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ON FINSLER SPACES WITH A QUATRIC METRIC

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Abstract. The so-called cubic $L^3 = a_{ijk}(x)y^iy^jy^k$ metric on a differential manifold with the local coordinates x^i has been defined by M.Matsumoto in the year 1979 [8]. In the paper, he has worked out the necessary and sufficient condition (n.a.s.c) for two and three dimensional Finsler space in terms of main scalars in order that the Finsler space is a with cubic metric. On the lines of cubic metric many authors have studied quartic metric as an example of Finsler metric. In the present paper we have work out the n.a.s.c in terms of main scalars of two and three dimensional Finsler space with quartic metric.

Keywords: Quartic Metric, Main Scalars, Two and three dimensional Finsler spaces

1. Introduction

The quartic metric on a differentiable manifold with local coordinates x is defined by

$$L^4(x,y) = a_{ijkl}(x)y^iy^jy^ky^l \tag{1.1}$$

where, $a_{ijkl}(x)$ are components of a symmetric tensor field of $(0,4)-$ type depending on the position x alone.

A Finsler space with a quartic metric is called the quartic Finsler space.

Many authors have studied Finsler spaces with cubic metric in the papers ([10, 4, 1, 5, 6, 2]). Quartic metric Finsler space has also been studied in the papers ([9, 7, 8, 3]), from the point of view different from what we are going to do in this paper. The purpose of the present paper is to study spaces with a quartic metric from the stand point of Finsler geometry.

M. Matsumoto and S. Numata in the paper [8] has studied cubic metric and found out the condition for a two dimensional Finsler space to be with a cubic metric. A necessary and sufficient condition has also been worked out on three dimensional cubic Finsler space in terms of the scalars of cubic Finsler space such that the three dimensional Finsler space is with a cubic metric. On the lines of the paper[7], we have tried to study a Finsler space with quartic metric.

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§2 is devoted to developing a fundamental treatment of quartic Finsler spaces and a characterization of such Finsler spaces is given in terms of well known tensors of Finsler geometry.

The fundamental treatment of quartic metric Finsler spaces and it's characterization is developed in Section 1, in terms of well known tensors of Finsler geometry and a proposition has been obtained. In Section 3, a theorem has been obtained for a two dimensional Finsler space and Section 4, deals with a three dimensional Finsler space. In the last in Section 5, we have obtained condition for a metric to be quartic metric Finsler space in terms of well known T-tensor ($\sharp eq^n(28.20)$).

Throughout the paper we shall confine ourselves to Cartan's connection, and the notations and terminology of the monograph [3] will be used without comment. In the paper monograph of M.Matsumoto[8] will be quoted by ($\sharp$).

2. Characterization of quartic metric

We consider an n-dimensional Finsler space F^n with a quatric metric defined by $L(x,y)$ in (1.1) Let us first define the tensors $a_{ijk}(x,y), a_{ij}(x,y)$ and $a_i(x,y)$ as follows:

$$(2.1)\qquad \begin{aligned} (i)\quad La_{ijk}(x,y) &= a_{ijkl}(x)y^l \\ (ii)\quad L^2a_{ij}(x,y) &= a_{ijkl}(x)y^ky^l \\ (iii)\quad L^3a_i(x,y) &= a_{ijkl}(x)y^jy^ky^l \end{aligned}$$

Some basic tensors used in Finsler geometry for the said metric :

the normalized supporting element $\quad l_i = \dfrac{\partial L}{\partial y^i}$,

the angular metric tensor $\quad h_{ij} = L\dfrac{\partial^2 L}{\partial y^i \partial y^j}$,

and the fundamental tensor $\quad g_{ij} = \frac{1}{2}\dfrac{\partial^2 L^2}{\partial y^i \partial y^j} = h_{ij} + l_i l_j$

For a quartic Finsler space these basic tensors have been obtained by using equation (2.1),

$$(2.2)\qquad i)\quad l_i = a_i \qquad ii)\quad h_{ij} = 3(a_{ij} - a_i a_j) \qquad iii)\quad g_{ij} = 3a_{ij} - 2a_i a_j$$

Detail calculations of above are given below.

Differentiating equation (1.1) with respect to(w.r.t)y^p, we get

$$L^3\frac{\partial L}{\partial y^p} = a_{pjkl}y^jy^ky^l$$

$$\Rightarrow \qquad l_p = \frac{\partial L}{\partial y^p} = a_{pjkl}\frac{y^j}{L}\frac{y^k}{L}\frac{y^l}{L}, \qquad i.e. l_p = a_p$$

$$\Rightarrow \qquad l_i = a_i$$

Differentiating again above equation w.r.t. y^q, we get

$$3L^2 a_p a_q + L^3 \frac{\partial^2 L}{\partial y^p \partial y^q} = 3a_{pqkl} y^k y^l$$

With the help of $\frac{\partial L}{\partial y^p} = l_p = a_p$ and $h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j}, (\because p \to i, q \to j)$ we get

$$\Rightarrow \qquad L^2 h_{ij} + 3L^2 a_i a_j = 3a_{ijkl} y^k y^l$$

$$\Rightarrow \qquad h_{ij} = 3(a_{ij} - a_i a_j)$$

Since, $g_{ij} = h_{ij} + l_i l_j = 3a_{ij} - 2a_i a_j$

Let us call $a_{ij}(x, y)$ the basic tensor, because this play an important role in the development of Quartic Finsler space.
The metric L is called regular, if the basic tensor has the non-vanishing determinant. Throughout our discussion of quartic metric we should suppose the regularity of the metrics.

A quatric Finsler space in some domain of the space is called regular, if the intrinsic metric tensor has non-vanishing determinant.

If a^{ij} is the inverse of tensor a_{ij} then the inverse g^{ij} of fundamental metric tensor g_{ij} for a quartic metric can be obtained as follows,

Since, $g^{ij} g_{jk} = \delta^i_k$

writing the value of g_{jk} in above, we have

$$g^{ij}(3a_{jk} - 2a_j a_k) = \delta^i_k \tag{2.3}$$

multiplying by a^{kl} on both side of above equation, we get

$$3g^{ij} a_{jk} a^{kl} - 2g^{ij} a_j a_k a^{kl} = \delta^i_k a^{kl}$$

$$3g^{il} = a^{il} + 2g^{ij} a_j a_k a^{kl} = a^{il} + 2g^{ij} a_j a^l \tag{2.4}$$

multiplying equation (2.3) by a^k and summing over k , we get

$$3g^{ij} a_j - 2g^{ij} a_j a^2 = a^i$$

this implies that

$$a_j g^{ij} = \frac{a^i}{3 - 2a^2} \tag{2.5}$$

Putting the value from equation (2.5) in equation (2.4), we have

$$\Rightarrow \qquad g^{ij} = \frac{1}{3}\left(a^{ij} + \frac{2a^i a^j}{3-2a^2}\right)$$

$$g^{ij} = \frac{a^{ij} + 2a^i a^j}{3} \qquad (\because a^2 = a_i a^i = l_i l^i = 1) \tag{2.6}$$

Since, $\quad h_{ij} = L\dfrac{\partial l_i}{\partial y^j} = L\dfrac{\partial a_i}{\partial y^j} = 3(a_{ij} - a_i a_j), \quad L^2\dfrac{\partial a_{ij}}{\partial y^k} = 2(a_{ijk}y^l - a_{ij}a_k)$

Differentiating g_{ij}, as given in (2.2)(iii), we have Cartan torsion tensor C_{ijk},

$$C_{ijk} = \frac{1}{2}(\frac{\partial g_{ij}}{\partial y^k}) = \frac{1}{2}\frac{\partial(3a_{ij} - 2a_i a_j)}{\partial y^k}$$

$$= \frac{3}{2}\frac{\partial a_{ij}}{\partial y^k} - (\frac{\partial a_i}{\partial y^k}a_j + a_i\frac{\partial a_j}{\partial y^k})$$

$$= \frac{3}{L^2}(a_{ijkl}y^l - La_k a_{ij}) - \frac{3}{L}(a_{ik} - a_i a_k)a_j - \frac{3}{L}(a_{jk} - a_j a_k)a_i$$

Thus,

$$LC_{ijk} = 3\left\{a_{ijk} - (a_{ij}a_k + a_{jk}a_i + a_{ki}a_j) + 2a_i a_j a_k\right\} \tag{2.7}$$

Since L^4 given in equation (1.1) is homogeneous of degree four in y^i and its fourth derivative w.r.t y^i will be function of x alone, due to fact its fifth derivative will be zero with respect to y^i,which is necessary for a Finsler space is with a quartic metric.

Thus, we have the proposition

Proposition 2.1. *A Finsler space is one with a quartic metric if*

$$L^4_{ijklm} = \frac{\partial^5 L^4}{\partial y^i \partial y^j \partial y^k \partial y^l \partial y^m} = \frac{\partial}{\partial y^m}a_{ijkl}(x) = 0$$

i.e. $\quad 2[L^2 L^2_{ijklm} + \odot_{(ijklm)}L^2_{ijkl}L^2_m + \odot_{(ijklm)}L^2_{ijk}L^2_{lm} + \odot_{(ikmjl)}L^2_{ikm}L^2_{jl}] = 0$

because $\qquad L^4 = a_{ijkl}(x)y^i y^j y^k y^l$

which can also be written as,

$$L^4(x,y) = L^2(x,y)L^2(x,y), \qquad \textit{where } L^2(x,y) = g_{ij}y^i y^j$$

On differentiating above equation five times, we get

$$L^4_{ijklm} = (L^2L^2)_{ijklm} = \frac{\partial^5 L^4}{\partial y^i \partial y^j \partial y^k \partial y^l \partial y^m} \tag{2.8}$$

$$= 2[L^2 L^2_{ijklm} + \odot_{(ijklm)}L^2_{ijkl}L^2_m + \odot_{(ijklm)}L^2_{ijk}L^2_{lm} + \odot_{(ikmjl)}L^2_{ikm}L^2_{jl}] = 0$$

where, $\odot_{(ijklm)}$ *stands for cyclic sum in i,j,k,l,m and*

$$\odot_{(ijklm)} L^2_{ijkl} L^2_m = L^2_{ijkl} L^2_m + L^2_{jklm} L^2_i + L^2_{klmi} L^j_m + L^2_{lmij} L^2_k + L^2_{mijk} L^2_l$$

$$\odot_{(ijklm)}\, L^2_{ijk} L^2_{lm} = L^2_{ijk} L^2_{lm} + L^2_{jkl} L^2_{mi} + L^2_{klm} L^2_{ij} + L^2_{lmi} L^2_{jk} + L^2_{mij} L^2_{kl}$$

$$\odot_{(ikmjl)}\, L^2_{ikm} L^2_{jl} = L^2_{ikm} L^2_{jl} + L^2_{kmj} L^2_{li} + L^2_{mjl} L^2_{ik} + L^2_{jli} L^2_{km} + L^2_{lik} L^2_{mj}$$

3. Characterization of quartic metric in two dimensional Finsler space

In a strongly non-Riemannian Finsler space F^2([8, 2, 3, 11]), refer to the Miron Moor frame (e_1, e_2) in this case is Berwald frame(l^i, m^i) and the main scalar I as given below *:*

(3.1) $$L^2_i = \frac{\partial L^2}{\partial y^i} = 2Ll_i, \qquad (\because l_i = \frac{\partial L}{\partial y^i})$$

(3.2) $$L^2_{ij} = \frac{\partial^2 L^2}{\partial y^i \partial y^j} = 2(l_i l_j + h_{ij}) = 2(l_i l_j + m_i m_j)$$

$$(\because h_{ij} = L\frac{\partial l_i}{\partial y^j} = g_{ij} - l_i l_j, g_{ij} = l_i l_j + m_i m_j)$$

(3.3) $$L^2_{ijk} = \frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k} = 4C_{ijk} = \frac{4I}{L} m_i m_j m_k = \frac{4I}{L} m_{ijk}, \quad (\sharp eq^n(28.3))$$

where, we have put $\quad m_{ijk} = m_i m_j m_k.$

(3.4) $$L^2_{ijkl} = 4(\frac{I_l}{L} - \frac{I}{L^2} l_l) m_{ijk} + 12\frac{I^2}{L^2} m_{ijklm} - 4\frac{I}{L^2}(l_i m_{jkl} + l_j m_{ikl} + l_k m_{ijl})$$

$$\Rightarrow \quad \frac{1}{4} L^2_{ijkl} = \frac{1}{L}(I_l m_{ijk} + 3\tfrac{I^2}{L} m_{ijklm} - \tfrac{I}{L} \odot_{(ijkl)} l_i m_{jkl})$$

$$(\because L\frac{\partial m_i}{\partial y^l} = I m_{il} - l_i m_l \textit{ and using } \frac{\partial I}{\partial y^l} = I_l)$$

(3.5) $$\frac{1}{4} L^2 L^2_{ijklm} = 4\frac{I}{L}(3I^2 - 1) m_{ijklm} + L I_{lm} m_{ijk} + I_m (6I m_{ijkl} - \odot_{(ijkl)} l_i m_{jkl})$$

$$+ I_l (3I m_{ijkm} - \odot_{(ijkm)} l_i m_{jkm}) - 6\frac{I^2}{L} \odot_{(ijklm)} l_i m_{jklm}$$

$$+ 2\frac{I}{L}(\odot_{(ijklm)} l_{ij} m_{klm} + \odot_{(ikmjl)} l_{ik} m_{mjl})$$

where $I_{lm} = \dfrac{\partial^2 I}{\partial y^l \partial y^m}, I_l = \dfrac{\partial I}{\partial y^l}$ and $m_{ijklm} = m_i m_j m_k m_l m_m$

Substituting the values from (3.1), (3.2), (3.3), (3.4) and (3.5) in equation (2.8), we get

$$L^4_{ijklm} = 8[4\frac{I}{L}(3I^2+4)m_{ijklm} + LI_{lm}m_{ijk} + I_m(6Im_{ijkl} - \odot_{(ijkl)} l_i m_{jkl})$$
$$+ I_l(3Im_{ijkm} - \odot_{(ijkm)} l_i m_{jkm} + 2\odot_{(ijklm)} l_m m_{ijk})$$
$$+ 2\tfrac{I}{L}(\odot_{(ijklm)} l_{ij} m_{klm} + \odot_{(ikmjl)} l_{ik} m_{mjl})]$$

Thus from proposition (2.1), we have

Theorem 3.1. *A two dimensional Finsler space with a quartic metric is characterized by the equation*

$$L^4_{ijklm} = 8[4\frac{I}{L}(3I^2+4)m_{ijklm} + LI_{lm}m_{ijk} + I_m(6Im_{ijkl} - \odot_{(ijkl)} l_i m_{jkl}) \tag{3.6}$$
$$+I_l(3Im_{ijkm} - \odot_{(ijkm)} l_i m_{jkm} + 2\odot_{(ijklm)} l_m m_{ijk})$$
$$+ 2\tfrac{I}{L}(\odot_{(ijklm)} l_{ij} m_{klm} + \odot_{(ikmjl)} l_{ik} m_{mjl}) = 0$$

Since, $\quad L\dfrac{\partial I}{\partial y^l} \quad = I_{;\alpha} e_{\alpha)l} \quad = I_{;1} l_l + I_{;2} m_l \qquad (\sharp eq^n (28.7))$

Multiplying both side by y^l in above equation, we get

$$0 = I_{;1} l_i y^i + 0 \quad \Rightarrow \quad I_{;1} L = 0 \quad \Rightarrow \quad I_{;1} = 0$$

Hence, $\qquad L\dfrac{\partial I}{\partial y^l} = I_{;2} m_l$

Differentiating above equation w.r.t. y^m, we get

$$L\frac{\partial^2 I}{\partial y^l \partial y^m} + l_m \frac{\partial I}{\partial y^l} \quad = \frac{\partial I_{;2}}{\partial y^m} m_l + \frac{\partial m_l}{\partial y^m} I_{;2} \quad = I_{;2;2} m_m \frac{m_l}{L} + \frac{I_{;2}}{L}(Im_l m_m - l_l m_m)$$

$$L\frac{\partial^2 I}{\partial y^l \partial y^m} \quad = (II_{;2} + I_{;2;2})\frac{m_l m_m}{L} - \frac{I_{;2}}{L}(l_l m_m + l_m m_l)$$

Using these results in equation (3.6), we get

$$L^4_{ijklm} = 8[4\frac{I}{L}(3I^2+4)m_{ijklm} + \{(II_{;2} + I_{;2;2})\frac{m_l m_m}{L} - \frac{I_{;2}}{L}(l_l m_m + l_m m_l)\} m_{ijk}$$
$$+ \frac{1}{L} I_{;2} m_m (6Im_{ijkl} - \odot_{(ijkl)} l_i m_{jkl}) + \frac{1}{L} I_{;2} m_l (3Im_{ijkm} - \odot_{(ijkm)} l_i m_{jkm}$$
$$+ 2\odot_{(ijklm)} l_m m_{ijk}) + 2\frac{I}{L}(\odot_{(ijklm)} l_{ij} m_{klm} + \odot_{(ikmjl)} l_{ik} m_{mjl})]$$

On contracting with $m^i m^j m^k m^l m^m$, we get

$$12I^3 + 16I + 10II_{;2} + I_{;2;2} = 0$$

Corollary 3.1. *A strongly non-Riemannian Finsler space of dimension two is with quartic metric, iff the condition*

$$12I^3 + 16I + 10II_{;2} + I_{;2;2} = 0,$$

holds good. where $I_{;2}$ and $I_{;2;2}$ are v-derivatives scalar of main scalar I as defined by M.Matsumoto *in* ($\sharp eq^n(28.7)$).

4. Characterization of quartic metric in three dimensional Finsler space with vanishing T tensor

In a strongly non-Riemannian Finsler space $F^3([10,\ 8,\ 2,\ 3], \sharp\S 29)$, we can refer to the Moor frame (l^i, m^i, n^i), main scalar H, I, J and v-connection vector v_i as given below :

$$(4.1) \qquad L^4_i = 4L^3 l_i \qquad (\because \frac{\partial L}{\partial y^i} = l_i)$$

$$(4.2) \qquad L^4_{ij} = 4(3L^2 l_i l_j + L^2 h_{ij}) = 4L^2(g_{ij} + 2l_i l_j) \qquad (\because h_{ij} = g_{ij} - l_i l_j)$$

$$(4.3) \quad L^4_{ijk} = 8L(LC_{ijk} + \odot_{(ijk)} g_{ij} l_k) = 8L(Hm_{ijk} - J \odot_{(ijk)} m_{ij} n_k + I \odot_{(ijk)} m_i n_{jk} + Jn_{ijk} + 3l_{ijk} + \odot_{(ijk)} m_{ij} l_k + \odot_{(ijk)} n_{ij} l_k)$$

where, $\quad m_{ijk} = m_i m_j m_k, \quad l_{ijk} = l_i l_j l_k, \quad m_{ij} n_k = m_i m_j n_k,$

$$LC_{ijk} = Hm_{ijk} - J \odot_{(ijk)} m_{ij} n_k + I \odot_{(ijk)} m_i n_{jk} + Jn_{ijk},$$

$$\odot_{(ijk)} g_{ij} l_k = 3l_{ijk} + \odot_{(ijk)} m_{ij} l_k + \odot_{(ijk)} n_{ij} l_k$$

Since, $L\frac{\partial m_i}{\partial y^l} = Hm_{il} - J(m_i n_l + m_l n_i) + In_{il} - (l_i m_l + n_i v_l)$

and $L\frac{\partial n_i}{\partial y^l} = J(n_{il} - m_{il}) + I(m_i n_l + m_i n_l) - (l_i m_l + m_i v_l)$

We consider a three dimensional Finsler space for which derivative of scalars H, I, J and v-connection vector v_i are zero. Actually, if H, I, J and v_i are zero then T-tensor T_{hijk} will be zero. ($\sharp eq^n(29.22^{'})$

So that, $L\dfrac{\partial m_i}{\partial y^l} = Hm_{il} - J(m_i n_l + m_l n_i) + In_{il} - l_i m_l$

and $L\dfrac{\partial n_i}{\partial y^l} = J(n_{il} - m_{il}) + I(m_i n_l + m_i n_l) - l_i m_l$

(4.4) $$\frac{1}{8}L^4_{ijkl} = 3(H^2 + J^2 + 1)m_{ijkl} + 3(J^2 + I^2 + 1)n_{ijkl} + 3l_{ijkl}$$

$-3J(H + I) \odot_{(ijkl)} m_{ijk}n_l + H \odot_{(ijkl)} m_{ijl}l_k + J \odot_{(ijkl)} n_{ijl}l_k + (J^2 + 2I^2 + HI + 1)\{\odot_{(ijkl)}m_{ij}n_{kl}+m_{ik}n_{jl}+m_{jl}n_{ik}\}+\{\odot_{(ijkl)}m_{ij}l_{kl}+m_{ik}l_{jl}+m_{jl}l_{ik}\}+\{\odot_{(ijkl)}n_{ij}l_{kl}+n_{ik}l_{jl}+n_{jl}l_{ik}\}-J\odot_{(ijkl)}\{m_{ij}n_k l_l+m_{jk}n_i l_l+m_{ki}n_j l_l\}+I\odot_{(ijkl)}\{n_{ij}m_k l_l+m_{jk}n_i l_l+m_{ki}n_j l_l\}$

where, $L\dfrac{\partial m_{ijk}}{\partial y^l} = 3(Hm_{ijkl} - J \odot_{(ijkl)} m_{ikl}n_j + I \odot_{(ijk)} n_{il}m_{jk} - \odot_{(ijk)} l_i m_{jkl})$,

$L\dfrac{\partial \odot_{(ijk)} m_{ij}n_k}{\partial y^l} = -3Jm_{ijkl}+3Im_{ijk}n_l+(2H+I)\odot_{(ijk)} m_{ijl}n_k-J\odot_{(ijk)} m_{ij}n_{kl}$
$-2J\odot_{(ijk)} m_{il}n_{jk}+2I\odot_{(ijk)} m_j n_{ikl}-\odot_{(ijk)}l_i n_k m_{jl}-\odot_{(ijk)}l_j n_k m_{il}-\odot_{(ijk)}l_k n_l m_{ij}$,

$L\dfrac{\partial \odot_{(ijk)} m_k n_{ij}}{\partial y^l} = 3In_{ijkl}-3Jn_{ijk}m_l+(2I+H)\odot_{(ijk)} m_{kl}n_{ij}+J\odot_{(ijk)} m_k n_{ijl}$
$-2J\odot_{(ijk)} m_{ikl}n_j+2I\odot_{(ijk)} m_{ik}n_{jl}-\odot_{(ijk)}l_i m_k n_{jl}-\odot_{(ijk)}l_j m_k n_{il}-\odot_{(ijk)}l_k m_l n_{ij}$,

$L\dfrac{\partial n_{ijk}}{\partial y^l} = 3(Im_l n_{ijk} + Jn_{ijkl}) - J \odot_{(ijk)} m_{il}n_{jk} + I \odot_{(ijk)} m_i n_{jkl} - \odot_{(ijk)} l_i n_{jkl}$

and $\dfrac{\partial g_{ij}l_k}{\partial y^l} = 2(Hm_{ijl} + Jn_{ijl} - J \odot_{(ijl)} m_{ij}n_l + I \odot_{(ijl)} m_i n_{jl})l_k + m_{ijkl} + n_{ijkl} + l_{ij}(m_{kl} + n_{kl}) + n_{ij}m_{kl} + m_{ij}n_{kl}$

(4.5) $$\frac{1}{8}LL^4_{ijklm} = 4\{3H(H^2 + J^2 + 1) + 3J^2(H + I) + 4H\}m_{ijklm}$$

$+4\{3J(J^2 + I^2 + 1) + 4J\}n_{ijklm} - 2J\{6(H^2 + J^2 + 1) + 6(H + I) - (J^2 + 2I^2 + HI + 1)\} \odot_{(ijklm)} m_{ijkl}n_m + 2I\{6(J^2 + I^2 + 1) + (J^2 + 2I^2 + HI + 1)\} \odot_{(ijklm)} n_{ijkl}m_m\{3I(H^2 + J^2 + 1) + 6J^2(H + I) + H + 2I(J^2 + 2I^2 + HI + 1)\} \odot_{(ijklm)} m_{jkl}n_{im} - \{3J(J^2+I^2+1)+2J(J^2+2I^2+HI+1)\}\odot_{(ijklm)} n_{jkl}m_{im}+\{J+(2H+I)(J^2+2I^2+HI+1)\}\odot_{(ijklm)} m_{ikm}n_{jl}-\{I+3J(J^2+2I^2+HI+1)\}\odot_{(ijklm)} n_{ilm}m_{jk}$

On contracting, we get

(1) $3H(H^2+J^2+1)+3J^2(H+I)+4H=0$

(2) $3J(J^2+I^2+1)+4J=0$

(3) $2J\{6(H^2+J^2+1)+6(H+I)-(J^2+2I^2+HI+1)\}=0$

(4) $2I\{6(J^2+I^2+1)+(J^2+2I^2+HI+1)\}=0$

(5) $3I(H^2+J^2+1)+6J^2(H+I)+H+2I(J^2+2I^2+HI+1)=0$

(6) $3J(J^2+I^2+1)+2J(J^2+2I^2+HI+1)=0$

(7) $J+(2H+I)(J^2+2I^2+HI+1)=0$

(8) $I+3J(J^2+2I^2+HI+1)=0$

Only solution of above set of equations is,

$$H=J=I=0$$

i.e. the space is Riemannian.

Thus, we have

Theorem 4.1. *For a three dimensional quartic metric Finsler space if T-tensor vanishes then it is Riemannian.*

5. Characterization of quartic metrics in terms of T-tensor

An n-dimensional Finsler space F^n with a quartic metric $L(x,y)$ defined by (1.1). To find out the fifth derivative of quartic metrics in terms of T-tensor, we have

$$L_i^4=2L^2L_i^2$$
$$L_{ij}^4=2(L_i^2L_j^2+L^2L_{ij}^2)$$
$$L_{ijk}^4=2(L_{jk}^2L_i^2+L_j^2L_{ik}^2+L_k^2L_{ij}^2+L^2L_{ijk}^2)$$
$$L_{ijkl}^4=2(L^2L_{ijkl}^2+\odot_{(ijkl)}L_i^2L_{jkl}^2+L_{ij}^2L_{kl}^2+L_{jk}^2L_{il}^2+L_{ki}^2L_{jl}^2)$$

Since, we know that

$$L_i^2=2Ll_i,\quad L_{ij}^2=2g_{ij}=2(l_il_j+h_{ij}),\quad L_{ijk}^2=4C_{ijk},$$

$$L_{ijkl}^2=4\frac{\partial C_{ijk}}{\partial y^l}=4(C_{ijk}|_l+C_{rjk}C_{il}^r+C_{rik}C_{jl}^r+C_{rij}C_{kl}^r)$$

Also, $$T_{ijkl}=LC_{ijk}|_l+l_iC_{jkl}+l_jC_{kli}+l_kC_{lij}+l_lC_{ijk}\quad(\sharp eq^n(28.20))$$

i.e. $$LC_{ijk}|_l=T_{ijkl}-\odot_{(ijkl)}l_iC_{jkl}$$

So that,

$$L^4_{ijkl} = 2\{4L^2(C_{ijk}|_l + C_{rjk}C^r_{il} + C_{rik}C^r_{jl} + C_{rij}C^r_{kl}) + \odot_{(ijkl)}2Ll_i4C_{jkl} + 4(g_{ij}g_{kl} + g_{jk}g_{il} + g_{kl}g_{ij})\}$$

$$\Rightarrow \quad L^4_{ijkl} = 8\{L(T_{ijkl} + \odot_{(ijkl)}l_iC_{jkl}) + L^2(C^r_{il}C_{rjk} + C^r_{jl}C_{rik} + C^r_{kl}C_{rij}) + (g_{ij}g_{kl} + g_{jk}g_{il} + g_{kl}g_{ij})\}$$

$$\frac{1}{8}L^4_{ijklm} = l_m(T_{ijkl} + \odot_{(ijkl)}l_iC_{jkl}) + 2Ll_m(C^r_{il}C_{rjk} + C^r_{jl}C_{rik} + C^r_{kl}C_{rij}) + L(T_{ijkl}|_m - \odot_{(ijkl)}T_{rjkl}C^r_{im}) + \odot_{(ijkl)}(g_{im} - l_il_m)C_{jkl} + \odot_{(ijkl)}l_i\{(T_{jklm} - l_jC_{klm} - l_kC_{jlm} - l_lC_{jkm} - l_mC_{jkl}) + L(C_{rkl}C^r_{jm} + C_{rlj}C^r_{km} + C_{rjk}C^r_{lm})\} + \odot_{(ijk)}C^r_{il}\{L(T_{rjkm} - l_rC_{jkm} - l_jC_{kmr} - l_kC_{mrj} - l_mC_{rjk}) + L^2(C_{prk}C^p_{jm} + C_{pjr}C^p_{km+C_{pjk}C^p_{rm}})\} + \odot_{(ijk)}C_{rjk}\{L(T^r_{ilm} + l^rC_{ilm} - l_iC^r_{lm} - l_lC^r_{im} - l_mC^r_{il}) + L^2(C^r_{pl}C^p_{im} + C^r_{pi}C^p_{lm} - C^p_{il}C^r_{pm})\} + 2(g_{ij}C_{klm} + g_{jk}C_{lmi} + g_{kl}C_{mij})$$

$$\textit{where,} \quad \frac{\partial T_{ijkl}}{\partial y^m} = T_{ijkl}|_m - T_{rjkl}C^r_{im} - T_{rkli}C^r_{jm} - T_{rlij}C^r_{km} - T_{rijk}C^r_{lm}$$

$$\Rightarrow \quad \frac{\partial T_{ijkl}}{\partial y^m} = T_{ijkl}|_m - \odot_{(ijkl)}T_{rjkm}C^r_{im}$$

$$\frac{\partial C_{jkl}}{\partial y^m} = C_{jkl}|_m + \odot_{(jkl)}C_{rjk}C^r_{lm} = \frac{1}{L}(T_{jklm} - \odot_{(jklm)}l_mC_{jkl}) + \odot_{(jkl)}C_{rjk}C^r_{lm}$$

$$L\frac{\partial l_i}{\partial y^m} = h_{im} = g_{im} - l_il_m$$

$$\frac{\partial C^r_{il}}{\partial y^m} = C^r_{il}|_m - C^p_{il}C^r_{pm} + C^r_{pl}C^p_{im} + C^r_{pi}C^p_{lm} \quad \textit{and}$$

$$C^r_{il}|_m = \frac{1}{L}(T^r_{ilm} + l^rC_{ilm} - l_iC^r_{lm} - l_lC^r_{im} - l_mC^r_{il})$$

Hence,

$$L^4_{ijklm} = 8[LT_{ijkl|m} + \odot_{(ijklm)}T_{ijkl}l_m + \odot_{(ijklm)}g_{ij}C_{klm} + \odot_{(ikmjl)}g_{ik}C_{jlm} + 2Ll_m(C^r_{il}C_{rjk} + C^r_{jl}C_{rik} + C^r_{kl}C_{rij}) - L(\odot_{(ijkl)}T_{rjkl}C^r_{im}) - \odot_{(ijkl)}l_i\{(l_jC_{klm} + l_kC_{jlm} + l_lC_{jkm} + l_mC_{jkl}) - L(C_{rkl}C^r_{jm} + C_{rlj}C^r_{km} + C_{rjk}C^r_{lm})\} + \odot_{(ijk)}C^r_{il}\{L(T_{rjkm} - l_rC_{jkm} - l_jC_{kmr} - l_kC_{mrj} - l_mC_{rjk}) + L^2(C_{prk}C^p_{jm} + C_{pjr}C^p_{km+C_{pjk}C^p_{rm}})\} + \odot_{(ijk)}C_{rjk}\{L(T^r_{ilm} + l^rC_{ilm} - l_iC^r_{lm} - l_lC^r_{im} - l_mC^r_{il}) + L^2(C^r_{pl}C^p_{im} + C^r_{pi}C^p_{lm} - C^p_{il}C^r_{pm})]$$

Thus from proposition (2.1), we have

Theorem 5.1. *A Finsler space is one with a quartic metric if and oly if the equation,*

$$L^4_{ijklm} = 8[LT_{ijkl|m} + \odot_{(ijklm)}T_{ijkl}l_m + \odot_{(ijklm)}g_{ij}C_{klm} + \odot_{(ikmjl)}g_{ik}C_{jlm}$$
$$+ 2Ll_m(C^r_{il}C_{rjk} + C^r_{jl}C_{rik} + C^r_{kl}C_{rij}) - L(\odot_{(ijkl)}T_{rjkl}C^r_{im}) - \odot_{(ijkl)}l_i\{(l_jC_{klm}$$
$$+ l_kC_{jlm} + l_lC_{jkm} + l_mC_{jkl}) - L(C_{rkl}C^r_{jm} + C_{rlj}C^r_{km} + C_{rjk}C^r_{lm})\}$$
$$+ \odot_{(ijk)}C^r_{il}\{L(T_{rjkm} - l_rC_{jkm} - l_jC_{kmr} - l_kC_{mrj} - l_mC_{rjk}) + L^2(C_{prk}C^p_{jm}$$
$$+ C_{pjr}C^p_{km+C_{pjk}C^p_{rm}})\} + \odot_{(ijk)}C_{rjk}\{L(T^r_{ilm} + l^rC_{ilm} - l_iC^r_{lm} - l_lC^r_{im} - l_mC^r_{il})$$
$$+ L^2(C^r_{pl}C^p_{im} + C^r_{pi}C^p_{lm} - C^p_{il}C^r_{pm})] = 0$$

holds.

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