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A FIXED POINT THEOREM FOR GENERALIZED CYCLIC CONTRACTIVE MAPPINGS IN B-METRIC SPACES

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Abstract. The purpose of this paper is to introduce the notion of generalized cyclic contractive mapping in b-metric spaces by adding four terms $\dfrac{d(T^2x, x) + d(T^2x, Ty)}{2s}$, $d(T^2x, Tx)$, $d(T^2x, y)$, $d(T^2x, Ty)$ to the contractive condition and state a fixed point theorem for this kind of mappings. Also, some corollaries are derived from this theorem. In addition, some examples are given to illustrate the obtained results.

Keywords: fixed point, b-metric space, generalized cyclic contractive mappings

1. Introduction and preliminaries

The Banach contraction principle is one of the most powerful and useful tools of modern mathematics. This principle was extended and improved in many ways and various fixed point theorems were obtained. One of the interesting generalizations of this basic principle was given by Kirk *et al.* [17] in 2003 by introducing the following notion of cyclic representation.

Definition 1.1. [17] Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \longrightarrow A \cup B$ be a mapping. Then T is called *a cyclic mapping* if $TA \subset B$ and $TB \subset A$.

The following interesting theorem for a cyclic mapping was given in [17].

Theorem 1.1. [17] *Let (X, d) be a complete metric space, A and B be non-empty closed subsets of X, $T : A \cup B \longrightarrow A \cup B$ be a cyclic mapping such that*

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x \in A$, $y \in B$ and for some $\alpha \in (0, 1)$. Then T has a unique fixed point in $A \cap B$.

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Notice that although a contraction is continuous, a cyclic contraction need not to be. This is one of the important gains of this approach. Following the work of Kirk *et al.*, several authors stated many fixed point results for cyclic mappings, satisfied various contractive conditions. The readers may refer to [1, 14, 16, 20] and references therein.

In recent times, there were some new approaches to generalizing the Banach contraction principle on complete metric spaces. In 2004, Ran and Reurings [23] stated a generalization of Banach contraction principle by using a partial order on a metric space. Some applications of this result to matrix equations were also established. In 2008, Suzuki [27] proved a generalization of Banach contraction principle by using a contraction condition depending on a non-increasing function $\theta : [0,1) \longrightarrow [\frac{1}{2}, 1]$. In 2015, Kumam *et al.* [19] introduced a new generalized quasi-contraction by adding four new values $d(T^2x, x), d(T^2x, Tx), d(T^2x, Ty), d(T^2x, y)$ to a quasi-contraction condition. Also, the authors stated a unique fixed point theorem which is the generalization of Ćirić fixed point theorem in [8].

There were many generalizations of a metric space and many fixed point theorems on generalized metric spaces were stated [4, 11, 15, 24]. The notion of b-metric space was introduced by Bakhtin in [6] and then extensively used by Czerwik in [9, 10] as follows.

Definition 1.2. [10] Let X be a non-empty set and $d : X \times X \longrightarrow [0, \infty)$ be a function such that for all $x, y, z \in X$ and some $s \geq 1$,

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \leq s\big(d(x, z) + d(z, y)\big)$.

Then d is called a *b-metric* on X and (X, d, s) is called a *b-metric space*.

Remark 1.1. *(X, d) is a metric space if and only if $(X, d, 1)$ is a b-metric space.*

The first important difference between a metric and a b-metric is that the b-metric need not be a continuous function in its two variables, see [18, Example 13]. In recent years, many fixed point theorems on b-metric spaces were stated, and the readers may refer to [2, 3, 5, 7, 12, 13, 14, 21, 22, 25, 26] and references therein.

The purpose of this paper is to introduce the notion of generalized cyclic contractive mapping in b-metric spaces by adding four terms $\dfrac{d(T^2x, x) + d(T^2x, Ty)}{2s}$, $d(T^2x, Tx)$, $d(T^2x, y)$, $d(T^2x, Ty)$ to the contractive condition and state a fixed point theorem for this kind of mappings. Also, some corollaries are derived from this theorem. In addition, some examples are given to illustrate the obtained results.

First, we recall some notions and lemmas which will be useful in what follows.

Definition 1.3. [10] Let (X, d, s) be a b-metric space.

1. A sequence $\{x_n\}$ is called *convergent* to x, written $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n, x) = 0$.
2. A sequence $\{x_n\}$ is called *Cauchy* in X if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.
3. (X, d, s) is called *complete* if each Cauchy sequence is a convergent sequence.

Aghajani *et al.* [2] proved the following simple lemma about the convergence in b-metric spaces.

Lemma 1.1. [2] *Let (X, d, s) be a b-metric space and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$. Then*

1. $\frac{1}{s^2} d(x, y) \le \liminf_{n\to\infty} d(x_n, y_n) \le \limsup_{n\to\infty} d(x_n, y_n) \le s^2 d(x, y)$. *In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*
2. *For each $z \in X$,* $\frac{1}{s} d(x, z) \le \liminf_{n\to\infty} d(x_n, z) \le \limsup_{n\to\infty} d(x_n, z) \le s d(x, z)$.

The following lemma is the equivalent condition for the Cauchy property of $\{x_n\}$ in b-metric spaces.

Lemma 1.2. *Let (X, d, s) be a b-metric space and $\{x_n\}$ be a sequence in (X, d, s). Then the following statements are equivalent.*

1. *$\{x_n\}$ is a Cauchy sequence in (X, d, s).*
2. *$\{x_{2n}\}$ is a Cauchy sequence in (X, d, s) and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.*

Proof. (1) $\Rightarrow$ (2). From the given assumption, we get $\{x_{2n}\}$ is a Cauchy sequence in (X, d, s) and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

(2) $\Rightarrow$ (1). For all $n, m \ge 0$, we only consider the following cases.

Case 1. $n = 2k + 1, m = 2l$ for all $k, l \ge 0$. Then

$$d(x_n, x_m) = d(x_{2k+1}, x_{2l}) \le sd(x_{2k+1}, x_{2k}) + sd(x_{2k}, x_{2l}).$$

Case 2. $n = 2k, m = 2l + 1$ for all $k, l \ge 0$. Then

$$d(x_n, x_m) = d(x_{2k}, x_{2l+1}) \le sd(x_{2k}, x_{2l}) + sd(x_{2l}, x_{2l+1}).$$

Case 3. $n = 2k + 1, m = 2l + 1$ for all $k, l \ge 0$. Then

$$d(x_n, x_m) = d(x_{2k+1}, x_{2l+1}) \le sd(x_{2k+1}, x_{2k}) + s^2 d(x_{2k}, x_{2l}) + s^2 d(x_{2l}, x_{2l+1}).$$

By the above cases, it is implied that $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence in (X, d, s). $\square$

2. Main results

First, we introduce the notion of a generalized cyclic contractive mapping in b-metric spaces. Denote by

1. Φ the family of all increasing, continuous functions $\varphi : [0, \infty) \longrightarrow [0, \infty)$ such that $\varphi(0) = 0$.
2. Ψ the family of all non-decreasing, right continuous functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ such that $\psi(t) < t$ for all $t > 0$.

Definition 2.1. Let (X, d, s) be a b-metric space, A and B be non-empty subsets of X, $Y = A \cup B$ and $T : Y \longrightarrow Y$ be a mapping. Then T is called *a generalized cyclic contractive mapping* if

1. $Y = A \cup B$ is a cyclic representation of Y with respect to T, that is, $TA \subset B$ and $TB \subset A$.
2. There exist $\varphi \in \Phi$, $\psi \in \Psi$ and a constant $L \geq 0$ such that
$$\varphi(s^4 d(Tx, Ty)) \leq \psi(\varphi(\overline{M}_s(x, y))) + L\varphi(N(x, y))$$
for all $(x, y) \in A \times B$ or $(x, y) \in B \times A$, where
$$\begin{aligned} \overline{M}_s(x, y) &= \max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}, \\ &\qquad \frac{d(T^2x, x) + d(T^2x, Ty)}{2s}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)\Big\}, \\ N(x, y) &= \min\big\{d(x, Tx), d(y, Tx), d(T^2x, T^2y)\big\}. \end{aligned}$$

The following theorem is a sufficient condition for the existence and uniqueness of the fixed point for a generalized cyclic contractive mapping in b-metric spaces.

Theorem 2.1. *Let (X, d, s) be a complete b-metric space, A and B be non-empty closed subsets of X, $Y = A \cup B$ and $T : Y \longrightarrow Y$ be a generalized cyclic contractive mapping. Then T has a unique fixed point in $A \cap B$.*

Proof. Let $x_0 \in A$. We construct the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. Since $x_0 \in A$, $x_1 = Tx_0 \in TA \subset B$. So, $x_2 = Tx_1 \in TB \subset A$. Continuing this process, we have

$$x_{2n} \in A, \quad x_{2n+1} \in B \tag{2.1}$$

for all $n \geq 0$. If there exists $k \geq 0$ such that $x_{k+1} = x_k$, then $Tx_k = x_k$, that is, x_k is a fixed point of T. Suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. From (2.1), we have $(x_{2n-1}, x_{2n}) \in B \times A$. Since T is a generalized cyclic contractive mapping, we have

$$\begin{aligned} \varphi(d(x_{2n}, x_{2n+1})) &\leq \varphi(s^4 d(Tx_{2n-1}, Tx_{2n})) \\ &\leq \psi(\varphi(\overline{M}_s(x_{2n-1}, x_{2n}))) + L\varphi(N(x_{2n-1}, x_{2n})) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned}\overline{M}_s(x_{2n-1}, x_{2n}) &= \max\Big\{d(x_{2n-1}, x_{2n}), d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}),\\ &\quad \frac{d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n})}{2s}, \frac{d(x_{2n+1}, x_{2n-1}) + d(x_{2n+1}, x_{2n+1})}{2s},\\ &\quad d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+1})\Big\},\end{aligned}$$

$$N(x_{2n-1}, x_{2n}) = \min\big\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n}), d(x_{2n+1}, x_{2n+2})\big\} = 0.$$

We also have

$$\begin{aligned}&\max\big\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\big\}\\ \leq\ & \overline{M}_s(x_{2n-1}, x_{2n})\\ =\ & \max\Big\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), \frac{d(x_{2n-1}, x_{2n+1})}{2s}\Big\}\\ \leq\ & \max\Big\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), \frac{d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})}{2}\Big\}\\ =\ & \max\big\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\big\}.\end{aligned}$$

It implies that $\overline{M}_s(x_{2n-1}, x_{2n}) = \max\big\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\big\}$. If there exists $n \geq 1$ such that $\max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\} = d(x_{2n}, x_{2n+1})$, then $\overline{M}_s(x_{2n-1}, x_{2n}) = d(x_{2n}, x_{2n+1})$ and hence (2.2) becomes

$$\varphi(d(x_{2n}, x_{2n+1})) \leq \psi(\varphi(d(x_{2n}, x_{2n+1}))) < \varphi(d(x_{2n}, x_{2n+1})).$$

It is a contradiction. Therefore, for all $n \geq 1$, we have $\overline{M}_s(x_{2n-1}, x_{2n}) = d(x_{2n-1}, x_{2n})$. Then, (2.2) becomes

$$\varphi(d(x_{2n}, x_{2n+1})) \leq \psi(\varphi(d(x_{2n-1}, x_{2n}))) < \varphi(d(x_{2n-1}, x_{2n})) \tag{2.3}$$

for all $n \geq 1$. On the other hand, from (2.1) we obtain $(x_{2n}, x_{2n-1}) \in A \times B$. Similarly, we also see that

$$\varphi(d(x_{2n+1}, x_{2n+2})) < \varphi(d(x_{2n}, x_{2n+1})). \tag{2.4}$$

for all $n \geq 0$. Therefore, from (2.3) and (2.4), we have $\varphi(d(x_{n+1}, x_n)) < \varphi(d(x_n, x_{n-1}))$ for all $n \geq 1$. It follows from the increasing property of φ that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of positive numbers. Then, there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(x_{n+1}, x_n) = r$. Taking the limit as $n \to \infty$ in (2.3), we get $\varphi(r) \leq \psi(\varphi(r))$. Therefore, by using the property of ψ, we have $\varphi(r) = 0$ and hence $r = 0$. Thus,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 1.2 and (2.5), it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose the contrary, that

$\{x_{2n}\}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}, \{x_{n(k)}\}$ of $\{x_{2n}\}$ where $m(k)$ is a smallest integer such that $m(k) > n(k) \geq k$ and

$$d(x_{2m(k)}, x_{2n(k)}) \geq \varepsilon. \tag{2.6}$$

It implies that

$$d(x_{2m(k)-2}, x_{2n(k)}) < \varepsilon. \tag{2.7}$$

Since $(x_{2n(k)-1}, x_{2m(k)}) \in B \times A$ and T is a generalized cyclic contractive mapping,

$$\begin{aligned} &\varphi(s^4 d(x_{2n(k)}, x_{2m(k)+1})) \\ = \;& \varphi(s^4 d(Tx_{2n(k)-1}, Tx_{2m(k)})) \\ \leq \;& \psi(\varphi(\overline{M}_s(x_{2n(k)-1}, x_{2m(k)}))) + L\varphi(N(x_{2n(k)-1}, x_{2m(k)})), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} &\overline{M}_s(x_{2n(k)-1}, x_{2m(k)}) \\ = \;& \max\Big\{d(x_{2n(k)-1}, x_{2m(k)}), d(x_{2n(k)-1}, x_{2n(k)}), d(x_{2m(k)}, x_{2m(k)+1}), \\ & \frac{d(x_{2n(k)-1}, x_{2m(k)+1}) + d(x_{2m(k)}, x_{2n(k)})}{2s}, \\ & \frac{d(x_{2n(k)+1}, x_{2n(k)-1}) + d(x_{2n(k)+1}, x_{2m(k)+1})}{2s}, \\ & d(x_{2n(k)+1}, x_{2n(k)}), d(x_{2n(k)+1}, x_{2m(k)}), d(x_{2n(k)+1}, x_{2m(k)+1})\Big\}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &N(x_{2n(k)-1}, x_{2m(k)}) \\ = \;& \min\big\{d(x_{2n(k)-1}, x_{2n(k)}), d(x_{2m(k)}, x_{2n(k)}), d(x_{2n(k)+1}, x_{2m(k)+2})\big\}. \end{aligned} \tag{2.10}$$

On the other hand, from (2.6), we get

$$\begin{aligned} \varepsilon \leq \;& d(x_{2m(k)}, x_{2n(k)}) \\ \leq \;& sd(x_{2m(k)}, x_{2m(k)+1}) + sd(x_{2m(k)+1}, x_{2n(k)}). \end{aligned} \tag{2.11}$$

Taking the upper limit as $k \to \infty$ in (2.11) and using (2.5), we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{2m(k)+1}, x_{2n(k)}). \tag{2.12}$$

From (2.7), we have

$$\begin{aligned} &d(x_{2m(k)}, x_{2n(k)}) \\ \leq \;& sd(x_{2m(k)}, x_{2m(k)-2}) + sd(x_{2m(k)-2}, x_{2n(k)}) \\ < \;& s^2 d(x_{2m(k)}, x_{2m(k)-1}) + s^2 d(x_{2m(k)-1}, x_{2m(k)-2}) + s\varepsilon. \end{aligned} \tag{2.13}$$

Taking the upper limit as $k \to \infty$ in (2.13) and using (2.5), we obtain

$$\limsup_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) \leq s\varepsilon. \tag{2.14}$$

Also, we have

$$d(x_{2m(k)}, x_{2n(k)-1}) \leq sd(x_{2m(k)}, x_{2n(k)}) + sd(x_{2n(k)}, x_{2n(k)-1}). \tag{2.15}$$

Taking the upper limit as $k \to \infty$ in (2.15) and using (2.14), we obtain

$$\limsup_{k\to\infty} d(x_{2m(k)}, x_{2n(k)-1}) \leq s^2\varepsilon. \tag{2.16}$$

Again, we have

$$\begin{aligned} & d(x_{2m(k)+1}, x_{2n(k)-1}) \\ \leq \quad & sd(x_{2m(k)+1}, x_{2m(k)}) + sd(x_{2m(k)}, x_{2n(k)-1}). \end{aligned} \tag{2.17}$$

Taking the upper limit as $k \to \infty$ in (2.17) and using (2.16), we obtain

$$\limsup_{k\to\infty} d(x_{2m(k)+1}, x_{2n(k)-1}) \leq s^3\varepsilon. \tag{2.18}$$

We also have

$$d(x_{2n(k)+1}, x_{2m(k)}) \leq sd(x_{2n(k)+1}, x_{2n(k)}) + sd(x_{2n(k)}, x_{2m(k)}). \tag{2.19}$$

Taking the upper limit as $k \to \infty$ in (2.19) and using (2.14), we obtain

$$\limsup_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)}) \leq s^2\varepsilon. \tag{2.20}$$

Again, we have

$$\begin{aligned} & d(x_{2m(k)+1}, x_{2n(k)+1}) \\ \leq \quad & sd(x_{2m(k)+1}, x_{2m(k)}) + sd(x_{2m(k)}, x_{2n(k)+1}). \end{aligned} \tag{2.21}$$

Taking the upper limit as $k \to \infty$ in (2.21) and using (2.20), we obtain

$$\limsup_{k\to\infty} d(x_{2m(k)+1}, x_{2n(k)+1}) \leq s^3\varepsilon. \tag{2.22}$$

Taking the upper limit as $k \to \infty$ in (2.9), (2.10) and using (2.14), (2.16), (2.18), (2.20), (2.22), we have

$$\begin{aligned} \limsup_{k\to\infty} \overline{M}_s(x_{2n(k)-1}, x_{2m(k)}) &\leq \max\left\{s^2\varepsilon, \frac{s^2\varepsilon+\varepsilon}{2}, \frac{s^2\varepsilon}{2}, s^3\varepsilon\right\} \\ &= s^3\varepsilon, \end{aligned} \tag{2.23}$$

$$\limsup_{k\to\infty} N(x_{2n(k)-1}, x_{2m(k)}) = 0. \tag{2.24}$$

Taking the upper limit as $k \to \infty$ in (2.8) and using (2.12), (2.23), (2.24), we get

$$\varphi(s^3\varepsilon) = \varphi\Big(s^4\frac{\varepsilon}{s}\Big) \le \psi(\varphi(s^3\varepsilon)) + L\varphi(0) = \psi(\varphi(s^3\varepsilon)) < \varphi(s^3\varepsilon).$$

It is a contradiction. Thus, $\{x_{2n}\}$ is a Cauchy sequence in (X, d, s). By Lemma 1.2, we conclude that $\{x_n\}$ is a Cauchy sequence. Since (X, d, s) is a complete b-metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

We shall prove that $u \in A \cap B$. Since $\{x_{2n}\} \subset A$, $\{x_{2n+1}\} \subset B$ and A, B are closed subsets of X, we have $u \in A$ and $u \in B$. Therefore, $u \in A \cap B$.

Now, we shall prove that u is a fixed point of T. Since $(x_{2n}, u) \in A \times B$ and T is a generalized cyclic contractive mapping,

$$\begin{aligned} \varphi(s^4 d(x_{2n+1}, Tu)) &= \varphi(s^4 d(Tx_{2n}, Tu)) \\ &\le \psi(\varphi(\overline{M}_s(x_{2n}, u))) + L\varphi(N(x_{2n}, u)) \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} &\overline{M}_s(x_{2n}, u) \\ &= \max\Big\{d(x_{2n}, u), d(x_{2n}, x_{2n+1}), d(u, Tu), \frac{d(x_{2n}, Tu) + d(u, x_{2n+1})}{2s}, \\ &\quad \frac{d(x_{2n+2}, x_{2n}) + d(d(x_{2n+2}, Tu)}{2s}, d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, u), d(x_{2n+2}, Tu)\Big\}, \end{aligned} \tag{2.26}$$

$$N(x_{2n}, u) = \min\{d(x_{2n}, x_{2n+1}), d(u, x_{2n+1}), d(x_{2n+2}, T^2u)\}. \tag{2.27}$$

Taking the upper limit as $n \to \infty$ in (2.26), (2.27) and using (2.5), Lemma 1.1, we have

$$\begin{aligned} \limsup_{n\to\infty} \overline{M}_s(x_{2n}, u) &\le \max\Big\{d(u, Tu), \frac{sd(u, Tu)}{2s}, sd(u, Tu)\Big\} \\ &= sd(u, Tu), \end{aligned} \tag{2.28}$$

$$\limsup_{n\to\infty} N(x_{2n}, u) = 0. \tag{2.29}$$

Taking the upper limit as $n \to \infty$ in (2.25) and using (2.28), (2.29), Lemma 1.1, we obtain

$$\varphi(s^3 d(u, Tu)) = \varphi(s^4 \frac{1}{s} d(u, Tu)) \le \psi(\varphi(sd(u, Tu))). \tag{2.30}$$

Suppose that $d(u, Tu) > 0$. Then, from (2.30) and the property of φ and ψ, we have

$$\varphi(s^3 d(u, Tu)) \le \psi(\varphi(sd(u, Tu))) < \varphi(sd(u, Tu)).$$

It is a contradiction. Therefore, $d(u, Tu) = 0$. It implies that u is a fixed point of T.

Finally, we prove that u is a unique fixed point of T. Suppose that v is also a fixed point of T, that is, $Tv = v$. Then, $v \in A \cap B$. Therefore, $(u, v) \in A \times B$. Since T is a generalized cyclic contractive mapping, we have

$$\varphi(d(u,v)) \le \varphi(s^4(d(Tu,Tv))) \le \psi(\varphi(\overline{M}_s(u,v))) + L\varphi(N(u,v)) \tag{2.31}$$

where

$$\begin{aligned}\overline{M_s}(u,v) &= \max\Big\{d(u,v), d(u,Tu), d(v,Tv), \frac{d(u,Tv)+d(v,Tu)}{2s},\\ &\qquad \frac{d(T^2u,u)+d(T^2u,Tv)}{2s}, d(T^2u,Tu), d(T^2u,v), d(T^2u,Tv)\Big\}\\ &= d(u,v),\end{aligned}$$

$$N(u,v) = \min\big\{d(u,Tu), d(v,Tu), d(T^2u,T^2v)\big\} = 0.$$

Suppose that $d(u,v) > 0$. Then, (2.31) becomes

$$\varphi(d(u,v)) \le \psi(\varphi(\overline{M}_s(u,v)) < \varphi(d(u,v)).$$

It is a contradiction. Therefore, $d(u,v) = 0$ and hence $u = v$. So, u is a unique fixed point of T. □

From Theorem 2.1, we obtain the following corollary.

Corollary 2.1. *Let (X, d, s) be a complete b-metric space, A and B be non-empty closed subsets of X, $Y = A \cup B$ and $T : Y \longrightarrow Y$ be a mapping such that*

1. *$Y = A \cup B$ is a cyclic representation of Y with respect to T.*
2. *There exist $\varphi \in \Phi, \psi \in \Psi$ and a constant $L \ge 0$ such that*

$$\varphi(s^4 d(Tx,Ty)) \le \psi(\varphi(M_s(x,y))) + L\varphi(N(x,y)) \tag{2.32}$$

for all $(x,y) \in A \times B$ or $(x,y) \in B \times A$, where

$$M_s(x,y) = \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\Big\},$$

$$N(x,y) = \min\big\{d(x,Tx), d(y,Tx), d(T^2x,T^2y)\big\}.$$

Then T has a unique fixed point in $A \cap B$.

Proof. Notice that $M_s(x,y) \leq \overline{M}_s(x,y)$ for all $(x,y) \in A \times B$ or $(x,y) \in B \times A$. By using the increasing property of φ and the non-decreasing property of ψ, it follows from (2.32) that T is a generalized cyclic contractive mapping. By Theorem 2.1, T has a unique fixed point in $A \cap B$. □

Since every metric space is a b-metric space with $s = 1$, from Theorem 2.1 and Corollary 2.1, we get the two following corollaries. Notice that Corollary 2.2 is a generalization of [20, Theorem 2.1] and Corollary 2.3 is an analogue of [20, Theorem 2.1].

Corollary 2.2. *Let (X,d) be a complete metric space, A and B be non-empty closed subsets of X, $Y = A \cup B$ and $T : Y \longrightarrow Y$ be a mapping such that*

1. *$Y = A \cup B$ is a cyclic representation of Y with respect to T.*
2. *There exist $\varphi \in \Phi, \psi \in \Psi$ and a constant $L \geq 0$ such that*

$$\varphi(d(Tx,Ty)) \leq \psi(\varphi(\overline{M}(x,y))) + L\varphi(N(x,y))$$

for all $(x,y) \in A \times B$ or $(x,y) \in B \times A$, where

$$\begin{aligned} \overline{M}(x,y) &= \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \\ &\quad \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\Big\}, \\ N(x,y) &= \min\big\{d(x,Tx), d(y,Tx), d(T^2x,T^2y)\big\}. \end{aligned}$$

Then T has a unique fixed point in $A \cap B$.

Corollary 2.3. *Let (X,d) be a complete metric space, A and B be non-empty closed subsets of X, $Y = A \cup B$ and $T : Y \longrightarrow Y$ be a mapping such that*

1. *$Y = A \cup B$ is a cyclic representation of Y with respect to T.*
2. *There exist $\varphi \in \Phi, \psi \in \Psi$ and a constant $L \geq 0$ such that*

$$\varphi(d(Tx,Ty)) \leq \psi(\varphi(M(x,y))) + L\varphi(N(x,y)) \tag{2.33}$$

for all $(x,y) \in A \times B$ or $(x,y) \in B \times A$, where

$$\begin{aligned} M(x,y) &= \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\Big\}, \\ N(x,y) &= \min\big\{d(x,Tx), d(y,Tx), d(T^2x,T^2y)\big\}. \end{aligned}$$

Then T has unique fixed point in $A \cap B$.

Finally, some examples are provided to support our results. The following example is an illustration of the existence of the fixed point of T in Theorem 2.1.

Example 2.1. Let $X = \mathbb{R}$ and a b-metric defined by $d(x,y) = (x-y)^2$ for all $x, y \in X$. Then, (X, d, s) be a complete b-metric space with $s = 2$. Let $A = \Big[-\frac{\pi}{2}, 0\Big]$, $B = \Big[0, \frac{\pi}{2}\Big]$, $Y = A \cup B$ and a mapping $T : Y \to Y$ be defined by

$$Tx = \begin{cases} -\dfrac{x}{16}\Big|\cos\dfrac{1}{x}\Big| & \text{if } x \in \Big[-\dfrac{\pi}{2}, 0\Big) \cup \Big(0, \dfrac{\pi}{2}\Big] \\ 0 & \text{if } x = 0. \end{cases}$$

Then, it is easy to check that $TA \subset B$ and $TB \subset A$. It implies that $Y = A \cup B$ is a cyclic representation of Y with respect to T. Let $\varphi \in \Phi$ and $\psi \in \Psi$ be defined by $\psi(t) = \frac{t}{4}$ and $\varphi(t) = t$ for all $t \geq 0$. Then, for all $(x,y) \in A \times B$ or $(x,y) \in B \times A$, we consider following cases.

Case 1. $x = y = 0$ with $(x,y) \in A \times B$ or $(x,y) \in B \times A$. Then

$$\varphi(s^4 d(T0, T0)) = 2^4 d(0,0) = 0 \leq \psi(\varphi(\overline{M}_s(0,0))) + L\varphi(N(0,0)) = 0.$$

Case 2. $x \neq 0, y = 0$ with $(x,y) \in A \times B$ or $(x,y) \in B \times A$. Then

$$\begin{aligned} \varphi(s^4 d(Tx, Ty)) &= 2^4 d\Big(-\frac{x}{16}\Big|\cos\frac{1}{x}\Big|, 0\Big) = 2^4\Big(-\frac{x}{16}\Big|\cos\frac{1}{x}\Big| - 0\Big)^2 \\ &\leq \frac{1}{16}x^2\cos^2\frac{1}{x} = \frac{x^2}{16} = \frac{1}{16}d(x,0) \leq \frac{1}{4}\overline{M}_s(x,0). \end{aligned}$$

Case 3. $x = 0, y \neq 0$ with $(x,y) \in A \times B$ or $(x,y) \in B \times A$. Then

$$\begin{aligned} \varphi(s^4 d(Tx, Ty)) &= 2^4 d\Big(0, -\frac{y}{16}\Big|\cos\frac{1}{y}\Big|\Big) = 2^4\Big(0 + \frac{y}{16}\Big|\cos\frac{1}{y}\Big|\Big)^2 \\ &\leq \frac{1}{16}y^2\cos^2\frac{1}{y} = \frac{y^2}{16} = \frac{1}{16}d(y,0) \leq \frac{1}{4}\overline{M}_s(0,y). \end{aligned}$$

Case 4. $x \neq 0, y \neq 0$ with $(x,y) \in A \times B$ or $(x,y) \in B \times A$. Then

$$\begin{aligned} \varphi(s^4 d(Tx, Ty)) &= 2^4 d\Big(-\frac{x}{16}\Big|\cos\frac{1}{x}\Big|, -\frac{y}{16}\Big|\cos\frac{1}{y}\Big|\Big) = \frac{1}{16}\Big(x\Big|\cos\frac{1}{x}\Big| - y\Big|\cos\frac{1}{y}\Big|\Big)^2 \\ &\leq \frac{1}{16}\Big(|x|\Big|\cos\frac{1}{x}\Big| + |y|\Big|\cos\frac{1}{y}\Big|\Big)^2 \leq \frac{1}{16}(|x| + |y|)^2 \\ &\leq \frac{1}{8}(x^2 + y^2) \leq \frac{1}{4}\max\{x^2, y^2\}. \end{aligned}$$

On the other hand, $x^2 \leq \Big(x + \frac{1}{16}x\Big|\cos\frac{1}{x}\Big|\Big)^2 = d(x, Tx)$ and $y^2 \leq \Big(y + \frac{1}{16}y\Big|\cos\frac{1}{y}\Big|\Big)^2 = d(y, Ty)$. Therefore,

$$\varphi(s^4 d(Tx, Ty)) \leq \frac{1}{4}\max\big\{d(x, Tx), d(y, Ty)\big\} \leq \psi(\varphi(\overline{M}_s(x,y))) + L\varphi(N(x,y)).$$

By the above cases, we conclude that T is a generalized cyclic contractive mapping. Therefore, all the assumptions of Theorem 2.1 are satisfied. So, Theorem 2.1 is applicable to T, b-metric space (X, d, s), ψ and φ.

The following example proves that Theorem 2.1 is a proper generalization of Corollary 2.1.

Example 2.2. Let $X = \{1, 2, 3, 4, 5\}$ and a b-metric d be defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 38 & \text{if } (x,y) \in \{(1,4),(4,1),(1,5),(5,1)\} \\ 1 & \text{if } (x,y) \in \{(1,2),(2,1),(1,3),(3,1)\} \\ 2 & \text{if } (x,y) \in \{(2,3),(3,2)\} \\ 18 & \text{otherwise.} \end{cases}$$

Then, (X, d, s) be a complete b-metric space with $s = 2$. Let $A = \{1, 2, 3, 4\}, B = \{1, 2, 3, 5\}, Y = A \cup B$ and a mapping $T : Y \longrightarrow Y$ be defined by

$$T1 = T2 = T3 = 1, T4 = 2, T5 = 3.$$

Then, $TA = \{1, 2\} \subset \{1, 2, 3, 5\} = B$ and $TB = \{1, 3\} \subset \{1, 2, 3, 4\} = A$. This implies that $A \cup B$ is a cyclic representation of Y with respect to T. By choosing $(x, y) = (4, 5) \in A \times B$, we have $d(Tx, Ty) = 2, M_s(x, y) = 18$ and $N(x, y) = 0$. Therefore, for all $\varphi \in \Phi, \psi \in \Psi$ and a constant $L \geq 0$, we have

$$\begin{aligned} \psi(\varphi(M_s(x,y))) + L\varphi(N(x,y)) &= \psi(\varphi(18)) + L\varphi(0)) \\ &< \varphi(18) < \varphi(32) = \varphi(s^4 d(Tx,Ty)). \end{aligned}$$

This implies that condition (2.32) in Corollary 2.1 is not satisfied. Therefore, Corollary 2.1 can not be applicable to T and b-metric space (X, d, s). Now, let $\varphi \in \Phi$ and $\psi \in \Psi$ be defined by $\psi(t) = \dfrac{8t}{9}$ and $\varphi(t) = t$ for all $t \geq 0$. For all $(x, y) \in A \times B$ or $(x, y) \in B \times A$, we consider following cases.

Case 1. $x, y \in \{1, 2, 3\}$. Then

$$\varphi(s^4 d(Tx,Ty)) = d(1,1) = 0 \leq \psi(\varphi(\overline{M}_s(x,y))) + L\varphi(N(x,y)).$$

Case 2. $x \in \{1, 2, 3\}, y \in \{4, 5\}$ or $x = 4, y \in \{1, 5\}$ or $x = 5, y \in \{1, 4\}$. Then

$$\varphi(s^4 d(Tx,Ty)) = 16 \leq \frac{304}{9} \leq \psi(\varphi(\overline{M}_s(x,y))) + L\varphi(N(x,y)).$$

Case 3. $x \in \{4, 5\}, y \in \{2, 3\}$. Then

$$\varphi(s^4 d(Tx,Ty)) = 16 \leq \psi(\varphi(\overline{M}_s(x,y))) + L\varphi(N(x,y)).$$

By the above cases, we conclude that T is a generalized cyclic contractive mapping. Therefore, all assumptions of Theorem 2.1 are satisfied. Thus, Theorem 2.1 is applicable to T, b-metric space (X, d, s), ψ and φ.

The following example proves that Corollary 2.2 is a proper generalization of Corollary 2.3.

Example 2.3. Let $X = \{1, 2, 3, 4, 5\}$ and a metric d be defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 3 & \text{if } (x,y) \in \{(1,4),(4,1),(1,5),(5,1)\} \\ 1 & \text{if } (x,y) \in \{(1,2),(2,1),(1,3),(3,1)\} \\ 2 & \text{otherwise.} \end{cases}$$

Then, (X, d) is a complete metric space. Let $A = \{1, 2, 3, 4\}, B = \{1, 2, 3, 5\}, Y = A \cup B$ and a mapping $T : Y \longrightarrow Y$ be defined by $T1 = T2 = T3 = 1, T4 = 2, T5 = 3$. Then, $TA = \{1, 2\} \subset \{1, 2, 3, 5\} = B$ and $TB = \{1, 3\} \subset \{1, 2, 3, 4\} = A$. This implies that $A \cup B$ is a cyclic representation of Y with respect to T. By choosing $(x, y) = (4, 5) \in A \times B$, we have $d(Tx, Ty) = 2, M(x, y) = 2$ and $N(x, y) = 0$. Therefore, for all $\varphi \in \Phi, \psi \in \Psi$ and a constant $L \geq 0$, we have

$$\begin{aligned} \psi(\varphi(M(x,y))) + L\varphi(N(x,y)) &= \psi(\varphi(2)) + L\varphi(0)) \\ &< \varphi(2) = \varphi(d(Tx,Ty)). \end{aligned}$$

This implies that condition (2.33) in Corollary 2.3 is not satisfied. Therefore, Corollary 2.3 can not be applicable to T and metric space (X, d). Now, let $\varphi \in \Phi$ and $\psi \in \Psi$ be defined by $\psi(t) = \dfrac{2t}{3}$ and $\varphi(t) = t$ for all $t \geq 0$. For all $(x, y) \in A \times B$ or $(x, y) \in B \times A$, we consider following cases.

Case 1. $x, y \in \{1, 2, 3\}$. Then

$$\varphi(d(Tx,Ty)) = d(1,1) = 0 \leq \psi(\varphi(\overline{M}(x,y))) + L\varphi(N(x,y)).$$

Case 2. $x \in \{1, 2, 3\}, y \in \{4, 5\}$ or $x = 4, y \in \{1, 5\}$ or $x = 5, y \in \{1, 4\}$. Then

$$\varphi(d(Tx,Ty)) = 1 \leq 2 \leq \psi(\varphi(\overline{M}(x,y))) + L\varphi(N(x,y)).$$

Case 3. $x = 4, y = 2$ or $x = 5, y = 3$. Then

$$\varphi(d(Tx,Ty)) = 1 \leq \frac{4}{3} \leq \psi(\varphi(\overline{M}(x,y))) + L\varphi(N(x,y)).$$

Case 4. $x = 4, y = 3$ or $x = 5, y = 2$. Then

$$\varphi(d(Tx,Ty)) = 1 \leq \frac{5}{3} \leq \psi(\varphi(\overline{M}(x,y))) + L\varphi(N(x,y)).$$

Thus, from the above cases, we conclude that T is a generalized cyclic contractive mapping. Therefore, all assumption of Corollary 2.2 are satisfied. Thus, Corollary 2.2 is applicable to T, metric space (X, d), ψ and φ.

Finally, we apply Theorem 2.1 to study the existence and uniqueness of solutions to the nonlinear integral equation.

Example 2.4. Let $C[a, b]$ be the set of all continuous functions on $[a, b]$ and the b-metric d with $s = 2^{p-1}$ defined by

$$d(x,y) = \sup_{t \in [a,b]} |x(t) - y(t)|^p$$

for all $x, y \in C[a, b]$ and for some $p > 1$. Consider the nonlinear integral equation

$$x(t) = g(t) + \int_a^b K(t,s,x(s))ds \tag{2.34}$$

where $t \in [a, b]$, $g : [a, b] \to \mathbb{R}$, $K : [a, b] \times [a, b] \times x[a, b] \to \mathbb{R}$ for each $x \in C[a, b]$. Suppose that the following statements hold.

1. g is continuous and $K(t,s,x(s))$ is integrable with respect to s on $[a,b]$.
2. There exist $\alpha,\beta\in C[a,b]$ and $(\alpha_0,\beta_0)\in\mathbb{R}^2$ such that $\alpha_0\le\alpha(t)\le\beta(t)\le\beta_0$ and

$$\begin{aligned}\alpha(t) &\le g(t)+\int_a^b K(t,s,\beta(s))ds,\\ \beta(t) &\ge g(t)+\int_a^b K(t,s,\alpha(s))ds\end{aligned}$$

for all $t\in[a,b]$.
3. If $x(s)\ge y(s)$ for all $s\in[a,b]$, then $K(t,s,x(s))\le K(t,s,y(s))$ for all $t,s\in[a,b]$.
4. For all $t,s\in[a,b]$ and all $x,y\in C[a,b]$ with $\alpha_0\le x(t)$ and $y(t)\le\beta_0$ or $x(t)\le\beta_0$ and $y(t)\ge\alpha_0$ for all $t\in[a,b]$,

$$\begin{aligned}&|K(t,s,x(s))-K(t,s,y(s))|^p\\ \le\ &\xi(t,s)\max\Big\{|x(s)-y(s)|^p,|x(s)-Tx(s)|^p,|y(s)-Ty(s)|^p,\\ &\frac{|x(s)-Ty(s)|^p+|y(s)-Tx(s)|^p}{2^p},\frac{|T^2x(s)-x(s)|^p+|T^2x(s)-Ty(s)|^p}{2^p},\\ &|T^2x(s)-Tx(s)|^p,|T^2x(s)-y(s)|^p,|T^2x(s)-Ty(s)|^p\Big\}\\ &+\min\big\{|x(s)-Tx(s)|^p,|y(s)-Tx(s)|^p,|T^2x(s)-T^2y(s)|^p\big\}\end{aligned}$$

where $Tx(t)=g(t)+\int_a^b K(t,s,x(s))ds$ for all $x\in C[a,b]$, $t\in[a,b]$ and $\xi:[a,b]\times[a,b]\longrightarrow[0,\infty)$ is a continuous function satisfying

$$\sup_{t\in[a,b]}\int_a^b\xi(t,s)ds\le\frac{1}{2^{4p-4}(b-a)^{p-1}}.$$

Then nonlinear integral equation (2.34) has a unique solution $u\in\big\{u\in C[a,b]:\alpha(t)\le u(t)\le\beta(t)\text{ for all }t\in[a,b]\big\}$.

Proof. (1). For α and β defined by assumption (2), we consider two closed subsets of $C[a,b]$ as follows.

$$A=\big\{u\in C[a,b]:u(t)\le\beta(t)\text{ for all }t\in[a,b]\big\}$$

and

$$B=\big\{u\in C[a,b]:u(t)\ge\ \alpha(t)\text{ for all }t\in[a,b]\big\}.$$

Define the mapping $T:A\cup B\longrightarrow A\cup B$ by

$$Tu(t)=g(t)+\int_a^b K(t,s,u(s))ds$$

for all $t\in[a,b]$ and all $u\in A\cup B$. We shall prove that $TA\subset B$ and $TB\subset A$. Let $u\in A$. Then $u(s)\le\beta(s)$ for all $s\in[a,b]$. By using assumption (3), we have $K(t,s,u(s))\ge K(t,s,\beta(s))$ for all t, $s\in[a,b]$. It follows from assumption (2) that

$$Tu(t)=g(t)+\int_a^b K(t,s,u(s))ds\ge g(t)+\int_a^b K(t,s,\beta(s))ds\ge\alpha(t)$$

for all $t \in [a,b]$. It implies that $Tu \in B$ and hence $TA \subset B$. Similarly, we also see that if $u \in B$, then $Tu \in A$ and hence $TB \subset A$.
(2). Let $(x,y) \in A \times B$ or $(x,y) \in B \times A$, that is, $\begin{cases} x(t) \le \beta(t) \\ y(t) \ge \alpha(t) \end{cases}$ or $\begin{cases} x(t) \ge \alpha(t) \\ y(t) \le \beta(t) \end{cases}$ for all $t \in [a,b]$. By assumption (2), we have $\begin{cases} x(t) \le \beta_0 \\ y(t) \ge \alpha_0 \end{cases}$ or $\begin{cases} x(t) \ge \alpha_0 \\ y(t) \le \beta_0 \end{cases}$ for all $t \in [a,b]$.

Now, let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. By using assumption (4), we have

$$\begin{aligned}
& 2^{4p-4}|Tx(t) - Ty(t)|^p \\
\le\ & 2^{4p-4}\Big[\int_a^b |K(t,s,x(s)) - K(t,s,y(s))|ds\Big]^p \\
\le\ & 2^{4p-4}\Big[\Big(\int_a^b ds\Big)^{\frac{1}{q}}\Big(\int_a^b \Big|K(t,s,x(s)) - K(t,s,y(s))\Big|^p ds\Big)^{\frac{1}{p}}\Big]^p \\
\le\ & 2^{4p-4}\big(b-a\big)^{p-1}\Big[\int_a^b \Big(\xi(t,s)\max\Big\{|x(s)-y(s)|^p, |x(s)-Tx(s)|^p, \\
& |y(s)-Ty(s)|^p, \frac{|x(s)-Ty(s)|^p + |y(s)-Tx(s)|^p}{2^p}, \\
& \frac{|T^2x(s)-x(s)|^p + |T^2x(s)-Ty(s)|^p}{2^p}, |T^2x(s)-Tx(s)|^p, \\
& |T^2x(s)-y(s)|^p, |T^2x(s)-Ty(s)|^p\Big\} + \min\big\{|x(s)-Tx(s)|^p, \\
& |y(s)-Tx(s)|^p, |T^2x(s)-T^2y(s)|^p\big\}\Big)ds\Big] \\
\le\ & 2^{4p-4}\big(b-a\big)^{p-1}\Big[\Big(\int_a^b \xi(t,s)ds\Big)\overline{M}_s(x,y) + N(x,y)\Big] \\
\le\ & 2^{4p-4}\big(b-a\big)^{p-1}\Big(\int_a^b \xi(t,s)ds\Big)\overline{M}_s(x,y) + 2^{4p-4}\big(b-a\big)^{p}N(x,y).
\end{aligned}$$

Put $\lambda = 2^{4p-4}(b-a)^{p-1}\sup_{t\in[a,b]}\int_a^b \xi(t,s)ds \in [0,1)$ and $L = 2^{4p-4}(b-a)^p \ge 0$. It implies that

$$(2^{p-1})^4 d(Tx,Ty) \le \lambda\overline{M}_s(x,y) + LN(x,y).$$

Therefore, T is a generalized cyclic contractive mapping with $\psi(t) = \lambda t$, $\varphi(t) = t$ for all $t \ge 0$ and $L = 2^{4p-4}(b-a)^p$. Thus, all assumptions of Theorem 2.1 are satisfied. By using Theorem 2.1, T has a unique fixed point in $A \cap B$ and hence nonlinear integral equation (2.34) has a unique solution in $\big\{u \in C[a,b] : \alpha(t) \le u(t) \le \beta(t)$ for all $t \in [a,b]\big\}$. □

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