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A NOTE ON CURVATURE TENSORS IN ALMOST HERMITIAN AND CONTACT GEOMETRY

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Abstract. In this note, we interrelate curvature properties between almost Hermitian and almost contact metric manifolds.
Key words: Almost Hermitian manifolds, almost contact metric manifolds, curvature tensors.

1. Introduction

Curvatures tensors in Riemannian geometry have been studied by [3, 5, 7, 8] among many others and recently by [1]. Almost Hermitian and almost contact geometries are two branches of Riemannian geometry. The first being furnished with an almost complex structure J, while the second is involving with the contact tensor fields φ of type $(1, 1)$.

On the other hand, it is known that an almost Hermitian manifold enjoys with the Riemannian curvature tensor called the Kähler identity, denoted by K_1 and defined by $\mathcal{R}(D, E, F, G) = \mathcal{R}(D, E, JF, JG)$. In [3], A. Gray studied curvature tensors for various classes of almost Hermitian manifolds considering the Nijenhuis tensor of J.

In the present note, we pursue the study of Gray by adding some other classes and their analogues in contact geometry. Examples of manifolds verifying the condition of Gray are given such as $\omega_1 \oplus \omega_3$ and G_1-manifolds. We have extended the results of Gray to Hermitian and Hermitian semi-Kählerian manifolds.

The case of almost contact manifolds deals with the classes of cosymplectic, closely cosymplectic, nearly cosymplectic and
nearly-K-cosymplectic. These classes have in common the property that $(\nabla_\phi)(D, E) = 0$ which is exploited. This property, used in the defining relation of the total space of an almost contact metric submersion of type I, leads to the Kähler identity on the fibres.

The paper is organized as follows. In Section §2, we treat the case of almost Hermitian manifolds. Section §3 is devoted to almost contact metric manifolds while

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Section §4 treats the case of Riemanniann curvature properties including the use of almost contact metric submersions.

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2. Almost Hermitian manifolds

By an almost Hermitian manifold, one understands a Riemannian manifold, (M, g), of even dimension $2m$, furnished with a tensor field , J, of type $(1,1)$ satisfying the following two conditions:

(i) $J^2D = -D$, and

(ii) $g(JD, JE) = g(D, E)$, for all $D, E \in \Gamma(M)$.

Any almost Hermitian manifold admits a differential 2-form, Ω, called the fundamental form or the Kähler form, defined by

$$\Omega(D, E) = g(D, JE).$$

Almost Hermitian structures have been completely classified by A. Gray and L.M. Hervella [4]. We just recall the defining relations of some classes which will be used in this note.

An almost Hermitian manifold (M^{2m}, g, J) is said to be :

(1) *Hermitian* if $(\nabla_D\Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G) = 0$;

(2) *quasi Kählerian* if $(\nabla_D\Omega)(E, G) + (\nabla_{JD}\Omega)(JE, G) = 0$;

(3) *Hermitian semi-Kähler manifold* if $(\nabla_D\Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G) = 0 = \delta\Omega$,

where the codifferential δ of Ω is defined by

$$\delta\Omega(D) = -\sum_{i=1}^{m} \left\{(\nabla_{E_i}\Omega)(E_i, D) + (\nabla_{JE_i}\Omega)(JE_i, D)\right\}.$$

Recall that Nijenhuis tensor, N_J, of J is defined by

$$N_J(D, E) = [D, E] + J\,[JD, E] + J\,[D, JE] - [JD, JE]\,.$$

With this in mind, one has $[\nabla_D, J] = \nabla_D J - J\nabla_D$.

Lemma 2.1. *([3]). If an almost Hermitian manifold* (M^{2m}, g, J) *is such that* $[\nabla_{JD}, J] = \pm J\,[\nabla_D, J]$, *then*

$$\left[\nabla_{N_J(D,E)}, J\right] = [R(D, E) - R(JD, JE), J] \pm J\,[R(JD, E) + R(D, E), J]\,.$$

Proof. It is known that $R(D,E) = \nabla_{[D,E]} - [\nabla_D, \nabla_E]$. Combining this with the definition of $N_J(D,E)$, we get
$\left[\nabla_{N_J(D,E)}, J\right] - [R(D,E) - R(JD,JE), J] \mp J\,[R(JD,E) + R(D,E), J] =$
$[[\nabla_D, \nabla_E] - [\nabla_{JD}, \nabla_{JE}, J]] \pm J\,[[\nabla_{JD}, \nabla_E] + [\nabla_D, \nabla_{JE}], J]$.

Consider the right hand-side, apply the Jacobi identity and use the fact that $[\nabla_{JD}, J] = \pm J[\nabla_D, J]$. Thus the right hand-side vanishes and the proof follows. □

Among manifolds satisfying the above condition, we note the following

(a) quasi Kählerian manifolds;

(b) Hermitian manifolds;

(c) Hermitian semi-Kählerian (or ω_3-manifolds) defined by
$(\nabla_D\Omega)(E,G) = (\nabla_{JD}\Omega)(JE,G)$ and $N_J = 0 = \delta\Omega$;

(d) $\omega_1 \oplus \omega_3-$manifolds in which
$(\nabla_D\Omega)(D,E) = (\nabla_{JD}\Omega)(JD,E)$ and $\delta\Omega = 0$;

(e) G_1-manifolds defined by $(\nabla_D\Omega)(D,E) = (\nabla_{JD}\Omega)(JD,E)$.

Proposition 2.1. *If M is Hermitian or a Hermitian semi-Kählerian manifold, then*

$[R(D,E), J] + J\,[R(JD,E), J] + J\,[R(D,JE), J] = [R(JD,JE), J]$.

Proof. It is known that the manifolds under consideration verify $N_J = 0$. Putting this in Lemma 2.1, we get the proof. □

Proposition 2.2. *([3]). Let (M^{2m}, g, J) be a quasi Kählerian manifold, then*

$[R(D,E) - R(JD,JE,J)] - J[R(JD,E) + R(D,JE), J] =$
$2J[\nabla_{[\nabla_D, J]D}, J] - 2J[\nabla_{[\nabla_E, J]D}, J]$.

Proof. Since the manifold is quasi Kählerian, we have

$$[\nabla_{JD,}, J] = -J[\nabla_D, J]$$

On the other hand, it can be shown that

$$N_J(D,E) = 2J\left\{[\nabla_D, J]E - [\nabla_E, J]D\right\}.$$

With this and the use of Lemma 2.1, we get the proof. □

Now, let us turn to

3. Almost contact metric manifolds

Recall that almost contact metric manifolds are extensively developed in [2]. Let M be a differentiable manifold of odd dimension $(2m+1)$. An almost contact structure on M is a triple (φ, ξ, η) where:

(1) ξ is a characteristic vector field,

(2) η is a 1-form such that $\eta(\xi) = 1$, and

(3) φ is a tensor field of type $(1, 1)$ satisfying

$$\varphi^2 = -\mathbb{I} + \eta \otimes \xi, \tag{3.1}$$

where $\mathbb{I}$ is the identity transformation.

If M is equipped with a Riemannian metric g such that

$$g(\varphi D, \varphi E) = g(D, E) - \eta(D)\eta(E), \tag{3.2}$$

then (g, φ, ξ, η) is called an *almost contact metric structure*. Therefore, the quintuple $(M^{2m+1}, g, \varphi, \xi, \eta)$ is an almost contact metric manifold. As in the case of almost Hermitian manifolds, any almost contact metric manifold admits a fundamental 2-form, ϕ, defined by

$$\phi(D, E) = g(D, \varphi E). \tag{3.3}$$

In this study, among the known classes of almost contact manifolds, we have settled the following.

An almost contact metric manifold is said to be :

(1) *cosymplectic* if $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection;

(2) *closely cosymplectic* if $(\nabla_D\varphi)D = 0$ and $d\eta = 0$;

(3) *nearly cosymplectic* if $(\nabla_D\varphi)D = 0$;

(4) *nearly-K-cosymplectic* if $(\nabla_D\varphi)E + (\nabla_E\varphi)D = 0$ and $\nabla_D\xi = 0$;

Proposition 3.1. *Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ be an almost contact metric manifold. If it satisfies $(\nabla_D\phi)(D, E) = 0$, then*

$$R(D, E, F, G) = R(D, E, \varphi F, \varphi G).$$

Proof. On an almost contact metric manifold, the Ricci identity is such that

$$R(D, E)\varphi - \varphi R(D, E) = \nabla_D\nabla_E\varphi - \nabla_E\nabla_D\varphi - \nabla_{[D,E]}\varphi \tag{3.4}$$

Since $(\nabla_D\phi)(D, E) = 0$, then $(\nabla_D\varphi)E = 0$. Thus the right hand side of (3.4) vanishes and, therefore the proof follows. □

As examples of almost contact metric manifolds verifying the above properties, we have cosymplectic, closely cosymplectic, nearly cosymplectic and nearly-K-cosymplectic.

4. Riemannian curvature properties

Recall that the Riemannian curvature tensor $\mathcal{R}$ of a Khlerian manifold satisfies the K_1-identity, named the Kähler identity, defined by

$$\mathcal{R}(D,E,F,G) = \mathcal{R}(D,E,JF,JG). \tag{4.1}$$

Other K_i-identities ($i = 1,2,3$) have been studied by A. Gray in [3].

Let (M^{2m}, g, J) be an almost Hermitian manifold. The K_i-curvature properties are defined in the following way.

$K_1 : \mathcal{R}(D,E,F,G) = \mathcal{R}(D,E,JF,JG)$,
$K_2 : \mathcal{R}(D,E,F,G) = \mathcal{R}(JD,E,JF,G) + \mathcal{R}(JD,JE,F,G) + \mathcal{R}(JD,E,F,JG)$,
$K_3 : \mathcal{R}(D,E,F,G) = \mathcal{R}(JD,JE,JF,JG)$.

In their study of curvature tensors of almost contact metric manifolds, D. Janssens and L. Vanhecke [5], have obtained the following properties of the Riemannian curvature tensor.

(1) the *cosymplectic curvature property,* defined by

$$\mathcal{R}(D,E,F,G) = \mathcal{R}(D,E,\varphi F,\varphi G);$$

(2) the *Kenmotsu curvature property,* defined by

$$\begin{aligned}\mathcal{R}(D,E,F,G) \quad &= \mathcal{R}(D,E,\varphi F,\varphi G) + g(D,F)g(E,G) - g(D,G)g(E,F) \\ &\quad -g(D,\varphi F)g(E,\varphi G) + g(D,\varphi G)g(E,\varphi F);\end{aligned}$$

(3) the *Sasakian curvature property,* defined by

$$\begin{aligned}\mathcal{R}(D,E,F,G) \quad &= \mathcal{R}(D,E,\varphi F,\varphi G) - g(D,F)g(E,G) + g(D,G)g(E,F) \\ &\quad +g(D,\varphi F)g(E,\varphi G) - g(D,\varphi G)g(E,\varphi F).\end{aligned}$$

The curvature tensors of an almost contact metric manifold are called $C(\alpha)$-curvature tensors where α is a real number. For instance, the cosymplectic curvature tensor is a $C(0)$-curvature tensor, the Kenmotsu curvature tensor is a $C(-1)$-curvature tensor and the Sasakian curvature tensor is a $C(1)$-curvature tensor. For more details, we refer to [5]. It is clear that the cosymplectic curvature tensor resembles to the Kähler identity.

Now, we want to determine the classes of almost contact metric manifolds satisfying the cosymplectic curvature property.

Theorem 4.1. [5] *Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ be an almost contact metric manifold. If M satisfies the condition*

$$(\nabla_D \varphi)E = 0,$$

then it has the cosymplectic curvature property.

Proof. For an almost contact metric manifold, the Ricci identity is given by

$$\mathcal{R}(D,E)\varphi - \varphi\mathcal{R}(D,E) = [\nabla_D, \nabla_E]\varphi - \nabla_{[D,E]}\varphi.$$

The condition on M being equivalent to $\nabla\varphi = 0$, the right hand side of the above relation vanishes. We get

$$\mathcal{R}(D,E)\varphi F - \varphi\mathcal{R}(D,E)F = 0$$

which gives

$$\begin{aligned} g(\mathcal{R}(D,E)\varphi F, \varphi G) &= g(\varphi\mathcal{R}(D,E)F, \varphi G) \\ &= -g(\mathcal{R}(D,E)F, \varphi^2 G) \end{aligned}$$

from which we get

$$g(\mathcal{R}(D,E)\varphi F, \varphi G) = -g(\mathcal{R}(D,E)F, -G) - g(\mathcal{R}(D,E)F, \eta(G)\xi).$$

It remains to show that $g(\mathcal{R}(D,E)F, \eta(G)\xi) = 0$. Indeed,
$g(\mathcal{R}(D,E)F, \eta(G)\xi) = g(\mathcal{R}(D,E)F, \xi)\eta(G)$,
but

$$\begin{aligned} g(\mathcal{R}(D,E)F, \xi) &= \mathcal{R}(D,E,F,\xi) \\ &= -\mathcal{R}(D,E,\xi,F) \\ &= -g(\mathcal{R}(D,E)\xi, F). \end{aligned}$$

Since, $\nabla_D\xi = 0$, we get $\mathcal{R}(D,E)\xi = 0$ from which we deduce
$g(\mathcal{R}(D,E)F, \xi) = 0$ so that $g(\mathcal{R}(D,E)\varphi F, \varphi G) = g(\mathcal{R}(D,E)F, G)$,
hence $\mathcal{R}(D,E,\varphi F,\varphi G) = \mathcal{R}(D,E,F,G)$ follows immediately. □

Now, let us examine the case of **almost contact metric submersions**, which are Riemannian submersions whose total space is an almost contact metric manifold studied by B. Watson [9].

Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ and $(M'^{2m'+1}, g', \varphi', \xi', \eta')$ be two almost contact metric manifolds. By an almost contact metric submersion of type I, in the sense of Watson [9], one understands a Riemannian submersion

$$\pi : M^{2m+1} \to M'^{2m'+1}$$

satisfying

(i) $\pi_*\varphi = \varphi'\pi_*$,

(ii) $\pi_*\xi = \xi'$.

Recall that in [6], O'Neill has defined two configuration tensors T and A, of the total space of a Riemannian submersion by setting

$$T_D E = \mathcal{H}\nabla_{\mathcal{V}D}\mathcal{V}E + \mathcal{V}\nabla_{\mathcal{V}D}\mathcal{H}E;$$

$$A_D E = \mathcal{V}\nabla_{\mathcal{H}D}\mathcal{H}E + \mathcal{H}\nabla_{\mathcal{H}D}\mathcal{V}E,$$

where $\mathcal{H}$ and $\mathcal{V}$ denote respectively horizontal and vertical projections.

Tensor T is used in the study of the fibres as it is in the following

Proposition 4.1. *Let $F^{2p} \longrightarrow M^{2m+1} \xrightarrow{\pi} M'^{2m'+1}$ be an almost contact metric submersion of type I. If the total space M is such that*

$$(\nabla_D \phi)(D, E) = \alpha.\eta(D)\phi(E, D),$$

then the fibres F^{2p} verify the NK_1-curvature identity.

Proof. Consider $\alpha = 0$, then we have $(\nabla_D \phi)(D, E) = 0$; this leads to the condition of Proposition 3.1.

Let U, V, W and S be vertical vector fields tangent to the fibres. In the light of Proposition 3.1, we have

$$R(U, V, W, S) = R(U, V, \varphi W, \varphi S) \tag{4.2}$$

Recall that the Gauss equation gives

$$R(U, V, W, S) = \hat{R}(U, V, \varphi W, \varphi S) - g(T_U W, T_V S) + g(T_V W, T_U S) \tag{4.3}$$

such that

$$R(U, V, \varphi W, \varphi S) = \hat{R}(U, V, \varphi W, \varphi S) - g(T_U \varphi W, T_V \varphi S) + g(T_V \varphi W, T_U \varphi S). \tag{4.4}$$

On the other hand, in this situation, we have $T_U \varphi W = \varphi T_U W$ and $T_V \varphi S = \varphi T_V S$ which gives

$$\begin{aligned} -g(T_U \varphi W, T_V \varphi S) &= -g(\varphi T_U W, \varphi T_V S) \\ &= g(T_U W, \varphi^2 T_V S) \\ &= -g(T_U W, T_V S) + g(T_U W, \eta(T_V S)\xi). \end{aligned}$$

Since $\eta(T_V S) = 0$, then $-g(T_U \varphi W, T_V \varphi S) = -g(T_U W, T_V S)$;

analogously $g(T_V \varphi W, T_U \varphi S) = g(T_V W, T_U S)$. In this way, equation (4.4) becomes

$$R(U, V, \varphi W, \varphi S) = \hat{R}(U, V, \varphi W, \varphi S) - g(T_U W, T_V S) + g(T_V W, T_U S) \tag{4.5}$$

Substracting (4.5) from (4.3) gives

(4.6) $R(U,V,W,S) - R(U,V,\varphi W,\varphi S) = \hat{R}(U,V,W,S) - \hat{R}(U,V,\varphi W,\varphi S)$

According to equation (4.2), the left hand side of (4.6) vanishes; thus the fibres verify the Kähler identity. Since the fibres are nearly Kahlerian, then they verify the NK_1−identity.

Suppose that $\alpha \neq 0$. It is known that η vanishes on vertical vector fields. Thus, $(\nabla_U \phi)(U,V) = 0$ which leads to (4.2) and gives the proof. □

Theorem 4.2. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I satisfying the following conditions*

(1) the total space satisfies the Kenmotsu curvature property,

(2) the configuration tensor T is φ-linear on the vertical distribution,

(3) $T_U\xi = 0$ for all vertical vector fields U.

Then the fibres have the K_2-curvature identity.

Proof. Since T is φ−linear and $T_U\xi = 0$, by calculation we get
$g(T_U\varphi W), T_V\varphi S) = g(T_UW, T_VS)$ and $g(T_{\varphi U}\varphi W, T_VS) = -g(T_UW, T_VS)$.
By virtue of the Kenmotsu curvature property, we have

(4.7) $\mathcal{R}(U,V,W,S) = \mathcal{R}(U,V,\varphi W,\varphi S) + \mathcal{R}(\varphi U,V,\varphi W,S) + \mathcal{R}(\varphi U,V,W,\varphi S).$

So, the Gauss equation gives

(i) $\mathcal{R}(U,V,\varphi W,\varphi S) = \hat{\mathcal{R}}(U,V,\hat{\varphi}W,\hat{\varphi}S) - g(T_UW,T_VS) + g(T_VW,T_US)$,

(ii) $\mathcal{R}(\varphi U,V,\varphi W,S) = \hat{\mathcal{R}}(\hat{\varphi}U,V,\hat{\varphi}W,S) + g(T_UW,T_VS) + g(T_VW,T_US)$,

(iii) $\mathcal{R}(\varphi U,V,W,\varphi S) = \hat{\mathcal{R}}(\hat{\varphi}U,V,W,\hat{\varphi}S) - g(T_UW,T_VS) - g(T_VW,T_US)$.

Therefore, summing (i), (ii) and (iii), we obtain a relation yielding to

$$\hat{\mathcal{R}}(U,V,W,S) = \hat{\mathcal{R}}(U,V,\hat{\varphi}W,\hat{\varphi}S) + \hat{\mathcal{R}}(\hat{\varphi}U,V,\hat{\varphi}W,S) + \hat{\mathcal{R}}(\hat{\varphi}U,V,W,\hat{\varphi}S),$$

which shows that the fibres verify the K_2-curvature identity. □

Theorem 4.3. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. Suppose that the total space satisfies the condition*

$$(\nabla_D\varphi)E = 0,$$

then the base space verifies the cosymplectic curvature property and, on the fibres, this property corresponds to the Kähler identity.

Proof. Let X and Y be basic vector fields. It is known that $\mathcal{H}(\nabla_X\varphi)Y$ is basic associated to $(\nabla'_{X_*}\varphi')Y_*$. Thus, since $(\nabla_X\varphi)Y = 0$, one deduces that $(\nabla'_{X_*}\varphi')Y_* = 0$. Therefore, according to the preceding Theorem 4.1, the base space verifies the cosymplectic curvature property. Now, consider the vector fields U, V, W and S tangent to the fibres. For a Riemannian submersion, the Gauss equation is given by

$$\mathcal{R}(U,V,W,S) = \hat{\mathcal{R}}(U,V,W,S) - g(T_UW,T_VS) + g(T_VW,T_US) \tag{4.8}$$

This equation can be transformed in

$$\mathcal{R}(U,V,\varphi W,\varphi S) = \hat{\mathcal{R}}(U,V,\hat{\varphi}W,\hat{\varphi}S) - g(T_U\varphi W,T_V\varphi S) + g(T_V\varphi W,T_U\varphi S). \tag{4.9}$$

Since T is $\varphi-$linear in the second variable, which means $T_U\varphi W = \varphi T_UW$, we have

$$\begin{aligned} g(T_U\varphi W,T_V\varphi S) &= g(\varphi T_UW,\varphi T_VS) \\ &= -g(T_UW,\varphi^2T_VS) \\ &= g(T_UW,T_VS) - g(T_UW,\eta(T_VS)\xi); \end{aligned}$$

Also

$$\begin{aligned} \eta(T_VS) &= g(\xi,T_VS) \\ &= -g(S,T_V\xi) \\ &= 0 \end{aligned}$$

because $T_V\xi = 0$.

Thus, $g(T_U\varphi W,T_V\varphi S) = g(T_UW,T_VS)$ and
$g(T_V\varphi W,T_U\varphi S) = g(T_VW,T_US)$.

In this case, (4.9) leads to

$$\mathcal{R}(U,V,\varphi W,\varphi S) = \hat{\mathcal{R}}(U,V,\hat{\varphi}W,\hat{\varphi}S) - g(T_UW,T_VS) + g(T_VW,T_US). \tag{4.10}$$

Subtracting (4.9) from (4.8), we get,

$$R(U,V,W,S) - R(U,V,\varphi W,\varphi S) = \hat{\mathcal{R}}(U,V,W,S) - \hat{\mathcal{R}}(U,V,\hat{\varphi}W,\hat{\varphi}S).$$

Since $R(U,V,W,S) = R(U,V,\varphi W,\varphi S)$,
then $\hat{\mathcal{R}}(U,V,W,S) = \hat{\mathcal{R}}(U,V,\hat{\varphi}W,\hat{\varphi}S)$
which shows that the fibres have the K_1-curvature identity. □

The above theorem can be viewed as a way to establish the following.

Corollary 4.1. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. Suppose the following conditions satisfied*

(1) the total space satisfies the cosymplectic curvature property,

(2) the configuration tensor T is φ-linear on the vertical distribution,

(3) $T_U\xi = 0$ for all vertical vector fields U.

Then the fibres have the Kähler identity.

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