FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 3 (2016), 629–644

# SOLVABILITY AND STABILITY FOR NONLINEAR FRACTIONAL INTEGRO-DIFFERENTIAL SYSTEMS OF HIGHT FRACTIONAL ORDER

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**Abstract.** In this paper, using Riemann-Liouville integral and Caputo derivative, we study an $n-$dimensional coupled system of nonlinear fractional integro-differential equations of hight arbitrary order. The contraction mapping principle and Schaefer fixed point theorem are applied to prove the existence and the uniqueness of solutions in Banach spaces. Furthermore, we derive the Ulam-Hyers and the generalized Ulam-Hyers stabilities of solutions. Some illustrative examples are also presented.
**Key words:** Riemann-Liouville integral, Caputo derivative, fractional integro-differential equations, Banach space.

## 1. Introduction and Preliminaries

The fractional calculus has attracted interest of researchers in several areas of mathematics, physics, chemistry and engineering sciences. For more details, we refer the reader to the monographs by Hilfer [19], Lakshmikantham [26], Podlubny [33] and the papers of [3, 5, 18, 20, 24, 30, 31, 32]. In addition, many authors paid much attention to the existence and uniqueness of solutions for some fractional differential equations and systems, see for example [1, 2, 4, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 21, 27, 28, 35, 38] and the references therein. Moreover, the study of Ulam type stability problems has grown to be one of the most important subjects in the mathematical analysis area. For instance, J. Wang et al. [17] obtained some results on Ulam type stability in the case of impulsive ordinary differential equations. Then, J. Wang, L. Lv and Y. Zhou [37] presented some interesting results on Ulam stability for fractional differential equations with Caputo derivative. Recently, R. Ibrahim [22, 23] obtained some results about Ulam-Hyers stability of solutions for fractional order dynamic equations. Very recently, in [12, 36], the Ulam-Hyers stability of solutions has been done for that problems of singular fractional differential equations.

Motivated by the above works, in this paper, we discuss the existence, the uniqueness and the Ulam-Hyers stability as well as the generalized Ulam-Hyers stability for the following fractional problem:

Received December 09, 2015; accepted June 03, 2016
2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 39B72, 39B82

$$
(1.1)\begin{cases}
D^{\alpha_1}x_1(t)=\sum\limits_{i=1}^{m}g_i^1(t,x_1(t),...,x_n(t))+\sum\limits_{i=1}^{m}J^{\delta_1}g_i^1(t,x_1(t),...,x_n(t)),\\
D^{\alpha_2}x_2(t)=\sum\limits_{i=1}^{m}g_i^2(t,x_1(t),...,x_n(t))+\sum\limits_{i=1}^{m}J^{\delta_2}g_i^2(t,x_1(t),...,x_n(t)),\\
\vdots\\
D^{\alpha_n}x_n(t)=\sum\limits_{i=1}^{m}g_i^n(t,x_1(t),...,x_n(t))+\sum\limits_{i=1}^{m}J^{\delta_n}g_i^n(t,x_1(t),...,x_n(t)),\\
\sum\limits_{k=1}^{n}|x_k(0)|=\sum\limits_{k=1}^{n}\left|x_k^{'}(0)\right|=...=\sum\limits_{k=1}^{n}\left|x_k^{(l-2)}(0)\right|=0,\\
x_k^{(l-1)}(1)=0,\ k=1,2,...,n,\ l\in\mathbb{N}^*-\{1\},
\end{cases}
$$

where, $t\in J=[0,1]$, $l\in\mathbb{N}^*\setminus\{1\}$, $n\in\mathbb{N}^*$, $m\in\mathbb{N}^*$, and for $k=1,...,n$, $l-1<\alpha_k<l$, $\delta_k$, is a nonnegative real number. The operator $J^{\alpha}$ is the Riemann-Liouville fractional integral given by:

$$
J^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)\,ds,\ \alpha>0,\ t\geq 0, \tag{1.2}
$$

where $\Gamma(\alpha):=\int_0^{\infty}e^{-x}x^{\alpha-1}dx$. The operator $D^{\alpha}$ is the derivative in the sense of Caputo defined by the following relation

$$
D^{\alpha}x(t)=\frac{1}{\Gamma(l-\alpha)}\int_0^t(t-s)^{l-\alpha-1}x^{(l)}(s)\,ds=J^{l-\alpha}x^{(l)}(t),\ l-1<\alpha<l. \tag{1.3}
$$

Finally, the functions $g_i^k:J\times\mathbb{R}^n\to\mathbb{R}$, for $k=1,...,n$, and $i=1,...,m$, will be specified later.

Now, let us recall some lemmas that we need to prove our main results [25, 29, 34].

**Lemma 1.1.** *For $l\in\mathbb{N}^*\setminus\{1\}$, and $l-1<\alpha<l$, the general solution of the fractional differential equation $D^{\alpha}x(t)=0$, is given by*

$$
x(t)=\sum_{j=0}^{l-1}c_j\,t^j, \tag{1.4}
$$

*where $c_j\in\mathbb{R}$, $j=0,...,l-1$.*

**Lemma 1.2.** *Given $l\in\mathbb{N}^*\setminus\{1\}$, and $l-1<\alpha<l$. Then*

$$
J^{\alpha}D^{\alpha}x(t)=x(t)+\sum_{j=0}^{l-1}c_j\,t^j, \tag{1.5}
$$

*where $c_j\in\mathbb{R}$, $j=0,1,...,l-1$.*

**Lemma 1.3.** *Let $q > p > 0$, $g \in L^1([a,b])$. Then,*

$$D^p J^q g(t) = J^{q-p} g(t), \qquad t \in [a,b].$$

**Lemma 1.4.** *Let $E$ be a Banach space and $T : E \to E$ be a completely continuous operator. If the set $V := \{x \in E : x = \nu T x,\ 0 < \nu < 1\}$ is bounded, then $T$ has a fixed point in $E$.*

Also, the following auxiliary result is important to give the integral solution of (1.1):

**Lemma 1.5.** *Let given $l \in \mathbb{N}^* \backslash \{1\}$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}^*$, and a family $\left(G_i^k\right) \in C(J,\mathbb{R})$ for $i = 1,...,m$, $k = 1,...,n$, and consider the problem*

$$\left\{ \begin{array}{l} D^{\alpha_1} x_1(t) = \sum\limits_{i=1}^{m} G_i^1(t) + \sum\limits_{i=1}^{m} \int_0^t \frac{(t-s)^{\delta_1 - 1}}{\Gamma(\delta_1)} G_i^1(s) ds,\ t \in J, \\ D^{\alpha_2} x_2(t) = \sum\limits_{i=1}^{m} G_i^2(t) + \sum\limits_{i=1}^{m} \int_0^t \frac{(t-s)^{\delta_2 - 1}}{\Gamma(\delta_2)} G_i^2(s) ds,\ t \in J, \\ \qquad \vdots \\ D^{\alpha_n} x_n(t) = \sum\limits_{i=1}^{m} G_i^n(t) + \sum\limits_{i=1}^{m} \int_0^t \frac{(t-s)^{\delta_n - 1}}{\Gamma(\delta_n)} G_i^n(s) ds,\ t \in J, \end{array} \right. \tag{1.6}$$

*where $\delta_k \in \mathbb{R}^+$, $l - 1 < \alpha_k < l$,*
*with the conditions:*

$$\begin{aligned} \sum_{k=1}^{n} |x_k(0)| &= \sum_{k=1}^{n} \left| x_k'(0) \right| = ... = \sum_{k=1}^{n} \left| x_k^{(l-2)}(0) \right| = 0, \\ x_k^{(l-1)}(1) &= 0,\ k = 1,2,...,n. \end{aligned} \tag{1.7}$$

*Then, the solution $(x_1, x_2, ..., x_n)$ of $(1.6) - (1.7)$ is given by*

$$\begin{aligned} x_k(t) = &\sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} G_i^k(s)\, ds + \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\alpha_k + \delta_k - 1}}{\Gamma(\alpha_k + \delta_k)} G_i^k(s)\, ds \\ &- \frac{t^{l-1}}{(l-1)!\Gamma(\alpha_k - l + 1)} \sum_{i=1}^{m} \int_0^1 (1-s)^{\alpha_k - l} G_i^k(s)\, ds \\ &- \frac{t^{l-1}}{(l-1)!\Gamma(\alpha_k + \delta_k - l + 1)} \sum_{i=1}^{m} \int_0^1 (1-s)^{\alpha_k + \delta_k - l} G_i^k(s)\, ds,\ k = 1,2,...,n. \end{aligned} \tag{1.8}$$

*Proof.* Using Lemma 1.1 and Lemma 1.2 and (1.6), we get

$$x_k(t) = \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} G_i^k(s)\, ds + \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\alpha_k + \delta_k - 1}}{\Gamma(\alpha_k + \delta_k)} G_i^k(s)\, ds - \sum_{j=0}^{l-1} c_j^k t^j, \tag{1.9}$$

where $c_j^k \in \mathbb{R}$, $j = 0, 1, 2, ..., l-1$ and $l - 1 < \alpha_k < l$, $k = 1, 2, ..., n$.

For all $k = 1, 2, ..., n$, $j = 0, 1, ..., l-2$, we can write

$$x_k^{(j)}(0) = -j!\, c_j^k.$$

Using Lemma 1.3 and applying the boundary conditions (1.7), we obtain:

$$c_j^k = 0, \qquad j = 0, 1, ..., l-2, \tag{1.10}$$

and

$$c_{l-1}^k = \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha_k - l}}{(l-1)!\,\Gamma(\alpha_k - l + 1)} G_i^k(s)\, ds$$
$$+ \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha_k + \delta_k - l}}{(l-1)!\,\Gamma(\alpha_k + \delta_k - l + 1)} G_i^k(s)\, ds, \tag{1.11}$$

Substituting the values of $c_j^k$ into (1.9), we end the proof of Lemma 1.5. $\square$

Now, to study the problem (1.1), we introduce the Banach space

$$S := \{(x_1, x_2, ..., x_n) : x_k \in C([0,1], \mathbb{R}),\ k = 1, 2, ..., n\},$$

equipped with the norm

$$\|(x_1, x_2, ..., x_n)\|_S = \max(\|x_1\|_\infty, \|x_2\|_\infty, ..., \|x_n\|_\infty). \tag{1.12}$$

## 2. Main Results

In this section, we will formulate and prove sufficient conditions for the existence and uniqueness of solutions for the system (1.1). Then, we study its Ulam-Hyers stability as well as its generalized Ulam-Hyers stability.

We begin by listing the following hypotheses:

$(H_1)$ : There exist nonnegative constants $\left(\omega_{i,\ i=1,...,m}^{k,\ k=1,2,...,n}\right)_j$, $j = 1, 2, ..., n$, such that for all $t \in [0,1]$ and all $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, we have

$$\left|g_i^k(t, x_1, x_2, ..., x_n) - g_i^k(t, y_1, y_2, ..., y_n)\right| \leq \sum_{j=1}^{n} \left(\omega_i^k\right)_j |x_j - y_j|,\ k = 1, ..., n, i = 1, ..., m.$$

$(H_2)$ : The functions $g_i^k : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are continuous for each $i = 1, 2, ..., m$ and $k = 1, 2, ..., n$, $m, n \in \mathbb{N}^*$.

$(H_3)$ : There exist nonnegative constants $\left(L_i^k\right)_{i=1,...,m}^{k=1,2...,n}$, such that for each $t \in J$ and all $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$, we have

$$\left|g_i^k(t, x_1, x_2, ..., x_n)\right| \leq L_i^k,\ i = 1, ..., m,\ k = 1, 2, ..., n.$$

We also need to define the following quantities:

$$\Delta_k = \frac{1}{\Gamma(\alpha_k+1)} + \frac{1}{\Gamma(\alpha_k+\delta_k+1)},\ k=1,2,...,n, \tag{2.1}$$

$$\Sigma_k = \sum_{i=1}^{m} \left( \left(\omega_i^k\right)_1 + \left(\omega_i^k\right)_2 + ... + \left(\omega_i^k\right)_n \right),\ k=1,2,...,n, \tag{2.2}$$

and

$$\theta_k = \frac{1}{(l-1)!\Gamma(\alpha_k+2-l)} + \frac{1}{(l-1)!\Gamma(\alpha_k+\delta_k+2-l)},\ k=1,2,...,n. \tag{2.3}$$

### 2.1. Existence and Uniqueness of Solutions

The first result is based on Banach contraction principle. We have:

**Theorem 2.1.** *Assume that* $(H_1)$ *holds and*

$$\max\left(\Sigma_1\Delta_1+\theta_1, \Sigma_2\Delta_2+\theta_2, ..., \Sigma_n\Delta_n+\theta_n\right) < 1. \tag{2.4}$$

*Then, the system* (1.1) *has a unique solution on* $J$.

*Proof.* We define the following nonlinear operator $T : S \to S$, by

$$T(x_1,...,x_n)(t) := \left(T_1(x_1,...,x_n)(t), T_2(x_1,...,x_n)(t), ..., T_n(x_1,...,x_n)(t)\right), \tag{2.5}$$

such that,

$$\begin{aligned} T_k(x_1,...,x_n)(t) &= \sum_{i=1}^{m}\int_0^t \tfrac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} G_i^k(s)\,ds + \sum_{i=1}^{m}\int_0^t \tfrac{(t-s)^{\alpha_k+\delta_k-1}}{\Gamma(\alpha_k+\delta_k)} G_i^k(s)\,ds \\ &\quad - \tfrac{t^{l-1}}{(l-1)!\Gamma(\alpha_k-l+1)} \sum_{i=1}^{m}\int_0^1 (1-s)^{\alpha_k-l} G_i^k(s)\,ds \\ &\quad - \tfrac{t^{l-1}}{(l-1)!\Gamma(\alpha_k+\delta_k-l+1)} \sum_{i=1}^{m}\int_0^1 (1-s)^{\alpha_k+\delta_k-l} G_i^k(s)\,ds, \end{aligned} \tag{2.6}$$

where,

$$G_i^k(s) = g_i^k\left(s, x_1(s), x_2(s), ..., x_n(s)\right),\ k=1,2,...,n, i=1,2,...,m. \tag{2.7}$$

We show that $T$ is a contractive operator.

Let $(x_1,...,x_n), (y_1,...,y_n) \in S$. Then, for each $k=1,2,...,n$ and $t\in J$, we have:

$$|T_k(x_1,...,x_n)(t) - T_k(y_1,...,y_n)(t)| \tag{2.8}$$

$$\begin{aligned}
&\leq \frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)} \sup_{s\in J} \sum_{i=1}^{m} \left| g_i^k(s, x_1(s), ..., x_n(s)) - g_i^k(s, y_1(s), ..., y_n(s)) \right| \\
&+ \frac{t^{\alpha_k+\delta_k}}{\Gamma(\alpha_k+\delta_k+1)} \sup_{s\in J} \sum_{i=1}^{m} \left| g_i^k(s, x_1(s), ..., x_n(s)) - g_i^k(s, y_1(s), ..., y_n(s)) \right| \\
&+ \frac{t^{l-1}}{(l-1)!\Gamma(\alpha_k+2-l)} \sup_{s\in J} \sum_{i=1}^{m} \left| g_i^k(s, x_1(s), ..., x_n(s)) - g_i^k(s, y_1(s), ..., y_n(s)) \right| \\
&+ \frac{t^{l-1}}{(l-1)!\Gamma(\alpha_k+\delta_k+2-l)} \sup_{s\in J} \sum_{i=1}^{m} \left| g_i^k(s, x_1(s), ..., x_n(s)) - g_i^k(s, y_1(s), ..., y_n(s)) \right|.
\end{aligned}$$

By the hypothesis $(H_1)$, for each $k = 1, 2, ..., n$, we can write

$$\begin{aligned}
&(2.9)\ \|T_k(x_1,...,x_n) - T_k(y_1,...,y_n)\| \\
&\leq \left( \frac{1}{\Gamma(\alpha_k+1)} + \frac{1}{\Gamma(\alpha_k+\delta_k+1)} + \frac{1}{(l-1)!\Gamma(\alpha_k-l+2)} \right. \\
&\quad \left. + \frac{1}{(l-1)!\Gamma(\alpha_k+\delta_k-l+2)} \right) \times \max(\|x_1-y_1\|, ..., \|x_n-y_n\|) \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\omega_i^k\right)_j.
\end{aligned}$$

Then, we obtain

$$\|T_k(x_1,...,x_n) - T_k(y_1,...,y_n)\| \leq \max_{1\leq k\leq n} (\Delta_k \Sigma_k + \theta_k) \|(x_1-y_1, ..., x_n-y_n)\|_S. \tag{2.10}$$

Therefore,

$$\|T(x_1,...,x_n) - T(y_1,...,y_n)\|_S \leq$$

$$\max(\Delta_1\Sigma_1 + \theta_1, ..., \Delta_n\Sigma_n + \theta_n) \|(x_1-y_1, ..., x_n-y_n)\|_S. \tag{2.11}$$

Using (2.4), we can see that $T$ is a contractive operator. Consequently, by Banach fixed point theorem, $T$ has a fixed point which is a solution of the system (1.1). Theorem 2.1 is thus proved. □

Our second main result is based on Lemma 1.4. We have:

**Theorem 2.2.** *Assume that the hypotheses* $(H_2)$ *and* $(H_3)$ *are satisfied. Then, the system* (1.1) *has at least a solution on* $J$.

*Proof.* $(i)$ : We prove that $T$ is completely continuous.

Let us take for $\mu > 0$, the set $B_\mu := \{(x_1,...,x_n) \in S,\ \|(x_1,...,x_n)\|_S \le \mu\}$. Then for any $(x_1,...,x_n) \in B_\mu$ and for each $k = 1,2,...,n$, we can write

$$\begin{aligned}
(2.12)\qquad & \|T_k(x_1,...,x_n)\| \\
\le\ & \left(\frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)} + \frac{t^{\alpha_k+\delta_k}}{\Gamma(\alpha_k+\delta_k+1)}\right)\sup_{s\in J}\sum_{i=1}^{m}\left|g_i^k(s,x_1(s),...,x_n(s))\right| \\
& +\theta_k \sup_{s\in J}\sum_{i=1}^{m}\left|g_i^k(s,x_1(s),...,x_n(s))\right| \\
\le\ & (\Delta_k+\theta_k)\sum_{i=1}^{m} L_i^k .
\end{aligned}$$

Thus,

$$(2.13)\ \|T(x_1,x_2,...,x_n)\|_S \le \max\left(\sum_{i=1}^{m} L_i^1(\Delta_1+\theta_1),...,\sum_{i=1}^{m} L_i^n(\Delta_n+\theta_n)\right) < \infty.$$

Using the above inequality $(2.13)$, we deduce that $T$ maps bounded sets into bounded sets in $S$.

The operator $T$ is continuous on $S$, in view of the continuity of $g_i^k$ given in the hypothesis $(H_2)$.

For any $0 \le t_1 < t_2 \le 1$, $(x_1,x_2,...,x_n) \in B_\mu$ and $k = 1,...,n$, we have:

$$|T_k(x_1,...,x_n)(t_2) - T_k(x_1,...,x_n)(t_1)| \le M_k \sup\nolimits_{s\in J}\sum_{i=1}^{m}\left|g_i^k(s,x_1(s),...,x_n(s))\right|,$$

where

$$\begin{aligned}
M_k =\ & \frac{1}{\Gamma(\alpha_k+1)}\left(2(t_2-t_1)^{\alpha_k} + (t_2^{\alpha_k} - t_1^{\alpha_k})\right) \\
& + \frac{1}{\Gamma(\alpha_k+\delta_k+1)}\left(2(t_2-t_1)^{\alpha_k+\delta_k} + \left(t_2^{\alpha_k+\delta_k} - t_1^{\alpha_k+\delta_k}\right)\right) \\
& + \left(\frac{1}{(l-1)!\Gamma(\alpha_k+2-l)} + \frac{1}{(l-1)!\Gamma(\alpha_k+\delta_k+2-l)}\right)\left(t_2^{l-1} - t_1^{l-1}\right).
\end{aligned}$$

Therefore,

$$(2.14)\qquad |T_k(x_1,...,x_n)(t_2) - T_k(x_1,...,x_n)(t_1)| \le M_k \sum_{i=1}^{m} L_i^k,$$

such that,

$$\|T(x_1,...,x_n)(t_2) - T(x_1,...,x_n)(t_1)\|_S = \max_{1\le k\le n}|T_k(x_1,...,x_n)(t_2) - T_k(x_1,...,x_n)(t_1)|.$$

The right-hand side of (26) is independent of $(x_1, x_2, ..., x_n) \in B_\mu$ and tends to zero as $t_2 - t_1 \to 0$. Therefore, $T$ is an equi-continuous operator. We conclude that $T$ is a completely continuous operator.

$(ii)$ : We shall show that the set $\Omega$ defined by

$$\Omega := \{(x_1, ..., x_n) \in S,\ (x_1, ..., x_n) = \lambda T(x_1, ..., x_n),\ 0 < \lambda < 1\}, \tag{2.15}$$

is bounded.
Let $(x_1, x_2, ..., x_n) \in \Omega$, then $(x_1, ..., x_n) = \lambda T(x_1, ..., x_n)$, for some $0 < \lambda < 1$. We have:

$$x_k(t) = \lambda T_k(x_1, ..., x_n)(t),\ k = 1, 2, ..., n. \tag{2.16}$$

Corresponding to (24), we get:

$$\|x_k\| \leq \lambda(\Delta_k + \theta_k) \sum_{i=1}^{m} L_i^k,\ k = 1, ..., n. \tag{2.17}$$

Thus,

$$\|(x_1, ..., x_n)\|_S \leq \lambda \max\left(\sum_{i=1}^{m} L_i^1(\Delta_1 + \theta_1), ..., \sum_{i=1}^{m} L_i^n(\Delta_n + \theta_n)\right) < \infty. \tag{2.18}$$

Consequently, the set $\Omega$ is bounded. So by Lemma 1.4, we deduce that the operator $T$ has at least one fixed point, which is a solution of the system $(1.1)$. Theorem 2.2 is thus proved. $\square$

## 2.2. Ulam-Hyers and Generalized Ulam-Hyers Stabilities

In this section, we prove some results on the Ulam-Hyers and the generalized Ulam-Hyers stabilities for the solutions of (1.1).

**Definition 2.1.** The fractional system (1.1) is Ulam-Hyers stable if there exists a real number $C > 0$, such that for each $\epsilon_k > 0$, $k = 1, 2, ..., n$, and for each solution $(x_1, x_2, ..., x_n) \in S$ of

$$\left| \begin{array}{c} D^{\alpha_k} x_k(t) - \sum\limits_{i=1}^{m} g_i^k(t, x_1(t), ..., x_n(t)) \\ - \sum\limits_{i=1}^{m} J^{\delta_k} g_i^k(t, x_1(t), ..., x_n(t)) \end{array} \right| \leq \epsilon_k, \tag{2.19}$$

there exists $(y_1, y_2, ..., y_n) \in S$ of (1) with $\sum\limits_{k=1}^{n} |y_k(0)| = \sum\limits_{k=1}^{n} \left|y_k^{'}(0)\right| = ... = \sum\limits_{k=1}^{n} \left|y_k^{(l-2)}(0)\right| = 0$, $y_k^{(l-1)}(1) = 0$, $k = 1, 2, ..., n$, satisfying

$$\|(x_1 - y_1, x_2 - y_2, ..., x_n - y_n)\|_S \leq C\epsilon,\ \epsilon > 0. \tag{2.20}$$

**Definition 2.2.** The fractional system (1.1) is generalized Ulam-Hyers stable if there exists $\Upsilon \in C(\mathbb{R}^{+}, \mathbb{R}^{+})$, such that for each $\epsilon > 0$ and for each solution $(x_1, x_2, ..., x_n) \in S$ of (2.19), there exists $(y_1, y_2, ..., y_n) \in S$ of (1.1) with

$$\|(x_1 - y_1, x_2 - y_2, ..., x_n - y_n)\|_S \leq \Upsilon(\epsilon).$$

We prove the following result:

**Theorem 2.3.** *Suppose that:*

$(a)$ : *The assumptions of Theorem 2.1 are satisfied.*

$(b)$ : *The hypotheses* $(H_2)$ *and* $(H_3)$ *are valid.*

$(c)$ : *The quantity* (2.2) *satisfies:* $0 < \Sigma_k < 1$.

$(d)$ : *For each* $k = 1, 2, ..., n$,

$$\sup_{t \in J} |D^{\alpha_k} x_k(t)| \geq (\Delta_k + \theta_k) \sum_{i=1}^{m} L_i^k. \tag{2.21}$$

*Then, the problem* (1.1) *has the generalized Ulam-Hyers stability in S.*

*Proof.* Let

$$\left| \begin{array}{c} D^{\alpha_k} x_k(t) - \sum\limits_{i=1}^{m} g_i^k(t, x_1(t), x_2(t), ..., x_n(t)) \\ - \sum\limits_{i=1}^{m} J^{\delta_k} g_i^k(t, x_1(t), x_2(t), ..., x_n(t)) \end{array} \right| < \epsilon_k, \tag{2.22}$$

where, $\epsilon_k > 0$ and $k = 1, 2, ..., n$.

Using Theorem 2.1, we conclude that (1.1) has a unique solution $(y_1, y_2, ..., y_n) \in S$ satisfying:

$$D^{\alpha_k} y_k(t) =$$

$$\sum_{i=1}^{m} g_i^k(t, y_1(t), y_2(t), ..., y_n(t)) + \sum_{i=1}^{m} J^{\delta_k} g_i^k(t, y_1(t), y_2(t), ..., y_n(t)), \tag{2.23}$$

with, $\sum\limits_{k=1}^{n} |y_k(0)| = \sum\limits_{k=1}^{n} \left|y_k'(0)\right| = ... = \sum\limits_{k=1}^{n} \left|y_k^{(l-2)}(0)\right| = 0,\ y_k^{(l-1)}(1) = 0,\ k = 1, 2, ..., n.$

Thanks to $(H_2)$ and $(H_3)$, we get

$$|x_k(t)| \leq (\Delta_k + \theta_k) \sum_{i=1}^{m} L_i^k,\ k = 1, 2, ..., n. \tag{2.24}$$

Combining (2.21) and (2.24), we obtain:

$$\sup_{t \in J} |x_k(t)| \leq \sup_{t \in J} |D^{\alpha_k} x_k(t)|. \tag{2.25}$$

Then,

$$\sup_{t\in J}|x_k(t)-y_k(t)| \le \sup_{t\in J}|D^{\alpha_k}(x_k(t)-y_k(t))| \tag{2.26}$$

$$\le \sup_{t\in J}\left|\begin{array}{c} \left(\begin{array}{c} D^{\alpha_k}x_k(t)-\sum\limits_{i=1}^{m}g_i^k(t,x_1(t),x_2(t),...,x_n(t)) \\ -\sum\limits_{i=1}^{m}J^{\delta_k}g_i^k(t,x_1(t),x_2(t),...,x_n(t)) \end{array}\right) \\ +\left(\begin{array}{c} -D^{\alpha_k}y_k(t)+\sum\limits_{i=1}^{m}g_i^k(t,y_1(t),y_2(t),...,y_n(t)) \\ +\sum\limits_{i=1}^{m}J^{\delta_k}g_i^k(t,y_1(t),y_2(t),...,y_n(t)) \end{array}\right) \\ +\left(\begin{array}{c} \sum\limits_{i=1}^{m}g_i^k(t,x_1(t),x_2(t),...,x_n(t)) \\ +\sum\limits_{i=1}^{m}J^{\delta_k}g_i^k(t,x_1(t),x_2(t),...,x_n(t)) \end{array}\right) \\ -\left(\begin{array}{c} \sum\limits_{i=1}^{m}g_i^k(t,y_1(t),y_2(t),...,y_n(t)) \\ +\sum\limits_{i=1}^{m}J^{\delta_k}g_i^k(t,y_1(t),y_2(t),...,y_n(t)) \end{array}\right) \end{array}\right|. \tag{2.27}$$

Using $(34)$ and $(35)$, we get

$$\sup_{t\in J}|x_k(t)-y_k(t)|$$

$$\le \epsilon_k+\sum_{i=1}^{m}\left(\left(\omega_i^k\right)_1+\left(\omega_i^k\right)_2+...+\left(\omega_i^k\right)_n\right)\max_{1\le k\le n}\|x_k(t)-y_k(t)\|. \tag{2.28}$$

By $(2.28)$, we obtain

$$\max_{1\le k\le n}\|x_k(t)-y_k(t)\| \le \max_{1\le k\le n}\frac{\epsilon_k}{1-\Sigma_k}=C\epsilon, \tag{2.29}$$

$$\epsilon=\max_{1\le k\le n}\epsilon_k,\ C=\max_{1\le k\le n}\frac{1}{1-\Sigma_k}, \tag{2.30}$$

which implies that

$$\|(x_1-y_1,x_2-y_2,...,x_n-y_n)\|_S=\max_{1\le k\le n}\|(x_k-y_k)\|\le C\epsilon. \tag{2.31}$$

Thus, the system (1.1) is stable in the sense of Ulam-Hyers.
Taking $\Upsilon(\epsilon)=C\epsilon$, we see that the system (1.1) is generalized Ulam-Hyers stable.
This completes the proof of Theorem 2.3. □

## 3. Examples

In this section, we present two examples to illustrate the application of our main result.

**Example 3.1.** Consider the following system:

$$(3.1)\left\{\begin{array}{l}
D^{\frac{5}{2}}x_1(t)=\frac{|x_1(t)+x_2(t)+x_3(t)|}{8\pi^2(1+|x_1(t)+x_2(t)+x_3(t)|)}\\
+\frac{1}{32\pi^2 e}\left(\frac{\sin(x_1(t))+\sin(x_2(t))}{2e^{t+1}}+\sin(x_3(t))\right)\\
+\int_0^t\frac{(t-s)^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)}\left(\begin{array}{c}\frac{|x_1(t)+x_2(t)+x_3(t)|}{8\pi^2(1+|x_1(t)+x_2(t)+x_3(t)|)}\\ +\frac{1}{32\pi^2 e}\left(\frac{\sin(x_1(t))+\sin(x_2(t))}{2e^{t+1}}+\sin(x_3(t))\right)\end{array}\right)ds,\ t\in[0,1],\\
D^{\frac{7}{3}}x_2(t)=\frac{|x_1(t)+x_2(t)+x_3(t)|}{8\pi^3e^2(1+|x_1(t)+x_2(t)+x_3(t)|)}\\
+\frac{t^2}{16\pi^2e^{t^2+1}}\left(\frac{\sin(x_1(t))+\cos(x_2(t))+\cos(x_3(t))}{1+|\sin(x_1(t))+\cos(x_2(t))+\cos(x_3(t))|}\right)\\
+\int_0^t\frac{(t-s)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left(\begin{array}{c}\frac{|x_1(t)+x_2(t)+x_3(t)|}{8\pi^3e^2(1+|x_1(t)+x_2(t)+x_3(t)|)}\\ +\frac{t^2}{16\pi^2e^{t^2+1}}\left(\frac{\sin(x_1(t))+\cos(x_2(t))+\cos(x_3(t))}{1+|\sin(x_1(t))+\cos(x_2(t))+\cos(x_3(t))|}\right)\end{array}\right)ds,\ t\in[0,1],\\
D^{\frac{9}{4}}x_3(t)=\frac{\cos(x_1(t))+\cos(x_2(t))+\cos(x_3(t))}{4\pi e^2}\\
+\frac{1}{16\pi\left(t^2+1\right)}\left(\sin x_1(t)+\frac{|x_2(t)+x_3(t)|}{3\pi^3(1+|x_2(t)+x_3(t)|)}\right)\\
+\int_0^t\frac{(t-s)^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)}\left(\begin{array}{c}\frac{\cos(x_1(t))+\cos(x_2(t))+\cos(x_3(t))}{4\pi e^2}\\ +\frac{1}{16\pi\left(t^2+1\right)}\left(\sin x_1(t)+\frac{|x_2(t)+x_3(t)|}{3\pi^3(1+|x_2(t)+x_3(t)|)}\right)\end{array}\right)ds,\ t\in[0,1],\\
|x_1(0)|+|x_2(0)|+|x_3(0)|=\left|x_1'(0)\right|+\left|x_2'(0)\right|+\left|x_3'(0)\right|=0,\\
x_1''(1)=x_2''(1)=x_3'(1)=0.
\end{array}\right.$$

In this example, we have:
$n=3,\ m=2,\ l=3,\ \alpha_1=\frac{5}{2},\ \alpha_2=\frac{7}{3},\ \alpha_3=\frac{9}{4},\ \delta_1=\frac{7}{2},\ \delta_2=\frac{5}{2},\ \delta_3=\frac{4}{3},\ J=[0,1],$

$$(3.2)\quad \begin{aligned}
g_1^1(t,x_1,x_2,x_3) &= \frac{|x_1+x_2+x_3|}{8\pi^2(1+|x_1+x_2+x_3|)},\\
g_2^1(t,x_1,x_2,x_3) &= \frac{1}{32\pi^2 e}\left(\frac{\sin x_1+\sin x_2}{2e^{t+1}}+\sin x_3\right),
\end{aligned}$$

$$(3.3)\quad \begin{aligned}
g_1^2(t,x_1,x_2,x_3) &= \frac{|x_1+x_2+x_3|}{8\pi^3e^2(1+|x_1+x_2+x_3|)},\\
g_2^2(t,x_1,x_2,x_3) &= \frac{t^2}{16\pi^2e^{t^2+1}}\left(\frac{\sin x_1+\cos x_2+\cos x_3}{1+|\sin x_1+\cos x_2+\cos x_3|}\right),
\end{aligned}$$

$$g_1^3(t,x_1,x_2,x_3) = \frac{\cos x_1+\cos x_2+\cos x_3}{4\pi e^2},$$

$$g_2^3(t,x_1,x_2,x_3) = \frac{1}{16\pi(t^2+1)}\left(\sin x_1+\frac{|x_2+x_3|}{3\pi^3(1+|x_2+x_3|)}\right). \tag{3.4}$$

So, for $t\in[0,1]$ and $(x_1,x_2,x_3),(y_1,y_2,y_3)\in\mathbb{R}^3$, we have:

$$\left|g_1^1(t,x_1,x_2,x_3)-g_1^1(t,y_1,y_2,y_3)\right| \leq$$
$$\frac{1}{8\pi^2}|x_1-y_1|+\frac{1}{8\pi^2}|x_2-y_2|+\frac{1}{8\pi^2}|x_3-y_3|, \tag{3.5}$$

$$\left|g_2^1(t,x_1,x_2,x_3)-g_2^1(t,y_1,y_2,y_3)\right| \leq$$
$$\frac{1}{64\pi^2e^2}|x_1-y_1|+\frac{1}{64\pi^2e^2}|x_2-y_2|+\frac{1}{32\pi^2e}|x_3-y_3|, \tag{3.6}$$

$$\left|g_1^2(t,x_1,x_2,x_3)-g_1^2(t,y_1,y_2,y_3)\right| \leq$$
$$\frac{1}{8\pi^3e^2}|x_1-y_1|+\frac{1}{8\pi^3e^2}|x_2-y_2|+\frac{1}{8\pi^3e^2}|x_3-y_3|, \tag{3.7}$$

$$\left|g_2^2(t,x_1,x_2,x_3)-g_2^2(t,y_1,y_2,y_3)\right| \leq$$
$$\frac{1}{16\pi^2e}|x_1-y_1|+\frac{1}{16\pi^2e}|x_2-y_2|+\frac{1}{16\pi^2e}|x_3-y_3|, \tag{3.8}$$

$$\left|g_1^3(t,x_1,x_2,x_3)-g_1^3(t,y_1,y_2,y_3)\right| \leq$$
$$\frac{1}{4\pi e^2}|x_1-y_1|+\frac{1}{4\pi e^2}|x_2-y_2|+\frac{1}{4\pi e^2}|x_3-y_3|, \tag{3.9}$$

$$\left|g_2^3(t,x_1,x_2,x_3)-g_2^3(t,y_1,y_2,y_3)\right| \leq$$
$$\frac{1}{16\pi}|x_1-y_1|+\frac{1}{48\pi^4}|x_2-y_2|+\frac{1}{48\pi^4}|x_3-y_3|. \tag{3.10}$$

We can take

$$\left(\omega_1^1\right)_1=\left(\omega_1^1\right)_2=\left(\omega_1^1\right)_3=\frac{1}{8\pi^2},\ \left(\omega_2^1\right)_1=\left(\omega_2^1\right)_2=\frac{1}{64\pi^2e^2},\ \left(\omega_2^1\right)_3=\frac{1}{32\pi^2e}, \tag{3.11}$$

$$\left(\omega_1^2\right)_1=\left(\omega_1^2\right)_2=\left(\omega_1^2\right)_3=\frac{1}{8\pi^3e^2},\ \left(\omega_2^2\right)_1=\left(\omega_2^2\right)_2=\left(\omega_2^2\right)_3=\frac{1}{16\pi^2e}, \tag{3.12}$$

$$(3.13)\ \left(\omega_1^3\right)_1 = \left(\omega_1^3\right)_2 = \left(\omega_1^3\right)_3 = \frac{1}{4\pi e^2},\ \left(\omega_2^3\right)_1 = \frac{1}{16\pi},\ \left(\omega_2^3\right)_2 = \left(\omega_2^3\right)_3 = \frac{1}{48\pi^4}.$$

It follows that

$$\Sigma_1 = 0.039589,\ \Sigma_2 = 0.008626,\ \Sigma_3 = 0.052631, \tag{3.14}$$

$$\Delta_1 = 0.302290,\ \Delta_2 = 0.370998,\ \Delta_3 = 0.468782, \tag{3.15}$$

$$\theta_1 = 0.585,\ \theta_2 = 0.419\,96,\ \theta_3 = 0.795\,44. \tag{3.16}$$

Which implies that the condition (2.4) holds

$$\max\left(\Sigma_1\Delta_1 + \theta_1, \Sigma_2\Delta_2 + \theta_2, \Sigma_3\Delta_3 + \theta_3\right) < 1. \tag{3.17}$$

Then by Theorem 2.1, we deduce that the fractional coupled system (3.1) has a unique solution on $[0,1]$.

**Example 3.2.** To illustrate the second main result, we consider the following system:

$$(3.18)\ \begin{cases} D^{\frac{3}{2}}x_1(t) = \frac{\pi e^t}{(t+2)+\sin(x_1(t)+x_2(t)+x_3(t))} + \frac{e^t\cos(x_2(t))}{2\pi+\sin(x_1(t))\sin(x_3(t))} \\ +\int_0^t \frac{(t-s)^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{\pi e^t}{(t+2)+\sin(x_1(t)+x_2(t)+x_3(t))} + \frac{e^t\cos(x_2(t))}{2\pi+\sin(x_1(t))\sin(x_3(t))}\right)ds,\ t\in[0,1], \\ D^{\frac{5}{3}}x_2(t) = \frac{\cos(x_1(t)+x_3(t))}{2\pi e+\cos(x_2(t)+x_3(t))} + \frac{(t+1)\sin(x_2(t)+x_3(t))}{e^{t^2+1}-\cos(x_1(t))} \\ +\int_0^t \frac{(t-s)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{\cos(x_1(t)+x_3(t))}{2\pi e+\cos(x_2(t)+x_3(t))} + \frac{(t+1)\sin(x_2(t)+x_3(t))}{e^{t^2+1}-\cos(x_1(t))}\right)ds,\ t\in[0,1], \\ D^{\frac{9}{5}}x_3(t) = \frac{\cos(x_3(t))}{\pi+t\sin(x_1(t)+x_2(t))} + \sin(x_1(t))\sin(x_2(t)+x_3(t)) \\ +\int_0^t \frac{(t-s)^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)}\left(\frac{\cos(x_3(t))}{\pi+t\sin(x_1(t)+x_2(t))} + \sin(x_1(t))\sin(x_2(t)+x_3(t))\right)ds,\ t\in[0,1], \\ |x_1(0)|+|x_2(0)|+|x_3(0)| = x_1'(1) = x_2'(1) = x_3'(1) = 0. \end{cases}$$

We have:

$n=3$, $m=2$, $l=2$, $\alpha_1=\frac{3}{2}$, $\alpha_2=\frac{5}{3}$, $\alpha_3=\frac{9}{5}$, $\delta_1=\frac{7}{2}$, $\delta_2=\frac{5}{2}$, $\delta_3=\frac{4}{3}$, $J=[0,1]$.
And for $i=1,2$, $k=1,2,3$, the functions $g_i^k$ are continuous.

It is clear that:

$$\left|g_1^1(t,x_1,x_2,x_3)\right| \le \pi e,\ \left|g_2^1(t,x_1,x_2,x_3)\right| \le \frac{e}{2\pi-1}, \tag{3.19}$$

$$\left|g_1^2(t,x_1,x_2,x_3)\right| \le \frac{1}{2\pi e-1},\ \left|g_2^2(t,x_1,x_2,x_3)\right| \le \frac{1}{e-1}, \tag{3.20}$$

$$\left|g_1^3(t,x_1,x_2,x_3)\right| \le \frac{1}{\pi-1},\ \left|g_2^3(t,x_1,x_2,x_3)\right| \le 1. \tag{3.21}$$

Also, the functions $g_i^k$ are also bounded on $[0,1]\times\mathbb{R}^3$. So, by Theorem 2.2, the system (3.18) has at least one solution on $[0,1]$.

## REFERENCES


1. M. A. Abdellaoui, Z. Dahmani and M. Houas: *On some boundary value problems for coupled system of arbitrary order.* Indian Journal of industrial and applied mathematics. Vol. 4(2), (2013), pp. 180–188.
2. M. A. Abdellaoui, Z. Dahmani: *Solvability for nonlinear systems of differential equations with two arbitrary orders.* New Zeland Journal of Mathematics-NZJM. Accepted 2014.
3. A. P. Agrawal, M. Benchora and B. A. Slimani: *Existence results for differential equations with fractional order and impulses.* Mem. Differential Equations Math. Phys. 44, (2008), pp. 1-21.
4. A. Anber, S. Belarbi and Z. Dahmani: *New existence and uniqueness results for fractional differential equations.* An. St. Univ. Ovidius Constanta. Vol. 21(3), (2013), pp. 33-41.
5. D. Baleanu, S. Z. Nazemi and S. Rezapour: *The existence of solution for a k dimensional system of multiterm fractional integrodifferential equations with antiperiodic boundary value problems.* Abstract And Applied Analysis. Vol. 2014, Article ID 896871, (2014), pp. 1-13.
6. S. Belarbi, Z. Dahmani: *Existence of solutions for multi-points fractional evolution equations.* Appl. Math., Vol. 9, (2014), pp. 416-427.
7. Y. Cao and C. Bai: *Existence results for a coupled system of nonlinear fractional differential equations in Banach spaces.* Journal of Applied Mathematics. Vol. 2014, Article ID. 463148, (2014), pp. 1-9.
8. Z. Cui, P. Yu and Z. Mao: *Existence of solutions for nonlocal boundary value problems of nonlinear fractional differential equations.* Advances in Dynamical Systems and Applications. Vol. 7(1), (2012), pp. 31-40.
9. Z. Dahmani, M. A. Abdellaoui: *New results for a weighted nonlinear system fractional integro-differential equations.* Facta Universitatis (NIS). Ser. Math. Inform. Vol. 29(3), (2014),pp. 233-242.
10. Z. Dahmani, M. A. Abdellaoui and M. Houas: *Coupled systems of fractional integro-differential equations involving several functions.* Theory and Applications of Mathematics & Computer Science. Vol. 5(1), (2015), pp. 53-61.
11. Z. Dahmani, L. Tabharit: *Fractional order differential equations involving Caputo derivative.* Theory and Applications of Mathematics & Computer Science. Vol. 4(1), (2014), pp. 40-55.
12. Z. Dahmani, A. Taïeb: *The hight order Lane-Emden fractional differential system: Existence, uniqueness and Ulam stabilities.* Kragujevac Journal of Mathematics., To Appear In 2015.
13. Z. Dahmani, A. Taïeb: *Solvability of a coupled system of fractional differential equations with periodic and antiperiodic boundary conditions.* PALM Letters., To Appear In 2015.
14. Z. Dahmani, A. Taïeb: *A coupled system of fractional differential equations involing two fractional orders.* To Appear in 2015.
15. Z. Dahmani, A. Taïeb: *New existence and uniqueness results for high dimensional fractional differential systems.* Facta Universitatis (NIS). Ser. Math. Inform., Vol. 30, No 3, (2015), pp. 281–293.

16. Z. Dahmani, A. Taïeb: *Solvability for high dimensional fractional differential systems with high arbitrary Orders.* Journal of Advanced Scientific Research in Dynamical and Control Systems., Vol. 7, 4, (2015), pp. 51 - 64.

17. M. Fečkan, J. Wang and Y. Zhou: *Ulam's type stability of impulsive ordinary differential equations.* J. Math. Anal. Appl. 395, (2012), pp. 258-264.

18. V. Ga ychuk, B. Datsko and V. Meleshko: *Mathematical modeling of time fractional reaction diffusion systems.* Journal Of Computational And Applied Mathematics. Vol. 220(1-2), (2008), pp. 215-225.

19. R. Hilfer: *Applications of fractional calculus in physics.* World Scientific. River Edge. NJ. USA. (2000).

20. R. Hilfer, R. Metzler, A. Blumen, J. Klafter: *Strange kinetics.* (Special Issue). Chemical Physics. Vol. 284(1-2), 2002.

21. M. Houas, Z. Dahmani: *New results for a system of two fractional differential equations involving $n$ Caputo derivatives.* Kragujevac Journal of Mathematics. Vol. 38(2), (2014), pp. 283-301.

22. R. W. Ibrahim: *Stability of a fractional differential equation.* International Journal of Mathematical. Computational. Physical and Quantum Engineering. Vol. 7(3), (2013), pp. 300-305.

23. R. W. Ibrahim: *Ulam stability of boundary value problem.* Kragujevac Journal of Mathematics. Vol. 37( 2), (2013), pp. 287-297.

24. W.H. Jiang: *Solvability for a coupled system of fractional differential equations at resonance.* Nonlinear Analysis: Real World Applications. 13, (2012), pp. 2285-2292.

25. A. A. Kilbas, S. A. Marzan: *Nonlinear differential equation with the Caputo fraction derivative in the space of continuously differentiable functions.* Differ. Equ. Vol. 41(1), (2005), pp. 84-89.

26. V. Lakshmikantham, S. Leela, and J. V. Devi: *Theory of fractional dynamic systems.* Cambridge Scientific Publishers. 2009.

27. M. Li, Y. Liu: *Existence and uniqueness of positive solutions for a coupled system of nonlinear fractional differential equations.* Open Journal of Applied Sciences. 3, (2013), pp. 53-61.

28. F. Mainardi: *Fractional calculus-Some basic problem in continuum and statistical mechanics. Fractals and fractional calculus in continuum mechanics.* Springer, Vienna. (1997), pp. 291-348.

29. K. S. Miller and B. Ross: *An Introduction to the fractional calculus and fractional differential equations.* John Wiley & Sons. New York. NY. USA. (1993).

30. P. Nasertayoob, S. M. Vaezpour and D. Baleanu: *Solvability for a coupled system of fractional integrodifferential equations with $m-$point boundary conditions on the Half-Line.* Abstract and Applied Analysis. Vol. 2014, Article ID. 297314, (2014), pp. 1-7.

31. N. Octavia: *Nonlocal initial value problems for first order differential systems.* Fixed Point Theory. Vol. 22(1), (2009), pp. 64-69.

32. M. D. Ortigueira, J. A. T. Machado: *Fractional calculus applications in signals and systems.* Signal Processing. Vol. 86(10), (2006), pp. 2503-3094.

33. I. Podlubny: *Fractional differential equations.* Academic Press. New York. (1999).
34. D. R. Smart: *Fixed point theorems.* Cambridge University Press. Cambridge. 31, 1980.
35. A. Taïeb, Z. Dahmani: *A coupled system of nonlinear differential equations involing m nonlinear terms.* Georgian Mathematical Journal. Accepted 2015.
36. A. Taïeb, Z. Dahmani: *A new problem of singular fractional differential equations.* Journal of Dynamical Systems and Geometric Theories. Accepted 2016.
37. J. Wang, L. Lv and Y. Zhou: *Ulam stability and data dependence for fractional differential equations with Caputo derivative.* Electronic Journal of Qualitative Theory of Differential Equations. 63, (2011), pp. 1-10.
38. C. X. Zhu, X. Zhang and Z. Q. Wu: *Solvablity for a coupled system of fractional differential equations with integer boundary conditions.* Taiwanese Journal of Mathematics. Vol. 17(6), (2013), pp. 2039-2054.

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