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A NEW COMPANION OF OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION

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Abstract. In this paper, we first define two functionals which are obtained using the Riemann-Stieltjes integral. Then, we establish a new companion of Ostrowski type inequalities for functions of two independent variables with bounded variation and give numerical cubature formulae for the Riemann-Stieltjes integral.

Keywords: Ostrowski type inequalities, bounded variation, Riemann-Stieltjes integrals

1. Introduction

Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty , \tag{1.1}$$

for all $x \in [a,b]$[26]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

In [18], Dragomir proved the following Ostrowski type inequalities for functions of bounded variation:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left| \int_a^b f(t)dt - (b-a) f(x) \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f)$$

holds for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible.

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In [5], Barnett et. al. proved the following inequalities for functions of bounded variation:

Theorem 1.2. *Assume that the function $f : [a,b] \to R$ is of bounded variation on $[a,b]$. Then we have the inequalities:*

(1.2)
$$|\Psi_f(t)| \leq \frac{1}{b-a}\left[(t-a)\bigvee_a^t(f) + (b-t)\bigvee_t^b(f)\right]$$

$$\leq \begin{cases} \left[\frac{1}{2} + \left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f), \\ \left[\left(\frac{t-a}{b-a}\right)^q + \left(\frac{b-t}{b-a}\right)^q\right]^{\frac{1}{q}}\left[\left(\bigvee_a^t(f)\right)^p + \left(\bigvee_t^b(f)\right)^p + \right]^{\frac{1}{p}} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^t(f) - \bigvee_t^b(f)\right| \end{cases}$$

where
$$\Psi_f(t) := f(t) - \frac{f(a)(t-a)+(b-t)f(b)}{b-a}.$$

The first inequality is sharp and the constant $\frac{1}{2}$ is also the best possible in both branches in (1.2).

2. Preliminaries and Lemmas

In 1910, Fréchet [23] provided the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x,y)$ and $g(x,y)$ are defined over the rectangle $Q = [a,b] \times [c,d]$; let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x = x_i$, $y = y_i$,

$$a = x_0 < x_1 < ... < x_n = b, \text{ and } c = y_0 < y_1 < ... < y_m = d;$$

let ξ_i, η_j be any numbers satisfying $\xi_i \in [x_{i-1}, x_i]$, $\eta_j \in [y_{j-1}, y_j]$, $(i = 1,2,...,n;$ $j = 1,2,...,m)$; and for all i,j let

$$\Delta_{11}g(x_i,y_j) = g(x_{i-1},y_{j-1}) - g(x_{i-1},y_j) - g(x_i,y_{j-1}) + g(x_i,y_j).$$

Then if the sum
$$S = \sum_{i=1}^{n}\sum_{j=1}^{m} f(\xi_i,\eta_j)\,\Delta_{11}g(x_i,y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to g is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b \int_c^d f(x,y) d_y d_x g(x,y). \tag{2.1}$$

If in the above formulation S is replaced by the sum

$$S^* = \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_{ij}, \eta_{ij}) \Delta_{11} g(x_i, y_j),$$

where ξ_{ij}, η_{ij} are numbers satisfying $\xi_{ij} \in [x_{i-1}, x_i]$, $\eta_{ij} \in [y_{j-1}, y_j]$, we call the limit, when it exists, an unrestricted integral, and designate it by the symbol

$$\int_a^b \int_c^d f(x,y) d_y d_x g(x,y). \tag{2.2}$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson [15] has shown that the existence of (2.1) does not imply the existence of (2.2).

In [14], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables:

2.1. Definitions

The function $f(x,y)$ is assumed to be defined in rectangle $R(a \le x \le b,\ c \le y \le d)$. By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned} x &= x_i (i = 0,1,2,...,m),\ a = x_0 < x_1 < ... < x_m = b; \\ y &= y_j (j = 0,1,2,...,n),\ c = y_0 < y_1 < ... < y_n = d. \end{aligned}$$

Each of the smaller rectangles into which R is devided by a net will be called a *cell*. We employ the notation

$$\Delta_{11} f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j)$$
$$\Delta f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_i, y_j)$$

The total variation function, $\phi(\overline{x})$ $[\psi(\overline{y})]$, is defined as the total variation of $f(\overline{x}, y)$ $[f(x, \overline{y})]$ considered as a function of y $[x]$ alone in interval (c,d) $[(a,b)]$,or as $+\infty$ if $f(\overline{x}, y)$ $[f(x, \overline{y})]$ is of unbounded variation.

Definition 2.1. (Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets.

Definition 2.2. (Fréchet). The function $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} \epsilon_i \overline{\epsilon_j} \left| \Delta_{11} f(x_i, y_j) \right|$$

is bounded for all nets and all possible choices of $\epsilon_i = \pm 1$ and $\overline{\epsilon_j} = \pm 1$.

Definition 2.3. (Hardy-Krause). The function $f(x,y)$ is said to be of bounded variation if it satisfies the condition of Definition 2.1 and if in addition $f(\overline{x}, y)$ is of bounded variation in y (i.e. $\phi(\overline{x})$ is finite) for at least one $\overline{x}$ and $f(x, \overline{y})$ is of bounded variation in y (i.e. $\psi(\overline{y})$ is finite) for at least one $\overline{y}$.

Definition 2.4. (Arzelà). Let (x_i, y_i) $(i = 0, 1, 2, ..., m)$ be any set of points satisfying the conditions

$$\begin{aligned} a &= x_0 < x_1 < ... < x_m = b; \\ c &= y_0 < y_1 < ... < y_m = d. \end{aligned}$$

Then $f(x,y)$ is said to be of bounded variation if the sum

$$\sum_{i=1}^{m} \left| \Delta f(x_i, y_i) \right|$$

is bounded for all such sets of points.

Therefore, one can define the consept of total variation of a function of variables, as follows:

Let f be of bounded variation on $Q = [a,b] \times [c,d]$, and let $\sum(P)$ denote the sum $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} \left| \Delta_{11} f(x_i, y_j) \right|$ corresponding to the partition P of Q. The number

$$\bigvee_Q (f) := \bigvee_c^d \bigvee_a^b (f) := \sup \left\{ \sum (P) : P \in P(Q) \right\},$$

is called the total variation of f on Q.

In [24], authors proved the following Lemmas for double Riemann-Stieltjes integral:

Lemma 2.1. *(Integrating by parts) If $f \in RS(g)$ on Q, then $g \in RS(f)$ on Q, and we have*

$$\begin{aligned} &\int_c^d \int_a^b f(t,s) d_t d_s g(t,s) + \int_c^d \int_a^b g(t,s) d_t d_s f(t,s) \\ &= f(b,d)g(b,d) - f(b,c)g(b,c) - f(a,d)g(a,d) + f(a,c)g(a,c). \end{aligned} \tag{2.3}$$

Lemma 2.2. *Assume that* $\Omega \in RS(g)$ *on* Q *and* g *is of bounded variation on* Q*, then*

$$\left|\int_c^d \int_a^b \Omega(x,y) d_x d_y g(x,y)\right| \leq \sup_{(x,y)\in Q} |\Omega(x,y)| \bigvee_Q (g). \tag{2.4}$$

In [24], Jawarneh and Noorani proved the following Ostrowski type inequality or functions of two variables with bounded variation:

Theorem 2.1. *Let* $f : Q \to\to \mathbb{R}$ *be mapping of bounded variation on* Q*. Then for all* $(x,y) \in Q$*, we have inequality*

$$\begin{aligned} &\left|(b-a)(d-c) f(x,y) - \int_c^d \int_a^b f(t,s) dt ds\right| \\ &\leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \left[\frac{1}{2}(d-c) + \left|y - \frac{c+d}{2}\right|\right] \bigvee_Q (f) \end{aligned} \tag{2.5}$$

where $\bigvee_Q (f)$ *denotes the total (double) variation of* f *on* Q*.*

In [7], Budak and Sarıkaya have proved the following generalization of the inequality (2.5):

Theorem 2.2. *Let* $f : Q \to \mathbb{R}$ *be mapping of bounded variation on* Q*. Then for all* $(x,y) \in Q$*, we have inequality*

$$\begin{aligned} &\Bigg|(b-a)(d-c)\left[\lambda\eta \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\ &+ (1-\lambda)\eta \frac{f(a,y)+f(b,y)}{2} + \lambda(1-\eta) \frac{f(x,c)+f(x,d)}{2} \\ &\left.(1-\lambda)(1-\eta) f(x,y)\right] - \int_a^b \int_c^d f(t,s) ds dt\Bigg| \\ &\leq \max\left\{\lambda \frac{b-a}{2}, \left(x - \frac{(2-\lambda)a + \lambda b}{2}\right), \left(\frac{(2-\lambda)b+\lambda a}{2} - x\right)\right\} \\ &\times \max\left\{\eta \frac{d-c}{2}, \left(y - \frac{(2-\eta)c + \eta d}{2}\right), \left(\frac{(2-\eta)d+\eta c}{2} - y\right)\right\} \bigvee_a^b \bigvee_c^d (f) \end{aligned} \tag{2.6}$$

for any $\lambda, \eta \in [0,1]$ *and* $a + \lambda\frac{b-a}{2} \leq x \leq b - \lambda\frac{b-a}{2}$, $c + \eta\frac{d-c}{2} \leq y \leq d - \eta\frac{d-c}{2}$*, where* $\bigvee_a^b \bigvee_c^d (f)$ *denotes he total variation of* f *on* Q*.*

In [10], authors have proved the following Ostrowski type inequality for mappings of bounded variation.

Theorem 2.3. *If the function $f : Q = [a,b] \times [c,d] \to R$ is of bounded variation on Q, then we have*

$$\left| \frac{1}{4}\left[f(x,y) + f(x,c+d-y) + f(a+b-x,y) + f(a+b-x,c+d-y)\right] - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right| \leq \left[\frac{1}{4} + \left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\left[\frac{1}{4} + \left|\frac{y-\frac{3c+d}{4}}{d-c}\right|\right] \bigvee_Q(f) \tag{2.7}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ and $y \in \left[c, \frac{c+d}{2}\right]$, where $\bigvee_Q(f)$ denotes the total (double) variation of f on Q.

Ostrowski type inequalities for function of two variables with bounded variation were first given in [24]. Then, Budak and Sarikaya established the generalization of these inequalities in [6] and [7]. A companion of Ostrowski type inequalities for functions of two variables with bounded variation were given by Budak and Sarıkaya in [10]. Then, in [11], authors gave the generalization of inequalities in [10]. In this paper, using inequalities in [10] and some functional, we establish a new companion of Ostrowski Ostrowski type inequalities for functions of two independent variables with bounded variation similar to inequalities in (1.2)

Recently, many of inequalities for functions of a single variable with bounded variation have been proved. For more information and recent developments on inequalities for mappings of single variable with bounded variation, please refer to ([1]-[4],[8],[12],[13],[16],[17],[19]-[22],[25],[27]-[32]). In the literature, there are a few study for functions of two variables with bounded variation(see [6],[7],[9]-[11],[24]).

3. Main Results

First of all, we give the following notations used to simplify the details of presentations of Theorem 3.1 and Theorem 3.2:

$$\begin{aligned} Q_1 &= [a,x] \times [c,y], \ Q_2 = [a,x] \times [y,d], \\ Q_3 &= [x,b] \times [c,y], \ Q_4 = [x,b] \times [y,d], \end{aligned}$$

$$\Psi_f(x,y) = \frac{1}{(b-a)(d-c)}$$
$$\times [(b-a)(d-c) f(x,y) - (b-a)(y-c) f(x,c) - (b-a)(d-y) f(x,d)$$
$$- (x-a)(d-c) f(a,y) - (b-x)(d-c) f(b,y) + (x-a)(y-c) f(a,c)$$
$$+ (x-a)(d-y) f(a,d) + (b-x)(y-c) f(b,c) + (b-x)(d-y) f(b,d)],$$

$$GS(f;u)$$
$$= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \frac{f(x,y) + f(x,c+d-y) + f(a+b-x,y) + f(a+b-x,c+d-y)}{4} d_y d_x u(x,y)$$
$$- \frac{u\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - u\left(\frac{a+b}{2}, c\right) - u\left(a, \frac{c+d}{2}\right) + u(a,c)}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s) ds dt.$$

We may state the following results.

Theorem 3.1. *If the function $f : Q = [a,b] \times [c,d] \to R$ is of bounded variation on Q, then we have*

$$(3.1) \qquad |\Psi_f(x,y)|$$
$$\leq \frac{1}{(b-a)(d-c)} \left[(x-a)(y-c) \bigvee_{Q_1}(f) + (x-a)(d-y) \bigvee_{Q_2}(f) \right.$$
$$\left. + (b-x)(y-c) \bigvee_{Q_3}(f) + (b-x)(d-y) \bigvee_{Q_4}(f) \right]$$
$$\leq \begin{cases} \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \left[\frac{1}{2} + \left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right] \bigvee_{Q}(f), \\ \left[\left(\frac{x-a}{b-a}\right)^\alpha \left(\frac{y-c}{d-c}\right)^\alpha + \left(\frac{x-a}{b-a}\right)^\alpha \left(\frac{d-y}{d-c}\right)^\alpha \right. \\ \left. + \left(\frac{b-x}{b-a}\right)^\alpha \left(\frac{y-c}{d-c}\right)^\alpha + \left(\frac{b-x}{b-a}\right)^\alpha \left(\frac{d-y}{d-c}\right)^\alpha\right]^{\frac{1}{\alpha}} \\ \times \left[\left[\bigvee_{Q_1}(f)\right]^\beta + \left[\bigvee_{Q_2}(f)\right]^\beta + \left[\bigvee_{Q_3}(f)\right]^\beta + \left[\bigvee_{Q_4}(f)\right]^\beta\right]^{\frac{1}{\beta}}, \ \text{if } \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \max\left\{\bigvee_{Q_1}(f), \bigvee_{Q_2}(f), \bigvee_{Q_3}(f), \bigvee_{Q_4}(f)\right\} \end{cases}$$

for all $(x,y) \in Q$.

Proof. Let us consider the mappings T defined by

$$T(x,t;y,s) = \begin{cases} (x-a)(y-c) & \text{if } (t,s) \in Q_1 \\ (x-a)(y-d) & \text{if } (t,s) \in Q_2 \\ (x-b)(y-c) & \text{if } (t,s) \in Q_3 \\ (x-b)(y-d) & \text{if } (t,s) \in Q_4. \end{cases}$$

Using the integrating by parts (Lemma 2.1), we have

$$\begin{aligned} &\int_a^b \int_c^d T(x,t;y,s) d_s d_t f(t,s) \\ = \;& (x-a)(y-c) \int_a^x \int_c^y d_s d_t f(t,s) + (x-a)(y-d) \int_a^x \int_y^d d_s d_t f(t,s) \\ & + (x-b)(y-c) \int_x^b \int_c^y d_s d_t f(t,s) + (x-b)(y-d) \int_x^b \int_y^d d_s d_t f(t,s) \\ = \;& (x-a)(y-c)\left[f(x,y) - f(x,c) - f(a,y) + f(a,c)\right] \\ & + (x-a)(y-d)\left[f(x,d) - f(x,y) - f(a,d) + f(a,y)\right] \\ & + (x-b)(y-c)\left[f(b,y) - f(b,c) - f(x,y) + f(x,c)\right] \\ & + (x-b)(y-d)\left[f(b,d) - f(b,y) - f(x,d) + f(x,y)\right] \\ = \;& (b-a)(d-c)\Psi_f(x,y). \end{aligned}$$

Hence,

$$\Psi_f(x,y) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d T(x,t;y,s) d_s d_t f(t,s). \tag{3.2}$$

On the other hand, taking modulus in (3.2), then we have

$$
\begin{aligned}
(3.3)\qquad & |\Psi_f(x,y)| \\
= {} & \frac{1}{(b-a)(d-c)}\left|\int_a^b\int_c^d T(x,t;y,s)d_sd_tf(t,s)\right| \\
\leq {} & \frac{1}{(b-a)(d-c)} \\
& \times\left|(x-a)(y-c)\int_a^x\int_c^y d_sd_tf(t,s)+(x-a)(y-d)\int_a^x\int_y^d d_sd_tf(t,s)\right. \\
& \left.+(x-b)(y-c)\int_x^b\int_c^y d_sd_tf(t,s)+(x-b)(y-d)\int_x^b\int_y^d d_sd_tf(t,s)\right| \\
\leq {} & \frac{1}{(b-a)(d-c)} \\
& \times\left[(x-a)(y-c)\left|\int_a^x\int_c^y d_sd_tf(t,s)\right|+(x-a)(d-y)\left|\int_a^x\int_y^d d_sd_tf(t,s)\right|\right. \\
& \left.+(b-x)(y-c)\left|\int_x^b\int_c^y d_sd_tf(t,s)\right|+(b-x)(d-y)\left|\int_x^b\int_y^d d_sd_tf(t,s)\right|\right].
\end{aligned}
$$

Applying Lemma 2.2 in (3.3), we obtain

$$
\begin{aligned}
(3.4)\qquad & |\Psi_f(x,y)| \\
\leq {} & \frac{1}{(b-a)(d-c)}\left[(x-a)(y-c)\bigvee_{Q_1}(f)+(x-a)(d-y)\bigvee_{Q_2}(f)\right. \\
& \left.+(b-x)(y-c)\bigvee_{Q_3}(f)+(b-x)(d-y)\bigvee_{Q_4}(f)\right]:=N(x,y)
\end{aligned}
$$

which completes the proof of the first inequality in (3.1).

$$
\begin{aligned}
& N(x,y) \\
\leq\ & \frac{1}{(b-a)(d-c)} \max_{x,y} \{(x-a)(y-c), (x-a)(d-y), \\
& (b-x)(y-c), (b-x)(d-y)\} \\
& \times \left\{ \bigvee_{Q_1}(f) + \bigvee_{Q_2}(f) + \bigvee_{Q_3}(f) + \bigvee_{Q_4}(f) \right\} \\
=\ & \frac{1}{(b-a)(d-c)} \max_{x} \left\{ (x-a) \max_{y} \{y-c, d-y\}, \right. \\
& \left. (b-x) \max_{y} \{y-c, d-y\} \right\} \bigvee_{Q}(f).
\end{aligned}
$$

Since $\max_{y}$ is independent of x, we have

$$
\begin{aligned}
& N(x,y) \\
\leq\ & \frac{1}{(b-a)(d-c)} \max_{x} \{x-a, b-x\} \max_{y} \{y-c, d-y\} \bigvee_{Q}(f) \\
=\ & \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left[\frac{1}{2} + \left| \frac{y - \frac{c+d}{2}}{d-c} \right| \right] \bigvee_{Q}(f).
\end{aligned}
$$

This finishes the proof of the first branch of the second inequality in (3.1).

For $\frac{1}{\alpha}+\frac{1}{\beta}=1$, using Hölder's discrete inequality in (3.4), then we have

$$
\begin{aligned}
& N(x,y) \\
\leq\ & \frac{1}{(b-a)(d-c)}\left[(x-a)^{\alpha}(y-c)^{\alpha}+(x-a)^{\alpha}(d-y)^{\alpha}\right. \\
& \left.+(b-x)^{\alpha}(y-c)^{\alpha}+(b-x)^{\alpha}(d-y)^{\alpha}\right]^{\frac{1}{\alpha}} \\
& \times\left[\left[\bigvee_{Q_1}(f)\right]^{\beta}+\left[\bigvee_{Q_2}(f)\right]^{\beta}+\left[\bigvee_{Q_3}(f)\right]^{\beta}+\left[\bigvee_{Q_4}(f)\right]^{\beta}\right]^{\frac{1}{\beta}} \\
=\ & \left[\left(\frac{x-a}{b-a}\right)^{\alpha}\left(\frac{y-c}{d-c}\right)^{\alpha}+\left(\frac{x-a}{b-a}\right)^{\alpha}\left(\frac{d-y}{d-c}\right)^{\alpha}\right. \\
& \left.+\left(\frac{b-x}{b-a}\right)^{\alpha}\left(\frac{y-c}{d-c}\right)^{\alpha}+\left(\frac{b-x}{b-a}\right)^{\alpha}\left(\frac{d-y}{d-c}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \\
& \times\left[\left[\bigvee_{Q_1}(f)\right]^{\beta}+\left[\bigvee_{Q_2}(f)\right]^{\beta}+\left[\bigvee_{Q_3}(f)\right]^{\beta}+\left[\bigvee_{Q_4}(f)\right]^{\beta}\right]^{\frac{1}{\beta}}
\end{aligned}
$$

which completes the proof of the second branch of the second inequality in (3.1).

Finally, using the function of maximum, we have

$$
\begin{aligned}
& N(x,y) \\
\leq\ & \frac{1}{(b-a)(d-c)}\max\left\{\bigvee_{Q_1}(f),\bigvee_{Q_2}(f),\bigvee_{Q_3}(f),\bigvee_{Q_4}(f)\right\} \\
& \times\left[(x-a)(y-c)+(x-a)(d-y)+(b-x)(y-c)+(b-x)(d-y)\right] \\
=\ & \max\left\{\bigvee_{Q_1}(f),\bigvee_{Q_2}(f),\bigvee_{Q_3}(f),\bigvee_{Q_4}(f)\right\}.
\end{aligned}
$$

Herewith, the proof is completed. □

Corollary 3.1. *Under the assumption Theorem 3.1, choosing $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, we have*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$

$$\left.-\frac{f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)}{2}+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|$$

$$\leq \frac{1}{4}\bigvee_{Q}(f).$$

Theorem 3.2. *Let $u : Q = [a,b]\times[c,d] \to R$ be a mapping of bounded variation on Q and $f : Q \to R$ be continuous and of bounded variation on Q. Then we have the inequality:*

$$|GS(f;u)| \leq \frac{1}{4}\bigvee_a^b\bigvee_c^d(f)\bigvee_a^{\frac{a+b}{2}}\bigvee_c^{\frac{c+d}{2}}(u).$$

Proof. Using Lemma 2.2, we have

$$|GS(f;u)|$$

$$= \left|\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left[\frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)\right.\right.\right.$$

$$\left.\left.\left.+f(a+b-x,c+d-y)\right]-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right]d_yd_xu(x,y)\right|$$

$$\leq \sup_{(x,y)\in Q}\left|\frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)\right.\right.$$

$$\left.\left.+f(a+b-x,c+d-y)\right]-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\bigvee_a^{\frac{a+b}{2}}\bigvee_c^{\frac{c+d}{2}}(u).$$

Since f is of bounded variation, using Theorem 2.3, we have

$$\left|\frac{f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)}{4}\right.$$

$$\left.-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq \left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\left[\frac{1}{4}+\left|\frac{y-\frac{3c+d}{4}}{d-c}\right|\right]\bigvee_a^b\bigvee_c^d(f).$$

Hence,

$$\begin{aligned} |GS(f;u)| &\le \sup_{(x,y)\in Q}\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\left[\frac{1}{4}+\left|\frac{y-\frac{3c+d}{4}}{d-c}\right|\right]\bigvee_a^b\bigvee_c^d(f)\bigvee_a^{\frac{a+b}{2}}\bigvee_c^{\frac{c+d}{2}}(u) \\ &\le \frac{1}{4}\bigvee_a^b\bigvee_c^d(f)\bigvee_a^{\frac{a+b}{2}}\bigvee_c^{\frac{c+d}{2}}(u). \end{aligned}$$

This completes the proof. □

Corollary 3.2. *If we take* $\bigvee_a^{\frac{a+b}{2}}\bigvee_c^{\frac{c+d}{2}}(u)=\bigvee_a^{\frac{a+b}{2}}\bigvee_{\frac{c+d}{2}}^{d}(u)=\bigvee_{\frac{a+b}{2}}^{b}\bigvee_c^{\frac{c+d}{2}}(u)=\bigvee_{\frac{a+b}{2}}^{b}\bigvee_{\frac{c+d}{2}}^{d}(u)$ *in Theorem 3.2, then we have*

$$|GS(f;u)|\le\frac{1}{16}\bigvee_a^b\bigvee_c^d(f)\bigvee_a^b\bigvee_c^d(u).$$

4. Application to Cubature Formulae

Let us consider an arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$, $h_i := x_{i+1} - x_i$, and $l_j := y_{j+1} - y_j$.

Then the following Theorem holds.

Theorem 4.1. *Let f and u be as in Theorem 3.2. Then we have the cubature formulae:*

$$\begin{aligned} &\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\frac{f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)}{4}d_yd_xu(x,y) \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\frac{u\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)-u\left(\frac{x_i+x_{i+1}}{2},y_j\right)-u\left(x_i,\frac{y_j+y_{j+1}}{2}\right)+u\left(x_i,y_j\right)}{h_il_j} \\ &\quad\times\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}f(t,s)dsdt+R(I_n,J_m,f,u) \end{aligned}$$

The remainder term $R(I_n,J_m,f,u)$ satisfies

$$|R(I_n,J_m,f,u)|\le\frac{1}{4}\max_{\substack{i=\overline{0,n-1}\\ j=\overline{0,m-1}}}\left\{\bigvee_{x_i}^{\frac{x_i+x_{i+1}}{2}}\bigvee_{y_j}^{\frac{y_j+y_{j+1}}{2}}(u)\right\}\bigvee_a^b\bigvee_c^d(f).$$

Proof. Applying Theorem 3.2 to bidimensional interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we have the inequality

$$
\begin{aligned}
(4.1)\quad & \Bigg| \int_{x_i}^{\frac{x_i+x_{i+1}}{2}} \int_{y_j}^{\frac{y_j+y_{j+1}}{2}} \frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)\right. \\
& \left.+f(a+b-x,c+d-y)\right] d_y d_x u(x,y) \\
& -\frac{u\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)-u\left(\frac{x_i+x_{i+1}}{2},y_j\right)-u\left(x_i,\frac{y_j+y_{j+1}}{2}\right)+u\left(x_i,y_j\right)}{h_i l_j} \\
& \times \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(t,s)dsdt \Bigg| \\
\leq\ & \frac{1}{4} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \bigvee_{x_i}^{\frac{x_i+x_{i+1}}{2}} \bigvee_{y_j}^{\frac{y_j+y_{j+1}}{2}} (u).
\end{aligned}
$$

Summing the inequality (4.1) over i from 0 to $n-1$ and j from 0 to $m-1$, then we get

$$
\begin{aligned}
& |R(I_n, J_m, f, u)| \\
\leq\ & \frac{1}{4} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \bigvee_{x_i}^{\frac{x_i+x_{i+1}}{2}} \bigvee_{y_j}^{\frac{y_j+y_{j+1}}{2}} (u) \\
\leq\ & \frac{1}{4} \max_{\substack{i=\overline{0,n-1} \\ j=\overline{0,m-1}}} \left\{ \bigvee_{x_i}^{\frac{x_i+x_{i+1}}{2}} \bigvee_{y_j}^{\frac{y_j+y_{j+1}}{2}} (u) \right\} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \\
=\ & \frac{1}{4} \max_{\substack{i=\overline{0,n-1} \\ j=\overline{0,m-1}}} \left\{ \bigvee_{x_i}^{\frac{x_i+x_{i+1}}{2}} \bigvee_{y_j}^{\frac{y_j+y_{j+1}}{2}} (u) \right\} \bigvee_{a}^{b} \bigvee_{c}^{d} (f).
\end{aligned}
$$

This completes the proof. □

REFERENCES

1. M. W. Alomari, *A Generalization of Weighted Companion of Ostrowski Integral Inequality for Mappings of Bounded Variation*, RGMIA Research Report Collection, 14(2011), Article 87, 11 pp.
2. M. W. Alomari and M.A. Latif, *Weighted Companion for the Ostrowski and the Generalized Trapezoid Inequalities for Mappings of Bounded Variation,* RGMIA Research Report Collection, 14(2011), Article 92, 10 pp.
3. M. W. Alomari, *A Generalization of Dragomir's generalization of Ostrowski Integral Inequality and applications in numerical integration*, Ukrainian Mathematical Journal, 2012, 64(4).
4. M. W. Alomari, *On approximation of Riemann-Stieltjes integral and application*, RGMIA Research Report Collection, 15(2012), Article 25, 14 pp.
5. N.S. Barnett, S.S. Dragomir, I. Gomm, *A companion for the Ostrowski and the generalized trapezoid inequalities,* Mathematical and Computer Modelling 50 (2009) 179 187.
6. H. Budak and M.Z. Sarıkaya, *On Ostrowski type inequalities for functions of two variables with bounded variation*,RGMIA Research Report Collection, 17(2014), Article 153, 18 pp.
7. H. Budak and M.Z. Sarıkaya, *On generalization Ostrowski type inequalities for functions of two variables with bounded variation,* Palestine J. Math., 5(1) (2016), 86-97.
8. H. Budak and M.Z. Sarıkaya, *On generalization of Dragomir's inequalities*, RGMIA Research Report Collection, 17(2014), Article 155, 10 pp.
9. H. Budak and M.Z. Sarıkaya, *On generalization trapezoid inequality for functions of two variables with bounded variation and applications*, Int. J. Nonlinear Anal. Appl.,7(1), (2016), 77-85.
10. H. Budak and M.Z. Sarıkaya, *A companion of Ostrowski type inequalities for functions of two variables with bounded Variation*, J. Adv. Math. Stud., 8(2), 2015 170-184.
11. H. Budak and M.Z. Sarıkaya, *A companion of generalization of Ostrowski type inequalities for functions of two variables with bounded variation,* RGMIA Research Report Collection, 18(2015), Article 44, 19 pp.
12. P. Cerone, W.S. Cheung, and S.S. Dragomir, *On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation,* Computers and Mathematics with Applications 54 (2007) 183–191.
13. P. Cerone, S. S. Dragomir, and C. E. M. Pearce, *A generalized trapezoid inequality for functions of bounded variation*, Turk J Math, 24 (2000), 147-163.
14. J.A Clarkson and C. R. Adams, *On definitions of bounded variation for functions of two variables*,Bull. Amer. Math. Soc. 35 (1933), 824-854.
15. J.A. Clarkson, *On double Riemann-Stieltjes integrals*, Bull. Amer. Math. Soc. 39 (1933), 929-936.
16. S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bull.Austral. Math. Soc., 60(1) (1999), 495-508.

17. S. S. Dragomir, *On the midpoint quadrature formula for mappings with bounded variation and applications*, Kragujevac J. Math. 22(2000) 13-19.

18. S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Inequal. Appl. 4 (2001), no. 1, 59–66.

19. S. S. Dragomir, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, Int. J. Nonlinear Anal. Appl. 5 (2014) No. 1, 89-97

20. S. S. Dragomir, *Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation.* Arch. Math. (Basel) 91 (2008), no. 5, 450–460.

21. S.S. Dragomir and E. Momoniat, *A Three point quadrature rule for functions of bounded bariation and applications*, RGMIA Research Report Collection, 14(2011), Article 33, 16 pp.

22. S. S. Dragomir, *Some perturbed Ostrowski type inequalities for functions of bounded variation*, Preprint RGMIA Res. Rep. Coll. 16 (2013), Art. 93.

23. M. Fréchet, *Extension au cas des intégrals multiples d'une définition de l'intégrale due á Stieltjes*, Nouvelles Annales de Math ematiques 10 (1910), 241-256.

24. Y. Jawarneh and M.S.M Noorani, *Inequalities of Ostrowski and Simpson Type for Mappings of Two Variables with Bounded Variation and Applications*, TJMM, 3 (2011), No. 2, 81-94

25. W. Liu and Y. Sun, *A Refinement of the Companion of Ostrowski inequality for functions of bounded variation and Applications*, arXiv:1207.3861v1, 2012.

26. A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.

27. K-L Tseng, G-S Yang, and S. S. Dragomir, *Generalizations of weighted trapezoidal inequality for mappings of bounded variation and their applications*, Mathematical and Computer Modelling 40 (2004) 77-84.

28. K-L Tseng, *Improvements of some inequalites of Ostrowski type and their applications*, Taiwan. J. Math. 12 (9) (2008) 2427–2441.

29. K-L Tseng, S-R Hwang, and S. S. Dragomir, *Generalizations of weighted Ostrowski type inequalities for mappings of bounded variation and applications,* Computers and Mathematics with Applications, 55(2008), 1785-1793.

30. K-L Tseng,S-R Hwang, G-S Yang, and Y-M Chou, *Improvements of the Ostrowski integral inequality for mappings of bounded variation I*, Applied Mathematics and Computation 217 (2010) 2348–2355.

31. K-L Tseng,S-R Hwang, G-S Yang, and Y-M Chou, *Weighted Ostrowski integral inequality for mappings of bounded variation*, Taiwanese J. of Math., Vol. 15, No. 2, pp. 573-585, April 2011.

32. K-L Tseng, *Improvements of the Ostrowski integral inequality for mappings of bounded variation II*, Applied Mathematics and Computation 218 (2012) 5841–5847.

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