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FRACTIONAL OSTROWSKI INEQUALITIES FOR HARMONIC h-PREINVEX FUNCTIONS

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Abstract. In this paper, we introduce a new class of harmonic preinvex functions, which are called harmonic $h-$preinvex functions. Several new Ostrowski-type inequalities for harmonic h-preinvex functions via Riemann-Liouville fractional integrals are established. Some special cases are also discussed, which appears to be new ones. The results obtained in this paper continue to hold for these cases. Interested readers are encouraged to find the applications of the harmonic h-preinvex functions in pure and applied sciences. This is an interesting topic for future research.
Keywords: Harmonic convex functions, preinvex functions, harmonic preinvex functions, h-convex functions, Ostrowski-type inequality

1. Introduction

In recent years, convexity theory has been extended and generalized in several directions using innovative ideas and techniques. A significant generalization of convex functions is that of invex function introduced by Hanson [16]. Ben-Israel and Mond [7] introduced the concept of invex set and preinvex functions. They have shown that the differentiable preinvex functions are invex functions. It is known that the converse is also true under certain conditions, see Noor and Noor [31]. Noor [29] proved that the minimum of the differentiable preinvex function on the invex sets can be characterized by a class of variational inequalities, called the variational-like inequalities. For the applications, formulation, numerical methods and other aspects of variational-like inequalities, see [30]. Pitea and Postolache [44, 45, 46] introduced the concept of quasi invexity and applied it to the theoretical mechanics and nonlinear optimization. This shows that the preinvexity and its variant generalizations play an important and significant role in the development of various fields of pure and applied sciences.

It is worth mentioning that the convex functions have closed relationship with the theory of integral inequalities. An important integral inequality, which has been studied extensively is called the Hermite-Hadamard inequalities. In [32], Noor has

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also established some Hermite-Hadamard type inequalities for preinvex functions. Another important class of convex functions, which is called harmonic function, was introduced and studied by Anderson et al. [2] and Iscan [18]. We would like to emphasize that preinvex functions and harmonic functions are two distinct classes of convex functions. It is natural to introduce a new class of convex functions, which unifies these concepts. Inspired and motivated by the ongoing research activities in this dynamic field, Noor et al. [41] introduced a new class of convex functions, which is called harmonic preinvex function. One can easily show that harmonic preinvex functions include harmonic functions as special case.

It is well known that harmonic mean has played an important and significant part in the development of various fields of pure and applied sciences. Using the concept of weighted harmonic means, one usually defines the harmonic convex functions. The harmonic convex functions can be regarded as significant and important generalization of the convex functions. The harmonic convex functions have been considered and studied by Anderson et al. [2] and Iscan [19, 21, 22]. Noor and Noor [33] have proved that the optimality conditions of the differentiable can be characterized by a class of variational inequalities, which is called harmonic variational inequalities. This may be starting point for future research in variational inequality theory. This is new concept, which needs further efforts to investigate various aspects of harmonic variational inequalities.

Varosanec [47] introduced the class of h-convex functions. She has shown that this class contains some previously known classes of convex functions as special cases. Motivated by the ongoing research in this field, we introduce a new class of harmonic preinvex functions with respect to an arbitrary function h, which is called the harmonic h-preinvex function. It can easily be shown that the class of harmonic h-preinvex is a unifying one and includes several class of convex functions as special cases such as harmonic s-preinvex functions, Godunova-Levin harmonic s-preinvex functions, etc. In this paper, we establish Ostrowski type inequalities for harmonic h-preinvex functions involving fractional integrals. Some special cases are discussed, which appear to be new ones. Results proved in this paper continue to hold for these cases.

We now recall the known concepts.

Definition 1.1. *[50].* A set $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$ is said to be a harmonic convex set, if

$$\frac{xy}{tx + (1-t)y} \in I, \qquad \forall x, y \in I, t \in [0, 1].$$

Definition 1.2. *[1, 13].* A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be a harmonic convex function, if and only if,

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0, 1].$$

Now we introduce several new concepts for harmonic preinvex functions. To be more precise, let I be a nonempty closed set in $\mathbb{R}^n \setminus \{0\}$. Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a continuous function and let $\eta(\cdot, \cdot) : I \times I \to \mathbb{R}$ be a continuous bifunction.

Definition 1.3. *[41].* A set $I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \setminus \{0\}$ is said to be a harmonic invex set with respect to the bifunction $\eta(\cdot, \cdot)$, if

$$\frac{x(x + \eta(y, x))}{x + (1 - t)\eta(y, x)} \in I, \quad \forall x, y \in I, t \in [0, 1].$$

If $\eta(y, x) = y - x$, then harmonic invex set reduces to harmonic convex set. Clearly, every harmonic convex set is invex set but the converse is not true.

Definition 1.4. *[42].* Let $h : [0, 1] \subseteq J \to \mathbb{R}$ be a non-negative function. A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a harmonic h-preinvex function with respect to $\eta(\cdot, \cdot)$, if

$$(1.1) \quad f\left(\frac{x(x + \eta(y, x))}{x + (1 - t)\eta(y, x)}\right) \leq h(1 - t)f(x) + h(t)f(y), \qquad \forall x, y \in I, t \in [0, 1].$$

Note that for $t = \frac{1}{2}$, we have Jensen type harmonic h-preinvex function.

$$f\left(\frac{2x(x + \eta(y, x))}{2x + \eta(y, x)}\right) \leq h\big(\frac{1}{2}\big)[f(x) + f(y)], \qquad \forall x, y \in I.$$

We now discuss some special cases of Definition 1.4.

1. If $h(t) = t$ in 1.1, then Definition 1.4 reduces to the definition of harmonic preinvex functions [41].

2. If $h(t) = t^s$ in 1.1, then Definition 1.4 reduces to the definition of Breckner type of harmonic s-preinvex functions.

3. If $h(t) = t^{-s}$ in 1.1, then Definition 1.4 reduces to the definition of Godunova-Levin type of harmonic s-preinvex functions.

4. If $h(t) = t^{-1}$ in 1.1, then Definition 1.4 reduces to the definition of Godunova-Levin type of harmonic preinvex functions.

5. If $h(t) = 1$ in 1.1, then Definition 1.4 reduces to the definition of harmonic P-preinvex functions.

Iscan and Wu [23] have established Hermite-Hadamard inequality for harmonic convex functions in fractional form as follow:

Theorem 1.1. *Let $f : I = [a,b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic convex function. If $f \in L[a,b]$, then*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{a}-}(f\circ g)(1/b) + J^{\alpha}_{\frac{1}{b}+}(f\circ g)(1/a)\right\} \\ &\leq \frac{f(a)+f(b)}{2}, \end{aligned}$$

where $\alpha > 0$ and $g(x) = \frac{1}{x}$.

The following result is due to Iscan [22]. He used this result to establish the Ostrowski type inequalities for harmonic s-convex functions.

Lemma 1.1. *Let $f : I = [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a,b]$ then for all $x \in [a,b]$, we have*

$$\begin{aligned} & f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}\mathrm{d}u \\ = \; & \frac{ab}{b-a}\left\{(x-a)^2\int_0^1 \frac{t}{[ta+(1-t)x]^2} f'\left(\frac{ax}{ta+(1-t)x}\right)\mathrm{d}t\right. \\ & \left. -(b-x)^2\int_0^1 \frac{t}{[tb+(1-t)x]^2} f'\left(\frac{bx}{tb+(1-t)x}\right)\mathrm{d}t\right\}. \end{aligned} \tag{1.2}$$

We also recall the well-known following concepts.

Definition 1.5. For the real or complex numbers a, b, c other than $0, -1, -2, ..$ the hypergeometric series is defined by

$${}_2F_1[a,b;c;z] = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + ... = \sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{z^m}{m!}.$$

Here $(\phi)_m$ is the Pochhammer symbol, which is defined by

$$(\phi)_m = \begin{cases} 1 & m = 0 \\ \phi(\phi+1)...(\phi+m-1), & m > 0, \end{cases}$$

which has the integral form

$${}_2F_1[a,b;c;z] = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t$$

where $|z| < 1$, $c > b > 0$ and

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

is Euler Beta function.

Definition 1.6. Let $f \in L[a, a+\eta(b,a)]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{[a+\eta(b,a)]-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \mathrm{d}t, \quad x > a$$

and

$$J_{[a+\eta(b,a)]-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{a+\eta(b,a)} (t-x)^{\alpha-1} f(t) \mathrm{d}t, \quad x < [a+\eta(b,a)]$$

, respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \mathrm{d}t$ and $J_{a+}^{0} f(x) = J_{[a+\eta(b,a)]-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

2. Main results

We need the following Lemma in order to prove our main results.

Lemma 2.1. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a+\eta(b,a)]$, then for all $x \in [a, a+\eta(b,a)]$ and $\alpha > 0$, we have*

$$\begin{aligned}
&\Psi_f(g; \alpha; x; a, a+\eta(b,a)) \\
&= \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \int_0^1 \frac{t^\alpha}{[ta+(1-t)x]^2} f'\left(\frac{ax}{ta+(1-t)x}\right) \mathrm{d}t \\
&\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}} \int_0^1 \frac{t^\alpha}{[t[a+\eta(b,a)]+(1-t)x]^2} f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right) \mathrm{d}t,
\end{aligned}$$

where

$$\begin{aligned}
&\Psi_f(g; \alpha; x; a, a+\eta(b,a)) \\
= \; &\left[\left(\frac{x-a}{ax}\right)^\alpha + \left(\frac{a+\eta(b,a)-x}{[a+\eta(b,a)]x}\right)^\alpha\right] f(x) \\
- \; &\Gamma(\alpha+1)\left\{J_{\frac{1}{x}-}^{\alpha} (f \circ g)\left(\frac{1}{a+\eta(b,a)}\right) + J_{\frac{1}{x}+}^{\alpha} (f \circ g)\left(\frac{1}{a}\right)\right\}
\end{aligned}$$

and $g(u) = \frac{1}{u}$ and $\Gamma(\cdot)$ is the Euler Gamma function.

Proof. By integration by parts, we have

$$
\begin{aligned}
&ax(x-a)\int_0^1 \frac{t^\alpha}{[ta+(1-t)x]^2} f'\left(\frac{ax}{ta+(1-t)x}\right)\mathrm{d}t \\
&= t^\alpha f\left(\frac{ax}{ta+(1-t)x}\right)\Big|_0^1 - \alpha\int_0^1 t^{\alpha-1} f\left(\frac{ax}{ta+(1-t)x}\right)\mathrm{d}t \\
&= f(x) - \alpha\left(\frac{ax}{x-a}\right)^\alpha \int_{\frac{1}{x}}^{\frac{1}{a}} \left(\frac{1}{a}-u\right)^{\alpha-1} f\left(\frac{1}{u}\right)\mathrm{d}u \\
&= f(x) - \Gamma(\alpha+1)\left(\frac{ax}{x-a}\right)^\alpha J^\alpha_{\frac{1}{x}+}(f\circ g)\left(\frac{1}{a}\right) \qquad (2.1)
\end{aligned}
$$

and

$$
\begin{aligned}
&-[a+\eta(b,a)]x([a+\eta(b,a)]-x) \\
&\times\int_0^1 \frac{t^\alpha}{[t[a+\eta(b,a)]+(1-t)x]^2} f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\mathrm{d}t \\
&= t^\alpha f\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\Big|_0^1 - \alpha\int_0^1 t^{\alpha-1} f\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\mathrm{d}t \\
&= f(x) - \alpha\left(\frac{[a+\eta(b,a)]x}{[a+\eta(b,a)]-x}\right)^\alpha \int_{\frac{1}{a+\eta(b,a)}}^{\frac{1}{x}} \left(u-\frac{1}{a+\eta(b,a)}\right)^{\alpha-1} f\left(\frac{1}{u}\right)\mathrm{d}u \\
&= f(x) - \Gamma(\alpha+1)\left(\frac{[a+\eta(b,a)]x}{[a+\eta(b,a)]-x}\right)^\alpha J^\alpha_{\frac{1}{x}-}(f\circ g)\left(\frac{1}{a+\eta(b,a)}\right). \qquad (2.2)
\end{aligned}
$$

Multiplying both sides of 2.1 and 2.2 by $(\frac{x-a}{ax})^\alpha$ and $(\frac{[a+\eta(b,a)]-x}{[a+\eta(b,a)]x})^\alpha$, respectively and adding the resultants, we obtain the required result. □

Remark 2.1. In Lemma 2.1, if we take $\alpha = 1$, then it reduces to the following result.

Corollary 2.1. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a+\eta(b,a)]$ then for all $x \in [a, a+\eta(b,a)]$, we have*

$$
\begin{aligned}
&f(x) - \frac{a[a+\eta(b,a)]}{\eta(b,a)}\int_a^{a+\eta(b,a)} \frac{f(u)}{u^2}\mathrm{d}u \\
= &\frac{a[a+\eta(b,a)]}{\eta(b,a)}\Bigg\{(x-a)^2\int_0^1 \frac{t}{[ta+(1-t)x]^2} f'\left(\frac{ax}{ta+(1-t)x}\right)\mathrm{d}t \\
&-([a+\eta(b,a)]-x)^2\int_0^1 \frac{t}{[t[a+\eta(b,a)]+(1-t)x]^2} f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\mathrm{d}t\Bigg\}.
\end{aligned}
$$

Remark 2.2. In Lemma 2.1, if we take $\alpha = 1$ and $\eta(b,a) = b-a$, then it reduces to the identity 1.2 of Lemma 1.1.

Theorem 2.1. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a+\eta(b,a)]$ and $|f'|^q$ is harmonic h-preinvex function on I for $q \geq 1$, then for all $x \in [a, a+\eta(b,a)]$, we have*

$$\begin{aligned}
&|\Psi_f(g;\alpha;x;a,a+\eta(b,a))| \\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\Big\{(\Psi_1(a,x,h,q,\alpha q)|f'(x)|^q + \Psi_2(a,x,h,q,\alpha q)|f'(a)|^q)^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}(\Psi_3(a+\eta(b,a),x,h,q,\alpha q)|f'(x)|^q \\
&+\Psi_4(a+\eta(b,a),x,h,q,\alpha q)|f'(a+\eta(b,a))|^q)^{\frac{1}{q}}\Big\}, \qquad (2.3)
\end{aligned}$$

where

$$\Psi_1(a,x,h,q,\alpha q) = \int_0^1 \frac{t^{\alpha q}}{(ta+(1-t)x)^{2q}}h(t)\mathrm{d}t, \qquad (2.4)$$

$$\Psi_2(a,x,h,q,\alpha q) = \int_0^1 \frac{t^{\alpha q}}{(ta+(1-t)x)^{2q}}h(1-t)\mathrm{d}t, \qquad (2.5)$$

$$\Psi_3(a+\eta(b,a),x,h,q,\alpha q) = \int_0^1 \frac{t^{\alpha q}}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}h(t)\mathrm{d}t, \qquad (2.6)$$

$$\Psi_4(a+\eta(b,a),x,h,q,\alpha q) = \int_0^1 \frac{t^{\alpha q}}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}h(1-t)\mathrm{d}t. \qquad (2.7)$$

Proof. Using Lemma 2.1 and the power mean inequality and harmonic h-preinvexity

of $|f'|^q$ on I, we have

$$
\begin{aligned}
&\left|\Psi_f(g;\alpha;x;a,a+\eta(b,a))\right| \\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\int_0^1 \frac{t^\alpha}{[ta+(1-t)x]^2}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|\mathrm{d}t \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}} \\
&\times\int_0^1 \frac{t^\alpha}{[t[a+\eta(b,a)]+(1-t)x]^2}\left|f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\right|\mathrm{d}t \\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\left(\int_0^1 1\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{t^{\alpha q}}{[ta+(1-t)x]^{2q}}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}\left(\int_0^1 1\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\left(\int_0^1 \frac{t^{\alpha q}}{[t[a+\eta(b,a)]+(1-t)x]^{2q}}\left|f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\left(\int_0^1 1\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \frac{t^{\alpha q}}{[ta+(1-t)x]^{2q}}[h(t)|f'(x)|^q+h(1-t)|f'(a)|^q]\mathrm{d}t\right)^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}\left(\int_0^1 1\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\left(\int_0^1 \frac{t^{\alpha q}}{[t[a+\eta(b,a)]+(1-t)x]^{2q}}[h(t)|f'(x)|^q+h(1-t)|f'(a+\eta(b,a))|^q]\mathrm{d}t\right)^{\frac{1}{q}} \\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\Big\{(\Psi_1(a,x,h,q,\alpha q)|f'(x)|^q+\Psi_2(a,x,h,q,\alpha q)|f'(a)|^q)^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}(\Psi_3(a+\eta(b,a),x,h,q,\alpha q)|f'(x)|^q \\
&+\Psi_4(a+\eta(b,a),x,h,q,\alpha q)|f'(a+\eta(b,a))|^q)^{\frac{1}{q}}\Big\},
\end{aligned}
$$

which is the required result. □

Corollary 2.2. *In Theorem 2.1, if $|f'(x)| \leq M$, $x \in [a, a+\eta(b,a)]$, then inequality*

$$\begin{aligned}
&\left|\Psi_f(g;\alpha;x;a,a+\eta(b,a))\right| \\
&\leq M\Bigg\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi_1(a,x,h,q,\alpha q)+\Psi_2(a,x,h,q,\alpha q)]^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi_3(a+\eta(b,a),x,h,q,\alpha q) \\
&+\Psi_4(a+\eta(b,a),x,h,q,\alpha q)]^{\frac{1}{q}}\Bigg\},
\end{aligned}$$

holds.

Remark 2.3. In Theorem 2.1, if we take $h(t) = 1$, then the identity 2.3 reduces to the following result.

Corollary 2.3. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a+\eta(b,a)]$ and $|f'|^q$ is harmonic P-preinvex function on I for $q \geq 1$, then for all $x \in [a, a+\eta(b,a)]$, we have*

$$\begin{aligned}
&|\Psi_f(g;\alpha;x;a,a+\eta(b,a))| \\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\Bigg\{(\Psi_1^*(a,x,0,q,\alpha q)[|f'(x)|^q+|f'(a)|^q])^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}(\Psi_2^*(a+\eta(b,a),x,0,q,\alpha q)[|f'(x)|^q+|f'(a+\eta(b,a))|^q])^{\frac{1}{q}}\Bigg\},
\end{aligned}$$

where an easy calculation gives

$$\begin{aligned}
\Psi_1^*(a,x,0,q,\alpha q) &= \int_0^1 \frac{t^{\alpha q}}{(ta+(1-t)x)^{2q}}\mathrm{d}t \\
&= \frac{1}{x^{2q}(\alpha q+1)}\,{}_2F_1\left(2q,\alpha q+1;\alpha q+2;1-\frac{a}{x}\right),
\end{aligned}$$

$$\begin{aligned}
\Psi_2^*(a+\eta(b,a),x,0,q,\alpha q) &= \int_0^1 \frac{t^{\alpha q}}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t \\
&= \frac{1}{[a+\eta(b,a)]^{2q}(\alpha q+1)}\,{}_2F_1\left(2q,1;\alpha q+2;1-\frac{x}{a+\eta(b,a)}\right),
\end{aligned}$$

Remark 2.4. In Theorem 2.1, if we take $h(t) = t^s$, then the identity 2.3 reduces to the following result.

Corollary 2.4. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a+\eta(b,a)]$ and $|f'|^q$ is harmonic s-preinvex function on*

I for $q \geq 1$, then for all $x \in [a, a+\eta(b,a)]$, we have

$$
\begin{aligned}
&|\Psi_f(g;\alpha;x;a,a+\eta(b,a))| \\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\Big\{(\Psi_1^{**}(a,x,s,q,\alpha q)|f'(x)|^q + \Psi_2^{**}(a,x,s,q,\alpha q)|f'(a)|^q)^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}(\Psi_3^{**}(a+\eta(b,a),x,s,q,\alpha q)|f'(x)|^q \\
&+\Psi_4^{**}(a+\eta(b,a),x,s,q,\alpha q)|f'(a+\eta(b,a))|^q)^{\frac{1}{q}}\Big\},
\end{aligned}
$$

where an easy calculation gives

$$
\begin{aligned}
\Psi_1^{**}(a,x,s,q,\alpha q) &= \int_0^1 \frac{t^{\alpha q+s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t \\
&= \frac{\beta(\alpha q+s+1,1)}{x^{2q}}\,{}_2F_1\left(2q,\alpha q+s+1;\alpha q+s+2;1-\frac{a}{x}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Psi_2^{**}(a,x,s,q,\alpha q) &= \int_0^1 \frac{t^{\alpha q}(1-t)^s}{(ta+(1-t)x)^{2q}}\mathrm{d}t \\
&= \frac{\beta(\alpha q+1,s+1)}{x^{2q}}\,{}_2F_1\left(2q,\alpha q+1;s+\alpha q+2;1-\frac{a}{x}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Psi_3^{**}(a+\eta(b,a),x,s,q,\alpha q) &= \int_0^1 \frac{t^{\alpha q+s}}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t \\
&= \frac{\beta(1,\alpha q+s+1)}{[a+\eta(b,a)]^{2q}}\,{}_2F_1\left(2q,1;\alpha q+s+2;1-\frac{x}{a+\eta(b,a)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Psi_4^{**}(a+\eta(b,a),x,s,q,\alpha q) &= \int_0^1 \frac{t^{\alpha q}(1-t)^s}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t \\
&= \frac{\beta(s+1,\alpha q+1)}{[a+\eta(b,a)]^{2q}}\,{}_2F_1\left(2q,s+1;s+\alpha q+2;1-\frac{x}{a+\eta(b,a)}\right).
\end{aligned}
$$

Remark 2.5. In Theorem 2.1, if we take $h(t) = t^{-s}$, then the identity 2.3 reduces to the following result.

Corollary 2.5. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a+\eta(b,a)]$ and $|f'|^q$ is harmonic s-Godunova-Levin-preinvex function on I for $q \geq 1$, then for all $x \in [a, a+\eta(b,a)]$, we have*

$$
\begin{aligned}
&|\Psi_f(g;\alpha;x;a,a+\eta(b,a))| \\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\Big\{(\Psi_1^{***}(a,x,-s,q,\alpha q)|f'(x)|^q + \Psi_2^{***}(a,x,-s,q,\alpha q)|f'(a)|^q)^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}(\Psi_3^{***}(a+\eta(b,a),x,-s,q,\alpha q)|f'(x)|^q \\
&+\Psi_4^{***}(a+\eta(b,a),x,-s,q,\alpha q)|f'(a+\eta(b,a))|^q)^{\frac{1}{q}}\Big\},
\end{aligned}
$$

where an easy calculation gives

$$\Psi_1^{***}(a,x,-s,q,\alpha q)=\int_0^1\frac{t^{\alpha q-s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t$$
$$=\quad\frac{\beta(\alpha q-s+1,1)}{x^{2q}}{}_2F_1\left(2q,\alpha q-s+1;\alpha q-s+2;1-\frac{a}{x}\right),$$

$$\Psi_2^{***}(a,x,-s,q,\alpha q)=\int_0^1\frac{t^{\alpha q}(1-t)^{-s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t$$
$$=\quad\frac{\beta(\alpha q+1,1-s)}{x^{2q}}{}_2F_1\left(2q,\alpha q+1;\alpha q-s+2;1-\frac{a}{x}\right),$$

$$\Psi_3^{***}(a+\eta(b,a),x,-s,q,\alpha q)=\int_0^1\frac{t^{\alpha q-s}}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t$$
$$=\quad\frac{\beta(1,\alpha q-s+1)}{[a+\eta(b,a)]^{2q}}{}_2F_1\left(2q,1;\alpha q-s+2;1-\frac{x}{a+\eta(b,a)}\right),$$

$$\Psi_4^{***}(a+\eta(b,a),x,-s,q,\alpha q)=\int_0^1\frac{t^{\alpha q}(1-t)^{-s}}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t$$
$$=\quad\frac{\beta(1-s,\alpha q+1)}{[a+\eta(b,a)]^{2q}}{}_2F_1\left(2q,1-s;\alpha q-s+2;1-\frac{x}{a+\eta(b,a)}\right).$$

Theorem 2.2. *Let $f:I=[a,a+\eta(b,a)]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,a+\eta(b,a)]$ and $|f'|^q$ is harmonic h-preinvex function on I for $q\geq 1$, then for all $x\in[a,a+\eta(b,a)]$, we have*

$$|\Psi_f(g;\alpha;x;a,a+\eta(b,a))|$$
$$\leq\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi_5(a,x,h,q,\alpha)|f'(x)|^q+\Psi_6(a,x,h,q,\alpha)|f'(a)|^q]^{\frac{1}{q}}\right.$$
$$+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi_7(a+\eta(b,a),x,h,q,\alpha)|f'(x)|^q$$
$$\left.+\Psi_8(a+\eta(b,a),x,h,q,\alpha)|f'(a+\eta(b,a))|^q]^{\frac{1}{q}}\right\},\tag{2.8}$$

where $\alpha>0$ and $\Psi_5(a,x,h,q,\alpha)$, $\Psi_6(a,x,h,q,\alpha)$, $\Psi_7(a+\eta(b,a),x,h,q,\alpha)$ and $\Psi_5(a+\eta(b,a),x,h,q,\alpha)$ can be deduced from 2.4, 2.5, 2.6 *and* 2.7 *respectively.*

Proof. Using Lemma 2.1 and the power mean inequality and harmonic h-preinvexity

of $|f'|^q$ on I, we have

$$\begin{aligned}
&\big|\Psi_f(g;\alpha;x;a,a+\eta(b,a))\big|\\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\int_0^1 \frac{t^\alpha}{[ta+(1-t)x]^2}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|\mathrm{d}t\\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}\int_0^1 \frac{t^\alpha}{[t[a+\eta(b,a)]+(1-t)x]^2}\\
&\times\left|f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\right|\mathrm{d}t\\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\left(\int_0^1 t^\alpha \mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{t^\alpha}{[ta+(1-t)x]^{2q}}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}\left(\int_0^1 t^\alpha \mathrm{d}t\right)^{1-\frac{1}{q}}\\
&\left(\int_0^1 \frac{t^\alpha}{[t[a+\eta(b,a)]+(1-t)x]^{2q}}\left|f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\left(\int_0^1 t^\alpha \mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{t^\alpha}{[ta+(1-t)x]^{2q}}\right.\\
&\left.\times[h(t)|f'(x)|^q+h(1-t)|f'(a)|^q]\mathrm{d}t\right)^{\frac{1}{q}}\\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}\left(\int_0^1 t^\alpha \mathrm{d}t\right)^{1-\frac{1}{q}}\\
&\left(\int_0^1 \frac{t^\alpha}{[t[a+\eta(b,a)]+(1-t)x]^{2q}}[h(t)|f'(x)|^q+h(1-t)|f'(a+\eta(b,a))|^q]\mathrm{d}t\right)^{\frac{1}{q}}\\
&\leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi_5(a,x,h,q,\alpha)|f'(x)|^q+\Psi_6(a,x,h,q,\alpha)|f'(a)|^q]^{\frac{1}{q}}\right.\\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi_7(a+\eta(b,a),x,h,q,\alpha)|f'(x)|^q\\
&\left.+\Psi_8(a+\eta(b,a),x,h,q,\alpha)|f'(a+\eta(b,a))|^q]^{\frac{1}{q}}\right\},
\end{aligned}$$

which is the required result. $\square$

Corollary 2.6. *In Theorem 2.2, if* $|f'(x)|\leq M$, $x\in[a,a+\eta(b,a)]$, *then inequality*

$$\begin{aligned}
&\big|\Psi_f(g;\alpha;x;a,a+\eta(b,a))\big|\\
&\leq M\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi_5(a,x,h,q,\alpha)+\Psi_6(a,x,h,q,\alpha)]^{\frac{1}{q}}\right.\\
&\left.+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi_7(a+\eta(b,a),x,h,q,\alpha)+\Psi_8(a+\eta(b,a),x,h,q,\alpha)]^{\frac{1}{q}}\right\},
\end{aligned}$$

holds.

Remark 2.6. In Theorem 2.2, if we take $h(t) = 1$, then the identity 2.8 reduces to the following result.

Corollary 2.7. *Let $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a + \eta(b, a)]$ and $|f'|^q$ is harmonic P-preinvex function on I for $q \geq 1$, then for all $x \in [a, a + \eta(b, a)]$, we have*

$$\begin{aligned}
&|\Psi_f(g; \alpha; x; a, a + \eta(b, a))| \\
&\leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left\{ \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} [\Psi_5^*(a, x, 0, q, \alpha)(|f'(x)|^q + |f'(a)|^q)]^{\frac{1}{q}} \right. \\
&\left. + \frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}} [\Psi_6^*(a+\eta(b,a), x, 0, q, \alpha)(|f'(x)|^q + |f'(a+\eta(b,a))|^q)]^{\frac{1}{q}} \right\},
\end{aligned}$$

where an easy calculation gives

$$\begin{aligned}
\Psi_5^*(a, x, 0, q, \alpha) &= \int_0^1 \frac{t^\alpha}{(ta + (1-t)x)^{2q}} \mathrm{d}t \\
&= \frac{1}{x^{2q}(\alpha+1)} {}_2F_1\left(2q, \alpha+1; \alpha+2; 1 - \frac{a}{x}\right),
\end{aligned}$$

$$\begin{aligned}
\Psi_6^*(a+\eta(b,a), x, 0, q, \alpha) &= \int_0^1 \frac{t^\alpha}{(t[a+\eta(b,a)] + (1-t)x)^{2q}} \mathrm{d}t \\
&= \frac{1}{[a+\eta(b,a)]^{2q}(\alpha+1)} {}_2F_1\left(2q, 1; \alpha+2; 1 - \frac{x}{a+\eta(b,a)}\right),
\end{aligned}$$

Remark 2.7. In Theorem 2.2, if we take $h(t) = t^s$, then the identity 2.8 reduces to the following result.

Corollary 2.8. *Let $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a + \eta(b, a)]$ and $|f'|^q$ is harmonic s-preinvex function on I for $q \geq 1$, then for all $x \in [a, a + \eta(b, a)]$, we have*

$$\begin{aligned}
&|\Psi_f(g; \alpha; x; a, a + \eta(b, a))| \\
&\leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left\{ \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} [\Psi_5^{**}(a, x, s, q, \alpha)|f'(x)|^q + \Psi_6^{**}(a, x, s, q, \alpha)|f'(a)|^q]^{\frac{1}{q}} \right. \\
&+ \frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}} [\Psi_7^{**}(a+\eta(b,a), x, s, q, \alpha)|f'(x)|^q \\
&\left. + \Psi_8^{**}(a+\eta(b,a), x, s, q, \alpha)|f'(a+\eta(b,a))|^q]^{\frac{1}{q}} \right\},
\end{aligned}$$

where an easy calculation gives

$$\begin{aligned}
\Psi_5^{**}(a, x, s, q, \alpha) &= \int_0^1 \frac{t^{\alpha+s}}{(ta + (1-t)x)^{2q}} \mathrm{d}t \\
&= \frac{\beta(\alpha+s+1, 1)}{x^{2q}} {}_2F_1\left(2q, \alpha+s+1; \alpha+s+2; 1 - \frac{a}{x}\right),
\end{aligned}$$

$$\Psi_6^{**}(a,x,s,q,\alpha) = \int_0^1 \frac{t^{\alpha}(1-t)^s}{(ta+(1-t)x)^{2q}}\mathrm{d}t$$
$$= \frac{\beta(\alpha+1,s+1)}{x^{2q}}\,{}_2F_1\left(2q,\alpha+1;s+\alpha+2;1-\frac{a}{x}\right),$$

$$\Psi_7^{**}(a+\eta(b,a),x,s,q,\alpha q) = \int_0^1 \frac{t^{\alpha+s}}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t$$
$$= \frac{\beta(1,\alpha+s+1)}{[a+\eta(b,a)]^{2q}}\,{}_2F_1\left(2q,1;\alpha+s+2;1-\frac{x}{a+\eta(b,a)}\right),$$

$$\Psi_8^{**}(a+\eta(b,a),x,s,q,\alpha) = \int_0^1 \frac{t^{\alpha}(1-t)^s}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t$$
$$= \frac{\beta(s+1,\alpha+1)}{[a+\eta(b,a)]^{2q}}\,{}_2F_1\left(2q,s+1;s+\alpha+2;1-\frac{x}{a+\eta(b,a)}\right).$$

Remark 2.8. In Theorem 2.2, if we take $h(t)=t^{-s}$, then the identity 2.8 reduces to the following result.

Corollary 2.9. *Let $f: I=[a,a+\eta(b,a)]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,a+\eta(b,a)]$ and $|f'|^q$ is harmonic s-Godunova-Levin-preinvex function on I for $q\geq 1$, then for all $x\in[a,a+\eta(b,a)]$, we have*

$$|\Psi_f(g;\alpha;x;a,a+\eta(b,a))|$$
$$\leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi_5^{***}(a,x,-s,q,\alpha)|f'(x)|^q+\Psi_6^{***}(a,x,-s,q,\alpha)|f'(a)|^q]^{\frac{1}{q}}\right.$$
$$+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi_7^{***}(a+\eta(b,a),x,-s,q,\alpha)|f'(x)|^q$$
$$\left.+\Psi_8^{***}(a+\eta(b,a),x,-s,q,\alpha)|f'(a+\eta(b,a))|^q]^{\frac{1}{q}}\right\},$$

where an easy calculation gives

$$\Psi_5^{***}(a,x,-s,q,\alpha) = \int_0^1 \frac{t^{\alpha-s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t$$
$$= \frac{\beta(\alpha-s+1,1)}{x^{2q}}\,{}_2F_1\left(2q,\alpha-s+1;\alpha-s+2;1-\frac{a}{x}\right),$$

$$\Psi_6^{***}(a,x,-s,q,\alpha) = \int_0^1 \frac{t^{\alpha}(1-t)^{-s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t$$
$$= \frac{\beta(\alpha+1,1-s)}{x^{2q}}\,{}_2F_1\left(2q,\alpha+1;\alpha-s+2;1-\frac{a}{x}\right),$$

$$\Psi_7^{***}(a+\eta(b,a),x,-s,q,\alpha) = \int_0^1 \frac{t^{\alpha-s}}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t$$
$$= \frac{\beta(1,\alpha-s+1)}{[a+\eta(b,a)]^{2q}}{}_2F_1\left(2q,1;\alpha-s+2;1-\frac{x}{a+\eta(b,a)}\right),$$

$$\Psi_8^{***}(a+\eta(b,a),x,-s,q,\alpha) = \int_0^1 \frac{t^{\alpha}(1-t)^{-s}}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t$$
$$= \frac{\beta(1-s,\alpha+1)}{[a+\eta(b,a)]^{2q}}{}_2F_1\left(2q,1-s;\alpha-s+2;1-\frac{x}{a+\eta(b,a)}\right).$$

Theorem 2.3. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a+\eta(b,a)]$ and $|f'|^q$ is harmonic h-preinvex function on I for $q \geq 1$, then for all $x \in [a, a+\eta(b,a)]$, we have*

$$\begin{aligned}
&|\Psi_f(g;\alpha;x;a,a+\eta(b,a))| \\
&\leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\Bigg\{\Psi_9^{1-\frac{1}{q}}(a,x,\alpha)\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi_{11}(a,x,h,1,\alpha)|f'(x)|^q \\
&+\Psi_{12}(a,x,h,1,\alpha)|f'(a)|^q]^{\frac{1}{q}} \\
&+\Psi_{10}^{1-\frac{1}{q}}(a+\eta(b,a),x,\alpha)\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi_{13}(a+\eta(b,a),x,h,1,\alpha)|f'(x)|^q \\
&+\Psi_{14}(a+\eta(b,a),x,h,1,\alpha)|f'(a+\eta(b,a))|^q]^{\frac{1}{q}}\Bigg\},
\end{aligned} \tag{2.9}$$

where

$$\Psi_9(a,x,\alpha) = \frac{1}{x^2}{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{x}\right), \tag{2.10}$$

$$\Psi_{10}(a+\eta(b,a),x,\alpha) = \frac{1}{[a+\eta(b,a)]^2}{}_2F_1\left(2,1;\alpha+2;1-\frac{x}{a+\eta(b,a)}\right), \tag{2.11}$$

and $\Psi_{11}(a,x,h,1,\alpha)$, $\Psi_{12}(a,x,h,1,\alpha)$, $\Psi_{13}(a+\eta(b,a),x,h,1,\alpha)$ and $\Psi_{14}(a+\eta(b,a),x,h,1,\alpha)$ can be deduced from 2.4, 2.5, 2.6 *and* 2.7 *respectively.*

Proof. Using Lemma 2.1 and the power mean inequality and harmonic h-preinvexity

of $|f'|^q$ on I, we have

$$\begin{aligned}
&\left|\Psi_f(g;\alpha;x;a,a+\eta(b,a))\right| \\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\int_0^1 \frac{t^\alpha}{[ta+(1-t)x]^2}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|\mathrm{d}t \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{(bx)^{\alpha-1}} \\
&\times\int_0^1 \frac{t^\alpha}{[t[a+\eta(b,a)]+(1-t)x]^2}\left|f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\right|\mathrm{d}t \\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\left(\int_0^1 \frac{t^\alpha}{[ta+(1-t)x]^2}\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\left(\int_0^1 \frac{t^\alpha}{[ta+(1-t)x]^2}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}\left(\int_0^1 \frac{t^\alpha}{[t[a+\eta(b,a)]+(1-t)x]^2}\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\left(\int_0^1 \frac{t^\alpha}{[t[a+\eta(b,a)]+(1-t)x]^2}\left|f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\left(\int_0^1 \frac{t^\alpha}{[ta+(1-t)x]^2}\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\left(\int_0^1 \frac{t^\alpha}{[ta+(1-t)x]^2}[h(t)|f'(x)|^q+h(1-t)|f'(a)|^q]\mathrm{d}t\right)^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}\left(\int_0^1 \frac{t^\alpha}{[t[a+\eta(b,a)]+(1-t)x]^2}\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\left(\int_0^1 \frac{t^\alpha}{[t[a+\eta(b,a)]+(1-t)x]^2}[h(t)|f'(x)|^q+h(1-t)|f'(a+\eta(b,a))|^q]\mathrm{d}t\right)^{\frac{1}{q}} \\
&\leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left\{\Psi_9^{1-\frac{1}{q}}(a,x,\alpha)\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\right. \\
&\times[\Psi_{11}(a,x,h,1,\alpha)|f'(x)|^q+\Psi_{12}(a,x,h,1,\alpha)|f'(a)|^q]^{\frac{1}{q}} \\
&+\Psi_{10}^{1-\frac{1}{q}}(a+\eta(b,a),x,\alpha)\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi_{13}(a+\eta(b,a),x,h,1,\alpha)|f'(x)|^q \\
&\left.+\Psi_{14}(a+\eta(b,a),x,h,1,\alpha)|f'(a+\eta(b,a))|^q]^{\frac{1}{q}}\right\},
\end{aligned}$$

which is the required result. □

Corollary 2.10. *In Theorem 2.2, if* $|f'(x)| \leq M$, $x \in [a, a+\eta(b,a)]$, *then inequal-*

ity

$$\begin{aligned}
&\left|\Psi_f(g;\alpha;x;a,a+\eta(b,a))\right| \\
&\leq M\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \\
&\times\Bigg\{\Psi_9^{1-\frac{1}{q}}(a,x,\alpha)\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi_{11}(a,x,h,1,\alpha)+\Psi_{12}(a,x,h,1,\alpha)]^{\frac{1}{q}} \\
&+\Psi_{10}^{1-\frac{1}{q}}(a+\eta(b,a),x,\alpha) \\
&\times\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi_{13}(b,x,h,1,\alpha)+\Psi_{14}(b,x,h,1,\alpha)]^{\frac{1}{q}}\Bigg\},
\end{aligned}$$

holds.

Remark 2.9. In Theorem 2.3, if we take $h(t)=1$, then the identity 2.9 reduces to the following result.

Corollary 2.11. *Let $f:I=[a,a+\eta(b,a)]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,a+\eta(b,a)]$ and $|f'|^q$ is harmonic P-preinvex function on I for $q\geq 1$, then for all $x\in[a,a+\eta(b,a)]$, we have*

$$\begin{aligned}
&|\Psi_f(g;\alpha;x;a,a+\eta(b,a))| \\
&\leq\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\Bigg\{\Psi_9^{1-\frac{1}{q}}(a,x,\alpha)\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi^*_{11}(a,x,0,1,\alpha)(|f'(x)|^q+|f'(a)|^q)]^{\frac{1}{q}} \\
&+\Psi_{10}^{1-\frac{1}{q}}(a+\eta(b,a),x,\alpha)\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi^*_{12}(a+\eta(b,a),x,0,1,\alpha)(|f'(x)|^q \\
&+|f'(a+\eta(b,a))|^q)]^{\frac{1}{q}}\Bigg\},
\end{aligned}$$

where $\Psi_9(a,x,\alpha)$ and $\Psi_{10}(a+\eta(b,a),x,\alpha)$ are given by 2.10 and 2.11 respectively and

$$\begin{aligned}
\Psi^*_{11}(a,x,0,1,\alpha) &= \int_0^1\frac{t^\alpha}{(ta+(1-t)x)^2}\mathrm{d}t \\
&= \frac{1}{x^2(\alpha+1)}{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{x}\right),
\end{aligned}$$

$$\begin{aligned}
&\Psi^*_{12}(a+\eta(b,a),x,0,1,\alpha)=\int_0^1\frac{t^\alpha}{(t[a+\eta(b,a)]+(1-t)x)^2}\mathrm{d}t \\
&= \frac{1}{[a+\eta(b,a)]^2(\alpha+1)}{}_2F_1\left(2,1;\alpha+2;1-\frac{x}{a+\eta(b,a)}\right),
\end{aligned}$$

Remark 2.10. In Theorem 2.3, if we take $h(t)=t^s$, then the identity 2.9 reduces to the following result.

Corollary 2.12. *Let $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a + \eta(b, a)]$ and $|f'|^q$ is harmonic s-preinvex function on I for $q \geq 1$, then for all $x \in [a, a + \eta(b, a)]$, we have*

$$
\begin{aligned}
&|\Psi_f(g; \alpha; x; a, a + \eta(b, a))| \\
&\leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \Bigg\{ \Psi_9^{1-\frac{1}{q}}(a, x, \alpha) \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \\
&\times [\Psi_{11}^{**}(a, x, s, 1, \alpha)|f'(x)|^q + \Psi_{12}^{**}(a, x, s, 1, \alpha)|f'(a)|^q]^{\frac{1}{q}} \\
&+ \Psi_{10}^{1-\frac{1}{q}}(a + \eta(b, a), x, \alpha) \frac{([a + \eta(b, a)] - x)^{\alpha+1}}{([a + \eta(b, a)]x)^{\alpha-1}} [\Psi_{13}^{**}(a + \eta(b, a), x, s, 1, \alpha)|f'(x)|^q \\
&+ \Psi_{14}^{**}(a + \eta(b, a), x, s, 1, \alpha)|f'(a + \eta(b, a))|^q]^{\frac{1}{q}} \Bigg\},
\end{aligned}
$$

where $\Psi_9(a, x, \alpha)$ and $\Psi_{10}(a + \eta(b, a), x, \alpha)$ are given by 2.10 *and* 2.11 *respectively and*

$$
\begin{aligned}
\Psi_{11}^{**}(a, x, s, 1, \alpha) &= \int_0^1 \frac{t^{\alpha+s}}{(ta + (1-t)x)^2} \mathrm{d}t \\
= \quad & \frac{\beta(\alpha + s + 1, 1)}{x^2} {}_2F_1\left(2, \alpha + s + 1; \alpha + s + 2; 1 - \frac{a}{x}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{12}^{**}(a, x, s, 1, \alpha) &= \int_0^1 \frac{t^{\alpha}(1-t)^s}{(ta + (1-t)x)^2} \mathrm{d}t \\
= \quad & \frac{\beta(\alpha + 1, s + 1)}{x^2} {}_2F_1\left(2, \alpha + 1; s + \alpha + 2; 1 - \frac{a}{x}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{13}^{**}(a + \eta(b, a), x, s, 1, \alpha) &= \int_0^1 \frac{t^{\alpha+s}}{(t[a + \eta(b, a)] + (1-t)x)^2} \mathrm{d}t \\
= \quad & \frac{\beta(1, \alpha + s + 1)}{[a + \eta(b, a)]^2} {}_2F_1\left(2, 1; \alpha + s + 2; 1 - \frac{x}{a + \eta(b, a)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{14}^{**}(a + \eta(b, a), x, s, 1, \alpha) &= \int_0^1 \frac{t^{\alpha}(1-t)^s}{(t[a + \eta(b, a)] + (1-t)x)^2} \mathrm{d}t \\
= \quad & \frac{\beta(s + 1, \alpha + 1)}{[a + \eta(b, a)]^2} {}_2F_1\left(2, s + 1; s + \alpha + 2; 1 - \frac{x}{a + \eta(b, a)}\right).
\end{aligned}
$$

Remark 2.11. In Theorem 2.3, if we take $h(t) = t^{-s}$, then the identity 2.9 reduces to the following result.

Corollary 2.13. *Let $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a + \eta(b, a)]$ and $|f'|^q$ is harmonic s-Godunova-Levin-*

preinvex function on I for $q \geq 1$, then for all $x \in [a, a+\eta(b,a)]$, we have

$$
\begin{aligned}
&|\Psi_f(g;\alpha;x;a,a+\eta(b,a))| \\
&\leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\Bigg\{\Psi_9^{1-\frac{1}{q}}(a,x,\alpha)\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi_{11}^{***}(a,x,-s,1,\alpha)|f'(x)|^q \\
+ \quad &\Psi_{12}^{***}(a,x,-s,1,\alpha)|f'(a)|^q]^{\frac{1}{q}} + \Psi_{10}^{1-\frac{1}{q}}(a+\eta(b,a),x,\alpha)\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}} \\
&[\Psi_{13}^{***}(a+\eta(b,a),x,-s,1,\alpha)|f'(x)|^q + \Psi_{14}^{***}(a+\eta(b,a),x,-s,1,\alpha)|f'(a+\eta(b,a))|^q]^{\frac{1}{q}}\Bigg\},
\end{aligned}
$$

where $\Psi_9(a,x,\alpha)$ and $\Psi_{10}(a+\eta(b,a),x,\alpha)$ are given by 2.10 and 2.11 respectively and

$$
\begin{aligned}
\Psi_{11}^{***}(a,x,-s,1,\alpha) &= \int_0^1 \frac{t^{\alpha-s}}{(ta+(1-t)x)^2}\mathrm{d}t \\
= \quad &\frac{\beta(\alpha-s+1,1)}{x^2}{}_2F_1\left(2,\alpha-s+1;\alpha-s+2;1-\frac{a}{x}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{12}^{***}(a,x,-s,1,\alpha) &= \int_0^1 \frac{t^{\alpha}(1-t)^{-s}}{(ta+(1-t)x)^2}\mathrm{d}t \\
= \quad &\frac{\beta(\alpha+1,1-s)}{x^2}{}_2F_1\left(2,\alpha+1;\alpha-s+2;1-\frac{a}{x}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{13}^{***}(a+\eta(b,a),x,-s,1,\alpha) &= \int_0^1 \frac{t^{\alpha-s}}{(t[a+\eta(b,a)]+(1-t)x)^2}\mathrm{d}t \\
= \quad &\frac{\beta(1,\alpha-s+1)}{[a+\eta(b,a)]^2}{}_2F_1\left(2,1;\alpha-s+2;1-\frac{x}{a+\eta(b,a)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{14}^{***}(a+\eta(b,a),x,-s,1,\alpha) &= \int_0^1 \frac{t^{\alpha}(1-t)^{-s}}{(t[a+\eta(b,a)]+(1-t)x)^2}\mathrm{d}t \\
= \quad &\frac{\beta(1-s,\alpha+1)}{[a+\eta(b,a)]^2}{}_2F_1\left(2,1-s;\alpha-s+2;1-\frac{x}{a+\eta(b,a)}\right).
\end{aligned}
$$

Theorem 2.4. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a+\eta(b,a)]$ and $|f'|^q$ is harmonic h-preinvex function on I for $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$
\begin{aligned}
&|\Psi_f(g;\alpha;x;a,a+\eta(b,a))| \\
&\leq \left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\Bigg\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi_{15}(a,x,h,q,0)|f'(x)|^q + \Psi_{16}(a,x,h,q,0)|f'(a)|^q]^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi_{17}(a+\eta(b,a),x,h,q,0)|f'(x)|^q \\
&+\Psi_{18}(a+\eta(b,a),x,h,q,0)|f'(a+\eta(b,a))|^q]^{\frac{1}{q}}\Bigg\}, \qquad (2.12)
\end{aligned}
$$

where $\alpha > 0$ and $\Psi_{15}(a,x,h,q,0)$, $\Psi_{16}(a,x,h,q,0)$, $\Psi_{17}(a+\eta(b,a),x,h,q,0)$ and $\Psi_{18}(a+\eta(b,a),x,h,q,0)$ can be deduced from 2.4, 2.5, 2.6 and 2.7 respectively.

Proof. Using Lemma 2.1 and Holder's inequality and harmonic h-preinvexity of $|f'|^q$ on I, we have

$$\begin{aligned}
&\left|\Psi_f(g;\alpha;x;a,a+\eta(b,a))\right| \\
&\le \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\int_0^1 \frac{t^\alpha}{[ta+(1-t)x]^2}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|\mathrm{d}t \\
&\quad+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{(bx)^{\alpha-1}} \\
&\times\int_0^1 \frac{t^\alpha}{[t[a+\eta(b,a)]+(1-t)x]^2}\left|f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\right|\mathrm{d}t \\
&\le \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\left(\int_0^1 t^{\alpha p}\mathrm{d}t\right)^{\frac{1}{p}} \\
&\left(\int_0^1 \frac{1}{[ta+(1-t)x]^{2q}}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}\left(\int_0^1 t^{\alpha p}\mathrm{d}t\right)^{\frac{1}{p}} \\
&\left(\int_0^1 \frac{1}{[t[a+\eta(b,a)]+(1-t)x]^{2q}}\left|f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\left(\int_0^1 t^{\alpha p}\mathrm{d}t\right)^{\frac{1}{p}} \\
&\left(\int_0^1 \frac{1}{[ta+(1-t)x]^{2q}}[h(t)|f'(x)|^q+h(1-t)|f'(a)|^q]\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}\left(\int_0^1 t^{\alpha p}\mathrm{d}t\right)^{\frac{1}{p}} \\
&\left(\int_0^1 \frac{1}{[t[a+\eta(b,a)]+(1-t)x]^{2q}}\right. \\
&\left.\times[h(t)|f'(x)|^q+h(1-t)|f'(a+\eta(b,a))|^q]\mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \\
&\times\left\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi_{15}(a,x,h,q,0)|f'(x)|^q+\Psi_{16}(a,x,h,q,0)|f'(a)|^q]^{\frac{1}{q}}\right. \\
&\quad+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi_{17}(a+\eta(b,a),x,h,q,0)|f'(x)|^q \\
&\quad\left.+\Psi_{18}(a+\eta(b,a),x,h,q,0)|f'(a+\eta(b,a))|^q]^{\frac{1}{q}}\right\},
\end{aligned}$$

which is the required result. □

Corollary 2.14. *In Theorem 2.2, if* $|f'(x)| \leq M$, $x \in [a, a+\eta(b,a)]$, *then inequality*

$$\begin{aligned}&\left|\Psi_f(g;\alpha;x;a,a+\eta(b,a))\right|\\ &\leq M\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi_{15}(a,x,h,q,0)+\Psi_{16}(a,x,h,q,0)]^{\frac{1}{q}}\right.\\ &\left.+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi_{17}(a+\eta(b,a),x,h,q,0)+\Psi_{18}(a+\eta(b,a),x,h,q,0)]^{\frac{1}{q}}\right\},\end{aligned}$$

holds.

Remark 2.12. In Theorem 2.4, if we take $h(t) = 1$, then the identity 2.12 reduces to the following result.

Corollary 2.15. *Let* $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be a differentiable function on the interior* I^o *of* I. *If* $f' \in L[a, a + \eta(b, a)]$ *and* $|f'|^q$ *is harmonic P-preinvex function on* I *for* $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\begin{aligned}&|\Psi_f(g;\alpha;x;a,a+\eta(b,a))|\\ &\leq\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi^*_{15}(a,x,0,q,0)(|f'(x)|^q+|f'(a)|^q)]^{\frac{1}{q}}\right.\\ &\left.+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi^*_{16}(a+\eta(b,a),x,0,q,0)(|f'(x)|^q+|f'(a+\eta(b,a))|^q)]^{\frac{1}{q}}\right\},\end{aligned}$$

where an easy calculation gives

$$\begin{aligned}\Psi^*_{15}(a,x,0,q,0) &= \int_0^1 \frac{1}{(ta+(1-t)x)^{2q}}\mathrm{d}t\\ &= \frac{1}{x^{2q}}{}_2F_1\left(2q,1;2;1-\frac{a}{x}\right),\end{aligned}$$

$$\begin{aligned}\Psi^*_{16}(a+\eta(b,a),x,0,q,0) &= \int_0^1 \frac{1}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t\\ &= \frac{1}{[a+\eta(b,a)]^{2q}}{}_2F_1\left(2q,1;2;1-\frac{x}{a+\eta(b,a)}\right),\end{aligned}$$

Remark 2.13. In Theorem 2.4, if we take $h(t) = t^s$, then the identity 2.12 reduces to the following result.

Corollary 2.16. *Let* $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be a differentiable function on the interior* I^o *of* I. *If* $f' \in L[a, a + \eta(b, a)]$ *and* $|f'|^q$ *is harmonic s-preinvex function on* I *for* $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\begin{aligned}&|\Psi_f(g;\alpha;x;a,a+\eta(b,a))|\\ &\leq\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi^{**}_{15}(a,x,s,q,0)|f'(x)|^q+\Psi^{**}_{16}(a,x,s,q,0)|f'(a)|^q]^{\frac{1}{q}}\right.\\ &+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi^{**}_{17}(a+\eta(b,a),x,s,q,0)|f'(x)|^q\\ &\left.+\Psi^{**}_{18}(a+\eta(b,a),x,s,q,0)|f'(a+\eta(b,a))|^q]^{\frac{1}{q}}\right\},\end{aligned}$$

where an easy calculation gives

$$\Psi_{15}^{**}(a,x,s,q,0)=\int_0^1 \frac{t^s}{(ta+(1-t)x)^{2q}}\mathrm{d}t$$
$$=\frac{1}{x^{2q}(s+1)}{}_2F_1\left(2q,s+1;s+2;1-\frac{a}{x}\right),$$

$$\Psi_{16}^{**}(a,x,s,q,0)=\int_0^1 \frac{(1-t)^s}{(ta+(1-t)x)^{2q}}\mathrm{d}t$$
$$=\frac{1}{x^{2q}(s+1)}{}_2F_1\left(2q,1;s+2;1-\frac{a}{x}\right),$$

$$\Psi_{17}^{**}(a+\eta(b,a),x,s,q,0)=\int_0^1 \frac{t^s}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t$$
$$=\frac{1}{[a+\eta(b,a)]^{2q}(s+1)}{}_2F_1\left(2q,1;s+2;1-\frac{x}{a+\eta(b,a)}\right),$$

$$\Psi_{18}^{**}(a+\eta(b,a),x,s,q,0)=\int_0^1 \frac{(1-t)^s}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t$$
$$=\frac{1}{[a+\eta(b,a)]^{2q}(s+1)}{}_2F_1\left(2q,s+1;s+2;1-\frac{x}{a+\eta(b,a)}\right).$$

Remark 2.14. In Theorem 2.4, if we take $h(t)=t^{-s}$, then the identity 2.12 reduces to the following result.

Corollary 2.17. *Let $f: I=[a,a+\eta(b,a)]\subseteq \mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,a+\eta(b,a)]$ and $|f'|^q$ is harmonic s-Godunova-Levin-preinvex function on I for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}
&|\Psi_f(g;\alpha;x;a,a+\eta(b,a))|\\
&\le\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}[\Psi_{15}^{***}(a,x,-s,q,0)|f'(x)|^q+\Psi_{16}^{***}(a,x,-s,q,0)|f'(a)|^q]^{\frac{1}{q}}\right.\\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}[\Psi_{17}^{***}(a+\eta(b,a),x,-s,q,0)|f'(x)|^q\\
&\left.+\Psi_{18}^{***}(a+\eta(b,a),x,-s,q,0)|f'(a+\eta(b,a))|^q]^{\frac{1}{q}}\right\},
\end{aligned}$$

where an easy calculation gives

$$\Psi_{15}^{***}(a,x,-s,q,0)=\int_0^1 \frac{t^{-s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t$$
$$=\frac{1}{x^{2q}(1-s)}{}_2F_1\left(2q,1-s;2-s;1-\frac{a}{x}\right),$$

$$\Psi_{16}^{***}(a,x,-s,q,0)=\int_0^1 \frac{(1-t)^{-s}}{(ta+(1-t)x)^{2q}}\mathrm{d}t$$
$$=\frac{1}{x^{2q}(1-s)}{}_2F_1\left(2q,1;2-s;1-\frac{a}{x}\right),$$

$$\Psi_{17}^{***}(a+\eta(b,a),x,-s,q,0)=\int_0^1 \frac{t^{-s}}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t$$
$$= \frac{1}{[a+\eta(b,a)]^{2q}(1-s)}\,{}_2F_1\left(2q,1;2-s;1-\frac{x}{a+\eta(b,a)}\right),$$

$$\Psi_{18}^{***}(a+\eta(b,a),x,-s,q,0)=\int_0^1 \frac{(1-t)^{-s}}{(t[a+\eta(b,a)]+(1-t)x)^{2q}}\mathrm{d}t$$
$$= \frac{1}{[a+\eta(b,a)]^{2q}(1-s)}\,{}_2F_1\left(2q,1-s;2-s;1-\frac{x}{a+\eta(b,a)}\right).$$

Theorem 2.5. *Let $f: I=[a,a+\eta(b,a)]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,a+\eta(b,a)]$ and $|f'|^q$ is harmonic h-preinvex function on I for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}
&|\Psi_f(g;\alpha;x;a,a+\eta(b,a))|\\
&\leq \Bigg\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}(\Psi_{19}(a,x,0,p,\alpha p))^{\frac{1}{p}}\Big([|f'(x)|^q+|f'(a)|^q]\int_0^1 h(t)\mathrm{d}t\Big)^{\frac{1}{q}}\\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}(\Psi_{20}(a+\eta(b,a),x,0,p,\alpha p))^{\frac{1}{p}}\Big([|f'(x)|^q\\
&+|f'(a+\eta(b,a))|^q]\int_0^1 h(t)\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg\},
\end{aligned}\tag{2.13}$$

where $\alpha>0$ and an easy calculation gives

$$\begin{aligned}
\Psi_{19}(a,x,0,p,\alpha p)&=\int_0^1 \frac{t^{\alpha p}}{(ta+(1-t)x)^{2p}}\mathrm{d}t\\
&= \frac{1}{x^{2p}(\alpha p+1)}\,{}_2F_1\left(2p,\alpha p+1;\alpha p+2;1-\frac{a}{x}\right),
\end{aligned}\tag{2.14}$$

$$\begin{aligned}
\Psi_{20}(a+\eta(b,a),x,0,p,\alpha p)&=\int_0^1 \frac{t^{\alpha p}}{(t[a+\eta(b,a)]+(1-t)x)^{2p}}\mathrm{d}t\\
&= \frac{1}{[a+\eta(b,a)]^{2p}(\alpha p+1)}\,{}_2F_1\left(2p,1;\alpha p+2;1-\frac{x}{a+\eta(b,a)}\right),
\end{aligned}\tag{2.15}$$

Proof. Using Lemma 2.1 and Holder's inequality and harmonic h-preinvexity of $|f'|^q$

on I, we have

$$\begin{aligned}
&\left|\Psi_f(g;\alpha;x;a,a+\eta(b,a))\right| \\
&\le \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\int_0^1 \frac{t^{\alpha}}{[ta+(1-t)x]^2}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|\mathrm{d}t \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}} \\
&\times\int_0^1 \frac{t^{\alpha}}{[t[a+\eta(b,a)]+(1-t)x]^2}\left|f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\right|\mathrm{d}t \\
&\le \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\left(\int_0^1 \frac{t^{\alpha p}}{[ta+(1-t)x]^{2p}}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}\left(\int_0^1 \frac{t^{\alpha p}}{[t[a+\eta(b,a)]+(1-t)x]^{2p}}\mathrm{d}t\right)^{\frac{1}{p}} \\
&\left(\int_0^1 \left|f'\left(\frac{[a+\eta(b,a)]x}{t[a+\eta(b,a)]+(1-t)x}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\left(\int_0^1 \frac{t^{\alpha p}}{[ta+(1-t)x]^{2p}}\mathrm{d}t\right)^{\frac{1}{p}}\left([|f'(x)|^q+|f'(a)|^q]\int_0^1 h(t)\mathrm{d}t\right)^{\frac{1}{q}} \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}\left(\int_0^1 \frac{t^{\alpha p}}{[t[a+\eta(b,a)]+(1-t)x]^{2p}}\mathrm{d}t\right)^{\frac{1}{p}} \\
&\left([|f'(x)|^q+|f'(a+\eta(b,a))|^q]\int_0^1 h(t)\mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \left\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}(\Psi_{19}(a,x,0,p,\alpha p))^{\frac{1}{p}}\left([|f'(x)|^q+|f'(a)|^q]\int_0^1 h(t)\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}(\Psi_{20}(a+\eta(b,a),x,0,p,\alpha p))^{\frac{1}{p}}\left([|f'(x)|^q\right. \\
&\left.\left.+|f'(a+\eta(b,a))|^q]\int_0^1 h(t)\mathrm{d}t\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

which is the required result. □

Corollary 2.18. *In Theorem 2.5, if* $|f'(x)|\le M$, $x\in[a,a+\eta(b,a)]$, *then inequality*

$$\begin{aligned}
&\left|\Psi_f(g;\alpha;x;a,a+\eta(b,a))\right| \\
&\le M\left\{\frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}}\Psi_{19}^{\frac{1}{p}}(a,x,0,p,\alpha p)\right. \\
&\left.+\frac{([a+\eta(b,a)]-x)^{\alpha+1}}{([a+\eta(b,a)]x)^{\alpha-1}}\Psi_{20}^{\frac{1}{p}}(a+\eta(b,a),x,0,p,\alpha p)\right\},
\end{aligned}$$

holds.

Remark 2.15. In Theorem 2.5, if we take $h(t) = 1$, then the identity 2.13 reduces to the following result.

Corollary 2.19. *Let $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a + \eta(b, a)]$ and $|f'|^q$ is harmonic P-preinvex function on I for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$|\Psi_f(g; \alpha; x; a, a + \eta(b, a))| \leq \left\{ \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \Psi_{19}^{\frac{1}{p}}(a, x, 0, p, \alpha p)\big(|f'(x)|^q + |f'(a)|^q\big)^{\frac{1}{q}} + \frac{([a + \eta(b, a)] - x)^{\alpha+1}}{([a + \eta(b, a)]x)^{\alpha-1}} \Psi_{20}^{\frac{1}{p}}(a + \eta(b, a), x, 0, p, \alpha p)\big(|f'(x)|^q + |f'(a + \eta(b, a))|^q\big)^{\frac{1}{q}} \right\},$$

where $\Psi_{19}(a, x, 0, p, \alpha p)$ and $\Psi_{20}(a + \eta(b, a), x, 0, p, \alpha p)$ are given by 2.14 *and* 2.15 *respectively.*

Remark 2.16. In Theorem 2.5, if we take $h(t) = t^s$, then the identity 2.13 reduces to the following result.

Corollary 2.20. *Let $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a + \eta(b, a)]$ and $|f'|^q$ is harmonic s-preinvex function on I for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$|\Psi_f(g; \alpha; x; a, a + \eta(b, a))| \leq \left\{ \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \Psi_{19}^{\frac{1}{p}}(a, x, 0, p, \alpha p)\left(\frac{|f'(x)|^q + |f'(a)|^q}{s+1}\right)^{\frac{1}{q}} + \frac{([a + \eta(b, a)] - x)^{\alpha+1}}{([a + \eta(b, a)]x)^{\alpha-1}} \Psi_{20}^{\frac{1}{p}}(a + \eta(b, a), x, 0, p, \alpha p)\left(\frac{|f'(x)|^q + |f'(a + \eta(b, a))|^q}{s+1}\right)^{\frac{1}{q}} \right\},$$

where $\Psi_{19}(a, x, 0, p, \alpha p)$ and $\Psi_{20}(a + \eta(b, a), x, 0, p, \alpha p)$ are given by 2.14 *and* 2.15 *respectively.*

Remark 2.17. In Theorem 2.5, if we take $h(t) = t^{-s}$, then the identity 2.13 reduces to the following result.

Corollary 2.21. *Let $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a + \eta(b, a)]$ and $|f'|^q$ is harmonic s-Godunova-Levin-preinvex function on I for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$|\Psi_f(g; \alpha; x; a, a + \eta(b, a))| \leq \left\{ \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \Psi_{19}^{\frac{1}{p}}(a, x, 0, p, \alpha p)\left(\frac{|f'(x)|^q + |f'(a)|^q}{1-s}\right)^{\frac{1}{q}} + \frac{([a + \eta(b, a)] - x)^{\alpha+1}}{([a + \eta(b, a)]x)^{\alpha-1}} \Psi_{20}^{\frac{1}{p}}(a + \eta(b, a), x, 0, p, \alpha p)\left(\frac{|f'(x)|^q + |f'(a + \eta(b, a))|^q}{1-s}\right)^{\frac{1}{q}} \right\},$$

where $\Psi_{19}(a, x, 0, p, \alpha p)$ and $\Psi_{20}(a + \eta(b, a), x, 0, p, \alpha p)$ are given by 2.14 *and* 2.15 *respectively.*

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