consider

$$\widetilde{D}_n^{p,q}(f,x) \;=\; D_n^{p,q}(f,x) + f(x) - f\left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}}\right),$$

where $f \in C[0,1]$. By Taylor's formula with $g \in W^2(\varphi)$, we have

$$g(t) = g(x) + (t-x)\, g'(x) + \int_x^t (t-u)\, g''(u)du,$$

with the applications of (4.2) and (4.3), we obtain

$$\begin{aligned}\widetilde{D}_n^{p,q}(g,x) \;&=\; g(x) + D_n^{p,q}\left(\int_x^t t-u)g''(u)\, du, x\right) \\ &\quad - \int_x^{\frac{p^n+q[n]_{p,q}x}{[n+2]_{p,q}}} \left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}} - u\right) g''(u)\, du.\end{aligned}$$

Thus we can write

$$\begin{aligned}|\widetilde{D}_n^{p,q}(g,x) - g(x)| \;&\leq\; D_n^{p,q}\left(\left|\int_x^t |t-u| \cdot |g''(u)|\, du\right|, x\right) \\ &\quad + \left|\int_x^{\frac{p^n+q[n]_{p,q}x}{[n+2]_{p,q}}} \left|\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}} - u\right| \cdot |g''(u)|\, du\right|. \end{aligned} \tag{4.12}$$

Also, the function δ_n^2 is concave on $[0,1]$, we have for $u = t + \tau(x-t)$, $\tau \in [0,1]$, the following estimate

$$\frac{|t-u|}{\delta_n^2(u)} = \frac{\tau|x-t|}{\delta_n^2(t + \tau(x-t))} \leq \frac{\tau|x-t|}{\delta_n^2(t) + \tau(\delta_n^2(x) - \delta_n^2(t))} \leq \frac{|t-x|}{\delta_n^2(x)}.$$

Hence, by (4.12), we obtain

$$\begin{aligned}&|\widetilde{D}_n^{p,q}(g,x) - g(x)| \\ &\leq D_n^{p,q}\left(\left|\int_x^t \frac{|t-u|}{\delta_n^2(u)} du\right|, x\right)\|\delta_n^2 g''\| + \left|\int_x^{\frac{p^n+q[n]_{p,q}x}{[n+2]_{p,q}}} \frac{\left|\frac{p^n+q[n]_{p,q}x}{[n+2]_{p,q}} - u\right|}{\delta_n^2(u)}\, du\right| \|\delta_n^2 g''\| \\ &\leq \frac{1}{\delta_n^2(x)} D_n^{p,q}((t-x)^2, x)\|\delta_n^2 g''\| + \frac{1}{\delta_n^2(x)}\left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}} - x\right)^2 \|\delta_n^2 g''\|.\end{aligned}$$

For $x \in [0,1]$, in view of (4.8) and

$$\delta_n^2(x)\cdot|g''(x)| = |\varphi^2(x)g''(x)| + \frac{1}{[n+2]_{p,q}}\cdot|g''(x)| \leqslant \|\varphi^2 g''\| + \frac{1}{[n+2]_{p,q}}\|g''\|,$$

we get

$$|\widetilde{D}_n^{p,q}(g,x) - g(x)| \leq \frac{5}{[n+2]_{p,q}}\cdot\left(\|\varphi^2 g''\| + \frac{1}{[n+2]_{p,q}}\cdot\|g''\|\right) \tag{4.13}$$

Obviously using $[n]_{p,q} \leq [n+2]_{p,q}$, (4.10) and (4.13), we find for $f \in C[0,1]$, that

$$\begin{aligned}|D_n^{p,q}(f,x) - f(x)| &\leqslant |\widetilde{D}_n^{p,q}(f-g,x)| \\ &\quad + |\widetilde{D}_n^{p,q}(g,x) - g(x)| + |g(x) - f(x)| + \left| f\left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}}\right) - f(x)\right| \\ &\leq 4\|f-g\| + \frac{5}{[n+2]_{p,q}}\|\varphi^2 g''\| + \frac{5}{[n+2]_{p,q}}\|g''\| + \left| f\left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}}\right) - f(x)\right|.\end{aligned}$$

Finally, taking the infimum on the right hand side over all $g \in W^2(\varphi)$, we obtain

$$\begin{aligned}|D_n^{p,q}(f,x) - f(x)| \quad &\leq \quad 5\overline{K}_{2,\varphi}\left(f, \frac{1}{[n+2]_{p,q}}\right) \\ &\quad + \left| f\left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}}\right) - f(x)\right|\end{aligned} \tag{4.14}$$

Also, we have

$$\begin{aligned}\left| f\left(\frac{p^n + q[n]_{p,q}x}{[n+2]_{p,q}}\right) - f(x)\right| &= \left| f\left(x + \psi(x)\frac{p^n - ([n+2]_{p,q} - q[n]_{p,q})x}{[n+2]_{p,q}\psi(x)}\right) - f(x)\right| \\ &\leq \sup_{t,\, t+\psi(t)\frac{p^n - ([n+2]_{p,q} - q[n]_{p,q})x}{[n+2]_{p,q}} \in [0,1]} \left| f\left(t + \psi(t)\frac{p^n - ([n+2]_{p,q} - q[n]_{p,q})x}{[n+2]_{p,q}\psi(x)}\right) - f(t)\right| \\ &\leq \vec{\omega}_\psi\left(f, \frac{|p^n - ([n+2]_{p,q} - q[n]_{p,q})x|}{[n+2]_{p,q}\psi(x)}\right) \\ &\leq \vec{\omega}_\psi\left(f, \frac{1-x}{[n+2]_{p,q}\psi(x)}\right) = \vec{\omega}_\psi\left(f, \frac{1}{[n+2]_{p,q}}\right).\end{aligned}$$

Hence, by (4.14) and (4.11), we get

$$\|D_n^{p,q}f - f\| \;\leq\; C\,\omega_2^\varphi(f, [n+2]_{p,q}^{-1/2}) \;+\; \omega_\psi(f, [n+2]_{p,q}^{-1}).$$

This completes the proof of the theorem. □

Remark 4.1. For $q \in (0,1)$ and $p \in (q,1]$ it is obvious that $\lim_{n\to\infty} [n]_{p,q} = 1/(p-q)$. Thus above theorems do not give a approximation result. If we choose $q_n = e^{-1/n}$ and $p_n = e^{-1/(n+1)}$ such that $0 < q_n < p_n \leq 1$, $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} p_n^n = \lim_{n\to\infty} q_n^n = 1/e$. Also we have

$$\lim_{n\to\infty} [n]_{p_n,q_n} = \infty.$$

Since $[n+2]_{p,q} = [2]_{p,q}\, p^n + q^2\, [n]_{p,q}$ we can write

$$\lim_{n\to\infty} \frac{1}{[n]_{p_n,q_n}} = \lim_{n\to\infty} \frac{1}{[n+2]_{p_n,q_n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{[n]_{p_n,q_n}}{[n+2]_{p_n,q_n}} = 1.$$

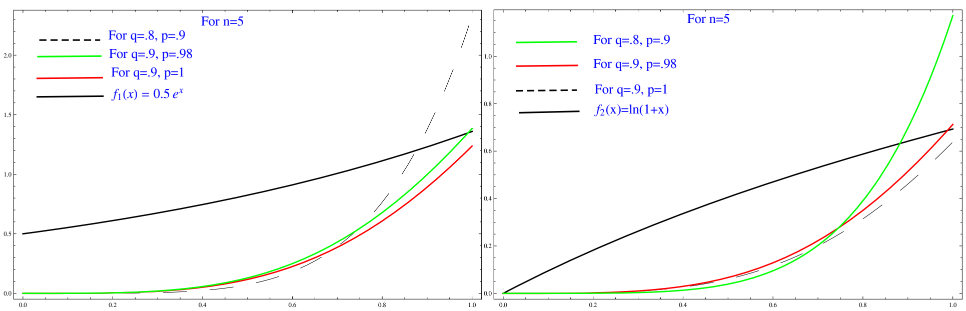

FIG. 4.1: Functions $f_1(x) = 0.5e^x$ and $f_2(x) = \ln(1+x)$

Example 4.1. With the help of MATHEMATICA, we show comparisons and some illustrative graphics for the convergence of (p,q)-Bernstein-Durrmeyer operators $D_n^{p,q}(f;x)$ for different values of the parameters p and q. We have considered different functions as shown in Figures 4.1.

REFERENCES

1. A. ARAL, V. GUPTA and R. P. AGARWAL: *Applications of q Calculus in Operator Theory*, Springer 2013.
2. R. A. DEVORE and G. G. LORENTZ: *Constructive Approximation*, Springer, Berlin 1993.
3. Z. DITZIAN and V. TOTIK: *Moduli of Smoothness*, Springer , New York 1987.
4. Z. FINTA and V. GUPTA: *Approximation by q-Durrmeyer operators*, J. Appl. Math. Computing 29 (1-2)(2009), 401-415.
5. V. GUPTA: *Some approximation properties on q-Durrmeyer operators*, Appl. Math. Computation 197 (1) (2008), 172-178.
6. V. GUPTA and R. P. AGARWAL: *Convergence Estimates in Approximation Theory*, Springer, 2014.
7. K. KHAN and D. K. LOBIYAL: *Bézier curves based on Lupas (p,q)-analogue of Bernstein polynomials in CAGD*, arXiv:1505.01810 [cs.GR].

8. M. Mursaleen, K. J. Ansari and A. Khan: *On* (p,q)*-analogue of Bernstein operators*, Appl. Math. Comput. 266 (2015) 874-882.

9. P. N. Sadjang: *On the* (p,q)*-Gamma and the* (p,q)*-Beta functions*, arXiv 1506.07394v1. 22 Jun 2015.

10. P. N. Sadjang: *On the fundamental theorem of* (p,q) *-calculus and some* (p,q)*-Taylor formulas*, arXiv:1309.3934 [math.QA].

11. V. Sahai and S. Yadav: *Representations of two parameter quantum algebras and* p,q*-special functions*, J. Math. Anal. Appl. 335(2007), 268-279.

Vijay Gupta
Department of Mathematics
Netaji Subhas Institute of Technology
Sector 3 Dwarka
New Delhi 110078, India
vijaygupta2001@hotmail.com

Ali Aral
Department of Mathematics
Kirikkale University
TR-71450 Yahsihan Kirikkale/ Turkey
aliaral73@yahoo.com