FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 2 (2016), 513–527

# A CLASSIFICATION OF CONFORMAL-WEYL MANIFOLDS IN VIEW OF NON-METRIC CONNECTIONS *

**Yanling Han, Tal Yun Ho, and Fengyun Fu**

**Abstract.** We give a classification of conformal-Weyl manifolds based on the perspective of semi-symmetric non-metric connections. This research is an extension of a geometrized theory of gravitation and electromagnetism with conformal-Weyl connections.

**Keywords**: Weyl manifolds, semi-symmetric projective conformal connection, $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric connection, conjugate symmetry, conjugate Ricci symmetry.

## 1. Introduction

It is well known that due to Einstein's theory of relativity Weyl [22] and Lyra [17] made an ingenious attempt to unify the gravitational and electromagnetic fields in geometrical arguments, respectively. They defined and studied the Weyl and Lyra manifolds in view of non-metric semi-symmetric connections.

The main purpose of this paper is to derive some further contribution for a classification of conformal-Weyl manifolds from the point of view of Weyl connections, which are a type of the so-called semi-symmetric essentially metric (non-metric) connections.

The concept of a semi-symmetric connection in a Riemannian manifold was firstly introduced by A. Hayden in [12]. K. Yano first in [24] introduced and investigated a semi-symmetric metric connection by using the idea of a metric connection with torsion. U. C. De et al studied some properties of a semi-symmetric metric connection on different manifolds, see [6, 7, 8]. A physical model of a semi-symmetric non-metric connection was studied by K. A. Dunn in [9]. P. Zhao et al in[26]studied a semi-symmetric connection in a sub-Riemannian manifold and arrived at some interesting invariant results. F. Unal and A. Uysal [21] considered Weyl manifold with a semi-symmetric connection. H. V. Le [16] regarded Amari-Chentsov connection as

Received January 08, 2016; Accepted February 15, 2016
2010 *Mathematics Subject Classification.* Primary 53C20; Secondary 53D11
*The authors were supported in part by NSFC(NO.11526055)

a geometrical structure of a statistical manifold and E. S. Stepanova [18] discovered a conjugate symmetric condition of the connection of the statistical manifold. S. K. Chaubeg, R. H. Ojha [2], J. P. Jaiswal, R. H. Ojha [15] investigated the properties of semi-symmetric non-metric connections. I. Suhendro [19] introduced the concept of a new semi-symmetric connection and studied its physical model.

A manifold associated with a semi-symmetric connection is exactly a Weyl manifold [20]. Many researchers have recently paid attention to Weyl manifolds and have produced some remarkable works (one can see [10], [21], [13], [23], [19] for details).

In this paper we will continue to consider Weyl manifolds and propose a classification of Weyl manifolds from a semi-symmetric projective conformal connection defined here that is a projective and conformal equivalent connection. We also study an $\alpha$-type semi-symmetric projective conformal connection as a special type of a semi-symmetric projective conformal connection. We further consider an $\alpha$-type $(\varphi, \omega)$ non-metric connection and discovered its geometrical properties and conjugate symmetric conditions.

The paper is organized as follows. The first two sections briefly introduce some necessary notations and terminologies. Section 3 proposes a projective conformal Weyl connection, and considers a geometrical nature of a conformal-Weyl manifold. The authors get Section 4 is devoted to an $\alpha-$type $(\varphi, \omega)$ semi-symmetric non-metric Weyl connection. Finally, we also study the mutual connection of an $\alpha-$type $(\varphi, \omega)$ semi-symmetric non-metric Weyl connection.

## 2. Preliminaries

A semi-symmetric projective conformal connection $\nabla$ is considered as a connection that is projective and conformal equivalent to a semi-symmetric metric connection $\hat{\nabla}$, namely, by a conformal transformation of the metric

$$g_{ij} \to \bar{g}_{ij} = e^{2\sigma(x)} g_{ij}, \tag{2.1}$$

and by a projective transformation of the connection

$$\hat{\Gamma}^k_{ij} \to \Gamma^k_{ij} = \hat{\Gamma}^k_{ij} + \psi_i \delta^k_j + \psi_j \delta^k_i, \tag{2.2}$$

which satisfies

$$\nabla_k \bar{g}_{ij} = 2(\sigma_k - \psi_k)\bar{g}_{ij} - \psi_i \bar{g}_{jk} - \psi_j \bar{g}_{ik}, T^k_{ij} = \varphi_j \delta^k_i - \varphi_i \delta^k_j, \tag{2.3}$$

where $\hat{\Gamma}^k_{ij}$ is the connection coefficient of a semi-symmetric metric connection $\hat{\nabla}$ that satisfies the following equation

$$\hat{\nabla}_k g_{ij} = 0, \hat{T}^k_{ij} = \varphi_j \delta^k_i - \varphi_i \delta^k_j, \tag{2.4}$$

And the coefficient of this connection is given as

$$\hat{\Gamma}^k_{ij} = \{{}^{\,k}_{ij}\} + \varphi_j \delta^k_i - g_{ij} \varphi^k, \tag{2.5}$$

where $g_{ij}$ is a component of a Riemannian metric $g$ and $\psi_i, \varphi_i$ are components of 1-form $\psi$ and $\varphi$, $\{{k \atop ij}\}$ is a Christoffel symbol of the metric $g_{ij}$, namely, the connection coefficient of a Levi-Civita connection $\nabla^\circ$ and $\sigma_k = \partial_k \sigma$. Also $T_{ij}^k$ and $\hat{T}_{ij}^k$ are respectively the torsion tensors of connections $\nabla$ and $\hat{\nabla}$.

**Remark 2.1.** *A Weyl manifold is characterized by a 1-form $\phi$ and the Weyl connection is determined by*

$$(\nabla_Z g)(X, Y) = -\phi(Z)g(X, Y), T(X, Y) = 0.$$

*thus it is torsion free but not metric preserving [20]. We call a manifold associated with (2.3) a semi-symmetric Weyl manifold.*

**Definition 2.1.** *A connection $\nabla$ is called a semi-symmetric projective conformal connection if there hold (2.1) and(2.3).*

The coefficient $\Gamma_{ij}^k$ of a semi-symmetric projective conformal connection by (2.3) is written as

$$\Gamma_{ij}^k = \overline{\{{k \atop ij}\}} + (\psi_i - \sigma_i)\delta_j^k + (\psi_j + \varphi_j - \sigma_j)\delta_i^k + \bar{g}_{ij}(\sigma^k - \varphi^k), \tag{2.6}$$

where $\overline{\{{k \atop ij}\}}$ is the Christoffel symbol for metric $\bar{g}_{ij}$. The relation between $\overline{\{{k \atop ij}\}}$ and $\{{k \atop ij}\}$ is

$$\overline{\{{k \atop ij}\}} = \{{k \atop ij}\} + \sigma_i\delta_j^k + \sigma_j\delta_i^k - g_{ij}\sigma^k. \tag{2.7}$$

If $\sigma = 0$ in (2.6), then the connection $\nabla$ is a semi-symmetric projective connection that is projective equivalent to $\hat{\nabla}$ and if $\psi_i = 0$ in (2.6), then the connection $\nabla$ is a semi-symmetric conformal connection that is conformal equivalent to $\hat{\nabla}$.

If $\psi_k = \alpha\sigma_k, \alpha \in \mathbb{R}$ in (2.3) and (2.6), then the semi-symmetric projective conformal connection $\nabla$ is called an $\alpha$-type semi-symmetric projective conformal connection. The $\alpha$-type semi-symmetric projective conformal connection satisfies

$$\nabla_k\bar{g}_{ij} = -2(\alpha - 1)\sigma_k\bar{g}_{ij} - \alpha\sigma_i\bar{g}_{jk} - \alpha\sigma_j\bar{g}_{ik}, T_{ij}^k = \varphi_j\delta_i^k - \varphi_i\delta_j^k, \tag{2.8}$$

from Equation (2.3), one obtains the coefficients

$$\Gamma_{ij}^k = \overline{\{{k \atop ij}\}} + (\alpha - 1)\sigma_i\delta_j^k + [(\alpha - 1)\sigma_j + \varphi_j]\delta_i^k + \bar{g}_{ij}(\sigma^k - \varphi^k), \tag{2.9}$$

If $\alpha = 0$ (namely, $\psi_k = 0$), then an $\alpha$-type semi-symmetric projective conformal connection is a semi-symmetric conformal connection. In this case, (2.8) and (2.9) are respectively given as

$$\begin{cases} \nabla_k\bar{g}_{ij} = 2\sigma_k\bar{g}_{ij}, T_{ij}^k = \varphi_j\delta_i^k - \varphi_i\delta_j^k, \\ \Gamma_{ij}^k = \overline{\{{k \atop ij}\}} - (\sigma_i\delta_j^k + \sigma_j\delta_i^k - \bar{g}_{ij}\sigma^k) + \varphi_j\delta_i^k - \bar{g}_{ij}\varphi^k, \end{cases} \tag{2.10}$$

**Remark 2.2.** *A manifold $M$ associated with (2.10) is said to be the first semi-symmetric Weyl manifold.*

In [9], this connection is used as a geometrical model for scalar-tensor theory of gravitation. If $\alpha = 1$, then (2.8) and (2.9) are respectively

$$\begin{cases} \nabla_k \bar{g}_{ij} = -\sigma_i \bar{g}_{jk} - \sigma_j \bar{g}_{ik}, T^k_{ij} = \varphi_j \delta^k_i - \varphi_i \delta^k_j, \\ \Gamma^k_{ij} = \{\bar{^k_{ij}}\} + \varphi_j \delta^k_i + \bar{g}_{ij}(\sigma^k - \varphi^k), \end{cases} \tag{2.11}$$

**Remark 2.3.** *A manifold $M$ associated with (2.11) is said to be the second semi-symmetric Weyl manifold.*

If $\alpha = 2$, then (2.8) and (2.9) are

$$\begin{cases} \nabla_k \bar{g}_{ij} = -2\sigma_k \bar{g}_{ij} - 2\sigma_j \bar{g}_{ik} - 2\sigma_i \bar{g}_{jk}, T^k_{ij} = \varphi_j \delta^k_i - \varphi_i \delta^k_j, \\ \Gamma^k_{ij} = \{\bar{^k_{ij}}\} + \sigma_i \delta^k_j + \sigma_j \delta^k_i + \bar{g}_{ij}\sigma^k + \varphi_j \delta^k_i - \bar{g}_{ij}\varphi^k, \end{cases} \tag{2.12}$$

**Remark 2.4.** *A manifold $M$ associated with (2.12) is said to be the third semi-symmetric Weyl manifold.*

These semi-symmetric Weyl manifolds are defined by the special type of the semi-symmetric projective conformal connections and a 1-form with component $\sigma_i$ being a closed form. Next we will consider for 1-form $\omega$ and $\varphi$ the $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric connection corresponding to the $\alpha$-type semi-symmetric projective conformal connection.

**Definition 2.2.** *A connection $\nabla$ is called the $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric connection if it satisfies*

$$\nabla_k g_{ij} = -2(\alpha - 1)\omega_k g_{ij} - \alpha\omega_i g_{jk} - \alpha\omega_j g_{ik}, T^k_{ij} = \varphi_j \delta^k_i - \varphi_i \delta^k_j. \tag{2.13}$$

*for 1-form $\omega$ and $\varphi$.*

**Remark 2.5.** *A manifold $M$ associated with (2.13) is said to be an $\alpha$-type $(\varphi, \omega)$ semi-symmetric Weyl manifold.*

When $\alpha = 0$, then it is the first $(\varphi, \omega)$ semi-symmetric non-metric connection, if $\alpha = 1$, then it is the second semi-symmetric non-metric connection and if $\alpha = 2$, then it is the third semi-symmetric non-metric connection. The coefficient of $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric connection is

$$\Gamma^k_{ij} = \{^k_{ij}\} + (\alpha - 1)\omega_i \delta^k_j + (\alpha - 1)\omega_j \delta^k_i + g_{ij}\omega^k + \varphi_j \delta^k_i - g_{ij}\varphi^k, \tag{2.14}$$

As you see, $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric connection is a connection family according to $\alpha$. By Definition 2.2, the first $(\varphi, \omega)$ semi-symmetric non-metric connection $\nabla$ satisfies

$$\nabla_k g_{ij} = 2\omega_k g_{ij}, T^k_{ij} = \varphi_j \delta^k_i - \varphi_i \delta^k_j. \tag{2.15}$$

and its the coefficient is

$$\Gamma_{ij}^{k} = \{{k \atop ij}\} - (\omega_i \delta_j^k + \omega_j \delta_i^k - g_{ij}\omega^k) + \varphi_j \delta_i^k - g_{ij}\varphi^k, \tag{2.16}$$

This connection was studied as a semi-symmetric recurrent connection in [28]. The second $(\varphi, \omega)$ semi-symmetric non-metric connection satisfies

$$\nabla_k g_{ij} = -\omega_i g_{jk} - \omega_j g_{ik}, T_{ij}^k = \varphi_j \delta_i^k - \varphi_i \delta_j^k. \tag{2.17}$$

and its coefficient is

$$\Gamma_{ij}^{k} = \{{k \atop ij}\} + \varphi_j \delta_i^k + g_{ij}(\omega^k - \varphi^k), \tag{2.18}$$

This connection was studied in [1] under the condition $\varphi = \omega$. The third $(\varphi, \omega)$ semi-symmetric non-metric connection satisfies

$$\nabla_k g_{ij} = -2\omega_k g_{ij} - 2\omega_i g_{jk} - 2\omega_j g_{ik}, T_{ij}^k = \varphi_j \delta_i^k - \varphi_i \delta_j^k, \tag{2.19}$$

and its coefficient is

$$\Gamma_{ij}^{k} = \{{k \atop ij}\} + \omega_i \delta_j^k + \omega_j \delta_i^k + g_{ij}\omega^k + \varphi_j \delta_i^k - g_{ij}\varphi^k, \tag{2.20}$$

This connection is a type of the Amari-Chentsov connection in the case $\varphi = 0$. By Definition 2.1 and Definition 2.2, the semi-symmetric projective conformal connection and $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric connection are different, but they have the same geometrical properties. In this paper this fact is discovered.

## 3. A Projective Conformal Weyl Connection

We studied the geometrical properties of the semi-symmetric projective conformal connection. By (2.6), the connection coefficient $\Gamma^{*k}_{ij}$ of dual connection $\nabla^*$ of a semi-symmetric projective conformal connection $\nabla$ is

$$\Gamma_{ij}^{*k} = \{{\bar{k} \atop ij}\} - (\psi_i - \sigma_i)\delta_j^k + (\varphi_j - \sigma_j)\delta_i^k - \bar{g}_{ij}(\psi^k + \varphi^k - \sigma^k), \tag{3.1}$$

Using (2.6) and (3.1), the curvature tensor $R_{ijk}^l$ of connection $\nabla$ and curvature tensor $R_{ijk}^{*l}$ of the dual connection $\nabla^*$ are respectively

$$\begin{cases} R_{ijk}^l = \bar{K}_{ijk}^l + \alpha_{ik}\delta_j^l - \alpha_{jk}\delta_i^l + \bar{g}_{jk}\beta_i^l - \bar{g}_{ik}\beta_j^l + \delta_k^l \psi_{ij}, \\ R_{ijk}^{*l} = \bar{K}_{ijk}^l - \beta_{ik}\delta_j^l + \beta_{jk}\delta_i^l - \bar{g}_{jk}\alpha_i^l + \bar{g}_{ik}\alpha_j^l - \delta_k^l \psi_{ij}, \end{cases} \tag{3.2}$$

where $\bar{\nabla}^\circ$ is Levi-Civita connection for $\bar{g}_{ij}$ and $\bar{K}_{ijk}^l$ is a curvature tensor field for $\bar{\nabla}^\circ$ and

$$\begin{cases} \alpha_{ik} = \bar{\nabla}_i^\circ(\psi_k + \varphi_k - \sigma_k) - (\psi_i + \varphi_i - \sigma_i)(\psi_k + \varphi_k - \sigma_k) + \bar{g}_{ik}(\psi_p + \varphi_p - \sigma_p)(\varphi^p - \sigma^p), \\ \beta_{ik} = \bar{\nabla}_i^\circ(\sigma_k - \varphi_k) + (\varphi_i - \sigma_i)(\varphi_k - \sigma_k), \\ \psi_{ik} = \bar{\nabla}_i^\circ \psi_k - \bar{\nabla}_k^\circ \psi_i. \end{cases} \tag{3.3}$$

From Equation (3.2), one gets

$$\begin{cases} R_{jk} = \bar{K}_{jk} - (n-1)\alpha_{jk} - \beta_{jk} + \bar{g}_{jk}\beta^h_h + \psi_{kj}, \\ P_{ij} = P^{\circ}_{ij} + (n+1)\psi_{ij}, \\ R = \bar{K} - (n-1)(\alpha^h_h - \beta^h_h) \end{cases} \tag{3.4}$$

and

$$\begin{cases} R^*_{jk} = \bar{K}_{jk} + (n-1)\beta_{jk} + \alpha_{jk} - \bar{g}_{jk}\alpha^h_h - \psi_{kj}, \\ P^*_{ij} = P^{\circ}_{ij} - (n+1)\psi_{ij}, \\ R^* = \bar{K} - (n-1)(\alpha^h_h - \beta^h_h) \end{cases} \tag{3.5}$$

The curvature tensors with respect to the semi-symmetric projective conformal connection $\nabla$ and dual connection $\nabla^*$ are related to the following relation: $P^*_{ij} = -P_{ij}$ and $R^* = R$ because of $P^{\circ}_{ij} = 0$.

**Theorem 3.1.** *The expression $C^l_{ijk} + C^{*l}_{ijk} = 2\bar{C}^{\circ l}_{ijk}$ is an invariant under the transformation of the connection $\bar{\nabla}^{\circ} \to \nabla$ and $\bar{\nabla}^{\circ} \to \nabla^*$, where $C^l_{ijk}, C^{*l}_{ijk}$ and $\bar{C}^l_{ijk}$ are Weyl conformal curvature tensors with respect to connections $\nabla, \nabla^*$ and $\bar{\nabla}^{\circ}$, namely,*

$$\begin{cases} C^l_{ijk} = R^l_{ijk} - \frac{1}{n-2}(\delta^l_i R_{jk} - \delta^l_j R_{ik} + g_{jk}R^l_i - g_{ik}R^l_j) - \frac{R}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_i g_{jk}), \\ C^{*l}_{ijk} = R^{*l}_{ijk} - \frac{1}{n-2}(\delta^l_i R^*_{jk} - \delta^l_j R^*_{ik} + g_{jk}R^{*l}_i - g_{ik}R^{*l}_j) - \frac{R^*}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_i g_{jk}), \\ \bar{C}^{\circ l}_{ijk} = \bar{K}^l_{ijk} - \frac{1}{n-2}(\delta^l_i K_{jk} - \delta^l_j K_{ik} + g_{jk}K^l_i - g_{ik}K^l_j) - \frac{K}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_i g_{jk}). \end{cases}$$

*Proof.* Let $\gamma_{ik} = \alpha_{ik} - \beta_{ik}$,

$$R^l_{ijk} + R^{*l}_{ijk} = 2\bar{K}^l_{ijk} + \delta^l_j\gamma_{ik} - \delta^l_i\gamma_{jk} + \bar{g}_{ik}\gamma^l_i - \bar{g}_{jk}\gamma^l_j, \tag{3.6}$$

Contracting the indices $i$ and $l$ of (3.6), then we find

$$R_{jk} + R^*_{jk} = 2\bar{K}_{jk} - (n-2)\gamma_{jk} - \bar{g}_{jk}\gamma^h_h, \tag{3.7}$$

Multiplying both sides of (3.7) by $\bar{g}^{jk}$,

$$R + R^* = 2\bar{K} - 2(n-1)\gamma^h_h,$$

From this expression,

$$\gamma^h_h = \frac{2K - (R + R^*)}{2(n-1)},$$

From expression (3.7) we find

$$\gamma_{jk} = \frac{1}{n-1}[2\bar{K}_{jk} - (R_{jk} + R^*_{jk}) - \frac{\bar{g}_{jk}}{2(n-1)}(2\bar{K} - (R + R^*))],$$

Substituting this expression into (3.6), by a direct computation and using (3.6), finishes the proof. □

**Theorem 3.2.** *A semi-symmetric Weyl manifold with a constant sectional curvature is conformal flat ($n \geq 3$).*

*Proof.* In fact, by $R^l_{ijk} = K(\delta^l_j \bar{g}_{ik} - \delta^l_i \bar{g}_{jk})$ and (3.6), one has $C^l_{ijk} = C^{*l}_{ijk} = 0$, then $\bar{C}^{\circ l}_{ijk} = 0$ based on Theorem 3.1. Considering $\bar{C}^{\circ l}_{ijk}$ keeping unchanged under a conformal transformation, one obtains $C^{\circ l}_{ijk} = 0$, where $C^{\circ l}_{ijk}$ is a Weyl conformal curvature tensor of a Riemannian metric $g_{ij}$. Hence if $n \geq 3$ ,then the Riemannian metric is conformal flat. □

**Theorem 3.3.** *In the Riemannian manifold $(M, g)$, the tensor*

$$\begin{aligned} V^{*l}_{ijk} &= R^{*l}_{ijk} + \frac{1}{n}(\delta^l_j R^*_{ik} - \delta^l_i R^*_{jk} + \bar{g}_{jk} R^{*l}_i - \bar{g}_{ik} R^{*l}_j) \\ &+ \frac{2}{n(n+4)}[\delta^l_i (R^*_{jk} - R^*_{kj}) - \delta^l_j (R^*_{ik} - R^*_{ki}) \\ &+ \bar{g}_{ik}(R^{*l}_j - R^{*l}_{.j}) - \bar{g}_{jk}(R^{*l}_i - R^{*l}_{.i}) + n\delta^l_k (R^*_{ij} - R^*_{ji})]. \end{aligned}$$

*is an invariant under the connection transformation $\nabla \to \nabla^*$.*

*Proof.* From (3.2), we find

$$R^{*l}_{ijk} = R^l_{ijk} + \delta^l_i \rho_{jk} - \delta^l_j \rho_{ik} + \bar{g}_{ik}\rho^l_j - \bar{g}_{jk}\rho^l_i - 2\delta^l_k \psi_{ij}, \tag{3.8}$$

where $\rho_{ik} = \alpha_{ik} + \beta_{ik}$. By using a contraction of the indices $i$ and $l$ of (3.8), we find

$$R^*_{jk} = R_{jk} + n\rho_{jk} - \bar{g}_{jk}\rho^h_h - 2\psi_{kj}, \tag{3.9}$$

Alternating the indices $j$ and $k$ of (3.9), using $\rho_{jk} - \rho_{kj} = \varphi_{jk}$, we have

$$R^*_{jk} - R^*_{kj} = R_{jk} - R_{kj} + (n+4)\psi_{jk},$$

From this relation we have

$$\psi_{jk} = \frac{1}{n+4}[(R^*_{jk} - R^*_{kj}) - (R_{jk} - R_{kj})],$$

From this equation and (3.9), we obtain

$$\rho_{jk} = \frac{1}{n}\{R^*_{jk} - R_{jk} + \bar{g}_{jk}\rho^h_h - \frac{2}{n+4}[(R^*_{jk} - R^*_{kj}) - (R_{jk} - R_{kj})]\},$$

Substituting the above results into (3.8), then we have $V^{*l}_{ijk} = V^l_{ijk}$, where $V^l_{ijk}$ is the tensor of $\nabla$, namely, we get

$$\begin{aligned} V^l_{ijk} &= R^l_{ijk} + \frac{1}{n}(\delta^l_j R_{ik} - \delta^l_i R_{jk} + \bar{g}_{jk} R^l_i - \bar{g}_{ik} R^l_j) \\ &+ \frac{2}{n(n+4)}[\delta^l_i (R_{jk} - R_{kj}) - \delta^l_j (R_{ik} - R_{ki}) \\ &+ \bar{g}_{ik}(R^l_j - R^l_{.j}) - \bar{g}_{jk}(R^l_i - R^l_{.i}) + n\delta^l_k (R_{ij} - R_{ji})]. \end{aligned}$$

□

From Theorem 3.3 the following theorem is proved.

**Theorem 3.4.** *A semi-symmetric projective conformal connection is conjugate symmetry, so it is necessary and sufficient that the corresponding Ricci tensors be equal.*

## 4. An $\alpha$-type $(\varphi, \omega)$ non-metric Weyl Connection

From (2.4) we find that the coefficient of dual connection $\nabla^*$ of $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric $\nabla$ is given by

$$\Gamma^{*k}_{ij} = \{^{k}_{ij}\} - (\alpha-1)\omega_i\delta^k_j - (\omega_j - \varphi_j)\delta^k_i + g_{ij}[(\alpha-1)\omega^k + \varphi^k], \tag{4.1}$$

Using (2.4) and (4.1), we find that the curvature tensor of $(M, g)$ with respect to $\nabla$ and $\nabla^*$ is given by

$$R^l_{ijk} = K^l_{ijk} + \delta^l_j\lambda_{ik} - \delta^l_i\lambda_{jk} + g_{jk}\nu^l_i - g_{ik}\nu^l_j + \delta^l_k\omega_{ij}, \tag{4.2}$$

and

$$R^{*l}_{ijk} = K^l_{ijk} - \delta^l_j\nu_{ik} + \delta^l_i\nu_{jk} - g_{jk}\lambda^l_i + g_{ik}\lambda^l_j - \delta^l_k\omega_{ij}, \tag{4.3}$$

where $K^l_{ijk}$ is a curvature tensor with respect to Levi-Civita connection $\nabla^\circ$ of a Riemannian metric $g_{ij}$ and

$$\begin{aligned} \lambda_{ik} &= \nabla^\circ_i[(\alpha-1)\omega_k + \varphi_k] - [(\alpha-1)\omega_i + \varphi_i][(\alpha-1)\omega_k + \varphi_k] \\ &\quad - g_{ik}[(\alpha-1)\omega_p + \varphi_p](\omega^p - \varphi^p), \\ \nu_{ik} &= \nabla^\circ_i(\omega_k - \varphi_k) + (\omega_i - \varphi_i)(\omega_k - \varphi_k), \\ \omega_{ik} &= (\alpha-1)(\nabla^\circ_i\omega_k - \nabla^\circ_k\omega_i). \end{aligned} \tag{4.4}$$

From (4.2) and (4.3), we have the Ricci tensor, the volume curvature tensor and scalar curvature, respectively

$$\begin{cases} R_{jk} = K_{jk} - (n-1)\lambda_{jk} - \nu_{jk} + g_{jk}\nu^h_h + \omega_{kj}, \\ P_{ij} = P^\circ_{ij} + (n + \frac{\alpha}{\alpha-1})\omega_{ij}, \\ R = K - (n-1)(\lambda^h_h - \nu^h_h), \end{cases} \tag{4.5}$$

and

$$\begin{cases} R^*_{jk} = K_{jk} + (n-1)\nu_{jk} + \lambda_{jk} - g_{jk}\lambda^h_h - \psi_{kj}, \\ P^*_{ij} = P^\circ_{ij} - (n + \frac{\alpha}{\alpha-1})\omega_{ij}, \\ R^* = K - (n-1)(\lambda^h_h - \nu^h_h), \end{cases} \tag{4.6}$$

where $R_{ij}, P_{ij}, R, R^*_{ij}, P^*_{ij}, R^*$ and $K_{ij}, K_{ij}, K$ are the Ricci tensor, volume curvature tensor, scalar curvature with respect to $\nabla, \nabla^*$ and $\nabla^\circ$.

Expressions (3.4) and (4.6) are different, but their types are equal. Expressions (3.3),(3.4),(3.5) and expressions (4.4),(4.5),(4.6) are formally equal. Hence the geometrical properties of the $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric connection and the semi-symmetric projective conformal connection are equal. From (4.5) and (4.6), we obtain $P^*_{ij} = P_{ij}, R^* = R$.

**Theorem 4.1.** *In a Riemannian manifold $(M, g)$, if $n \geq 3$, then the expression*

$$C^l_{ijk} + C^{*l}_{ijk} = 2C^{\circ l}_{ijk}$$

*is an invariant under the transformation of the connection $\nabla^\circ \to \nabla$ and $\nabla^\circ \to \nabla^*$, where $C^l_{ijk}, C^{*l}_{ijk}$ and $C^{\circ l}_{ijk}$ are Weyl conformal curvature tensors with respect to connections $\nabla, \nabla^*$ and $\nabla^\circ$, namely,*

$$\begin{cases} C^l_{ijk} = R^l_{ijk} - \frac{1}{n-2}(\delta^l_i R_{jk} - \delta^l_j R_{ik} + g_{jk} R^l_i - g_{ik} R^l_j) - \frac{R}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_i g_{jk}), \\ C^{*l}_{ijk} = R^{*l}_{ijk} - \frac{1}{n-2}(\delta^l_i R^*_{jk} - \delta^l_j R^*_{ik} + g_{jk} R^{*l}_i - g_{ik} R^{*l}_j) - \frac{R^*}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_i g_{jk}), \\ C^{\circ l}_{ijk} = K^l_{ijk} - \frac{1}{n-2}(\delta^l_i K_{jk} - \delta^l_j K_{ik} + g_{jk} K^l_i - g_{ik} K^l_j) - \frac{K}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_i g_{jk}). \end{cases}$$

*Proof.* Let $\mu_{ik} = \lambda_{ik} - \nu_{ik}$,

$$R^l_{ijk} + R^{*l}_{ijk} = 2K^l_{ijk} + \delta^l_j \mu_{ik} - \delta^l_i \mu_{jk} + g_{ik}\mu^l_i - g_{jk}\mu^l_j, \tag{4.7}$$

Contracting the indices $i$ and $l$ of (4.7), then we find

$$R_{jk} + R^*_{jk} = 2K_{jk} - (n-2)\gamma_{jk} - g_{jk}\mu^h_h, \tag{4.8}$$

Multiplying both sides of (4.8) by $g^{jk}$,

$$R + R^* = 2K - 2(n-1)\mu^h_h,$$

From this expression,

$$\mu^h_h = \frac{2K - (R + R^*)}{2(n-1)},$$

From expression (4.8) we find

$$\mu_{jk} = \frac{1}{n-1}[2K_{jk} - (R_{jk} + R^*_{jk}) - \frac{g_{jk}}{2(n-1)}(2K - (R + R^*))],$$

Substituting this expression into (4.7), by a direct computation and using (3.6), finishes the proof. □

**Theorem 4.2.** *An $\alpha$-type $(\varphi, \omega)$ semi-symmetric Weyl manifold with a constant sectional curvature is conformal flat ($n \geq 3$).*

*Proof.* By using the same argument as the proof of Theorem 3.2, one can prove Theorem 4.2. □

**Theorem 4.3.** *In the Riemannian manifold $(M, g)$, the tensor*

$$\begin{aligned} V^{*l}_{ijk} \;\; = \;\; & R^{*l}_{ijk} + \frac{1}{n}(\delta^l_j R^*_{ik} - \delta^l_i R^*_{jk} + g_{jk} R^{*l}_i - g_{ik} R^{*l}_j) \\ & + \frac{2(\alpha-1)}{n[n\alpha + 4(\alpha-1)]}[\delta^l_j (R^*_{ik} - R^*_{ki}) \\ & - \delta^l_i (R^*_{jk} - R^*_{kj}) + g_{jk}(R^{*l}_i - R^{*l}_{\cdot i}) - g_{ik}(R^{*l}_j - R^{*l}_{\cdot j}) + n\delta^l_k (R^*_{ij} - R^*_{ji})]. \end{aligned}$$

*is an invariant under the connection transformation of the $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric $\nabla \to \nabla^*$ .*

*Proof.* From (4.2) and (4.3) we find

$$R^{*l}_{ijk} = R^l_{ijk} + \delta^l_i \tau_{jk} - \delta^l_j \tau_{ik} + g_{ik}\tau^l_j - g_{jk}\tau^l_i - 2\delta^l_k \omega_{ij}, \tag{4.9}$$

where $\tau_{ik} = \nu_{ik} + \lambda_{ik}$. By using a contraction of the indices $i$ and $l$ of (4.9), we find

$$R^*_{jk} = R_{jk} + n\tau_{jk} - g_{jk}\tau^h_h - 2\omega_{kj}, \tag{4.10}$$

Alternating the indices $j$ and $k$ of (4.10), we have

$$R^*_{jk} - R^*_{kj} = R_{jk} - R_{kj} + n(\tau_{jk} - \tau_{kj}) + 4\omega_{jk}, \tag{4.11}$$

From this relation we have

$$\tau_{jk} - \tau_{kj} = (\nu_{jk} - \nu_{kj}) + (\lambda_{jk} - \lambda_{kj}) = \frac{\alpha}{\alpha - 1}\omega_{jk}, \tag{4.12}$$

Using (4.12), we have

$$\omega_{jk} = \frac{\alpha - 1}{n\alpha + 4(\alpha - 1)}[(R^*_{jk} - R^*_{kj}) - (R_{jk} - R_{kj})], \tag{4.13}$$

if $\alpha = 1$, then $\omega_{jk} = 0$. Using (4.13), from (4.10) we find

$$\tau_{jk} = \frac{1}{n}\{R^*_{jk} - R_{jk} + g_{jk}\tau^h_h - \frac{2(\alpha - 1)}{n\alpha + 4(\alpha - 1)}[(R^*_{jk} - R^*_{kj}) - (R_{jk} - R_{kj})]\}, \tag{4.14}$$

Substituting (4.13) and (4.14) into (4.9), then we have $V^{*l}_{ijk} = V^l_{ijk}$, where $V^l_{ijk}$ is the tensor of $\nabla$, namely

$$\begin{aligned} V^l_{ijk} &= R^l_{ijk} + \frac{1}{n}(\delta^l_j R_{ik} - \delta^l_i R_{jk} + g_{jk}R^l_i - g_{ik}R^l_j) \\ &+ \frac{2(\alpha - 1)}{n[n\alpha + 4(\alpha - 1)]}[\delta^l_j (R_{ik} - R_{ki}) \\ &- \delta^l_i (R_{jk} - R_{kj}) + g_{jk}(R^l_i - R^l_{.i}) - g_{ik}(R^l_j - R^l_{.j}) + n\delta^l_k (R_{ij} - R_{ji})]. \end{aligned}$$

□

**Remark 4.1.** *If $\alpha = 0$, then*

$$\begin{aligned} V^{*l}_{ijk} &= R^{*l}_{ijk} + \frac{1}{n}(\delta^l_j R^*_{ik} - \delta^l_i R^*_{jk} + g_{jk}R^{*l}_i - g_{ik}R^{*l}_j) + \frac{1}{2n}[\delta^l_j (R^*_{ik} - R^*_{ki}) \\ &- \delta^l_i (R^*_{jk} - R^*_{kj}) + g_{jk}(R^{*l}_i - R^{*l}_{.i}) - g_{ik}(R^{*l}_j - R^{*l}_{.j}) + n\delta^l_k (R^*_{ij} - R^*_{ji})], \end{aligned}$$

*If $\alpha = 1$, then*

$$V^{*l}_{ijk} = R^{*l}_{ijk} + \frac{1}{n}(\delta^l_j R^*_{ik} - \delta^l_i R^*_{jk} + g_{jk}R^{*l}_i - g_{ik}R^{*l}_j),$$

*If $\alpha = 2$, then*

$$\begin{aligned} V^{*l}_{ijk} &= R^{*l}_{ijk} + \frac{1}{n}(\delta^l_j R^*_{ik} - \delta^l_i R^*_{jk} + g_{jk}R^{*l}_i - g_{ik}R^{*l}_j) + \frac{1}{n(n+2)}[\delta^l_j (R^*_{ik} - R^*_{ki}) \\ &- \delta^l_i (R^*_{jk} - R^*_{kj}) + g_{jk}(R^{*l}_i - R^{*l}_{.i}) - g_{ik}(R^{*l}_j - R^{*l}_{.j}) + n\delta^l_k (R^*_{ij} - R^*_{ji})]. \end{aligned}$$

Using Theorem 4.3 the following theorem is proved without difficulty.

**Theorem 4.4.** *In order that a $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric connection is conjugate symmetry, it is necessary and sufficient that corresponding Ricci tensor is equal.*

## 5. A Mutual Weyl Connection of An $\alpha$-type $(\varphi, \omega)$ non-metric Weyl Connection

We studied geometrical properties and conjugate symmetry condition of the mutual connection of an $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric connection. From (2.13) and (2.14), the mutual connection $\bar{\nabla}$ of an $\alpha$-type semi-symmetric non-metric connection satisfies the relations

(5.1)
$$\bar{\nabla}_k g_{ij} = -2[(\alpha-1)\omega_k + \varphi_k]g_{ij} - (\alpha\omega_i - \varphi_i)g_{jk} - (\alpha\omega_j - \varphi_j)g_{ik}, \bar{T}^k_{ij} = \varphi_j\delta^k_i - \varphi_i\delta^k_j,$$

The coefficient of the mutual connection is

$$\bar{\Gamma}^k_{ij} = \left\{ {k \atop ij} \right\} - [(\alpha-1)\omega_i + \varphi_i]\delta^k_j + (\alpha-1)\omega_j\delta^k_i + g_{ij}(\omega^k - \varphi^k), \tag{5.2}$$

The dual connection $\bar{\nabla}^*$ of the mutual connection $\bar{\nabla}$ satisfies the relations

$$\begin{cases} \bar{\nabla}^*_k g_{ij} = 2[(\alpha-1)\omega_k + \varphi_k]g_{ij} + (\alpha\omega_i - \varphi_i)g_{jk} + (\alpha\omega_j - \varphi_j)g_{ik}, \\ \bar{T}^{*k}_{ij} = (\omega_i - \varphi_i)\delta^k_j - (\omega_j - \varphi_j)\delta^k_i + \alpha(\omega_j\delta^k_i - \omega_i\delta^k_j), \end{cases} \tag{5.3}$$

The coefficient of the dual connection of the mutual connection is

$$\bar{\Gamma}^{*k}_{ij} = \left\{ {k \atop ij} \right\} - [(\alpha-1)\omega_i + \varphi_i]\delta^k_j - (\omega_j - \varphi_j)\delta^k_i - (\alpha-1)g_{ij}\omega^k, \tag{5.4}$$

Using (5.2) and (5.4), we find the curvature tensor with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ is given respectively by

$$\bar{R}^l_{ijk} = K^l_{ijk} + \delta^l_j\omega_{ik} - \delta^l_i\omega_{jk} + g_{jk}\alpha^l_i - g_{ik}\alpha^l_j + \delta^l_k\beta_{ij}, \tag{5.5}$$

and

$$\bar{R}^{*l}_{ijk} = K^l_{ijk} - \delta^l_j\alpha_{ik} + \delta^l_i\alpha_{jk} - g_{jk}\omega^l_i + g_{ik}\omega^l_j - \delta^l_k\beta_{ij}, \tag{5.6}$$

where

$$\begin{cases} \omega_{ik} = (\alpha-1)(\nabla^\circ_i\omega_k - (\alpha-1)\omega_i\omega_k), \\ \alpha_{ik} = \nabla^\circ_i(\omega_k - \varphi_k) + (\omega_i - \varphi_i)(\omega_k - \varphi_k) + (\alpha-1)g_{ik}\omega_p(\omega^p - \varphi^p), \\ \beta_{ik} = \nabla^\circ_i[(\alpha-1)\omega_k + \varphi_k] - \nabla^\circ_k[(\alpha-1)\omega_i + \varphi_i], \end{cases} \tag{5.7}$$

**Theorem 5.1.** *In a Riemannian manifold $(M, g)$, if $n \geq 3$, then the expression $\bar{C}^l_{ijk} + \bar{C}^{*l}_{ijk} = 2C^{\circ l}_{ijk}$ is an invariant under the transformation of the connection*

*$\nabla^\circ \to \bar\nabla$ and $\nabla^\circ \to \bar\nabla^*$, where $\bar C^l_{ijk}$ and $\bar C^{*l}_{ijk}$ are Weyl conformal curvature tensors with respect to connections $\bar\nabla$and$\bar\nabla^*$, namely,*

$$(5.8)\quad \begin{cases} \bar C^l_{ijk} = \bar R^l_{ijk} - \frac{1}{n-2}(\delta^l_i \bar R_{jk} - \delta^l_j \bar R_{ik} + g_{jk}\bar R^l_i - g_{ik}\bar R^l_j) - \frac{\bar R}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_i g_{jk}), \\ \bar C^{*l}_{ijk} = \bar R^{*l}_{ijk} - \frac{1}{n-2}(\delta^l_i \bar R^*_{jk} - \delta^l_j \bar R^*_{ik} + g_{jk}\bar R^{*l}_i - g_{ik}\bar R^{*l}_j) - \frac{\bar R^*}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_i g_{jk}), \end{cases}$$

*Proof.* Adding (5.5) and (5.6), we have

$$(5.9)\qquad \bar R^l_{ijk} + \bar R^{*l}_{ijk} = 2K^l_{ijk} + \delta^l_j \gamma_{ik} - \delta^l_i \gamma_{jk} + g_{jk}\gamma^l_i - g_{ik}\gamma^l_j,$$

where $\gamma_{ik} = \alpha_{ik} - \omega_{ik}$, Contracting the indices $i$ and $l$ of (5.9), then we find

$$(5.10)\qquad \bar R_{jk} + \bar R^*_{jk} = 2K_{jk} + (n-2)\gamma_{jk} + g_{jk}\gamma^h_h,$$

Multiplying both sides of (5.10) by $g^{jk}$,

$$\bar R + \bar R^* = 2K + 2(n-1)\gamma^h_h,$$

From this expression,

$$\gamma^h_h = \frac{(\bar R + \bar R^*) - 2K}{2(n-1)},$$

Substituting this expression into (5.10) we find

$$\gamma_{jk} = \frac{1}{n-2}[(\bar R_{jk} + \bar R{*}_{jk}) - 2K_{jk} - \frac{g_{jk}}{2(n-1)}((\bar R + \bar R^*) - 2K)],$$

Substituting this expression into (5.9), by a direct computation and using (5.8), then this finishes the proof. □

**Theorem 5.2.** *If a Riemannian metric admits the mutual connection of an $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric connection with a constant curvature, then the Riemannian metric is conformal flat ($n \geq 3$).*

*Proof.* By using the same method as in the proof of Theorem 3.2, one can prove Theorem 5.2. □

**Theorem 5.3.** *In the Riemannian manifold $(M, g)$, the tensor*

$$\begin{aligned} \bar V^{*l}_{ijk} \;=\; & \bar R^{*l}_{ijk} + \frac{1}{n}(\delta^l_j \bar R^*_{ik} - \delta^l_i \bar R^*_{jk} + g_{jk}\bar R^{*l}_i - g_{ik}\bar R^{*l}_j) \\ & + \frac{1}{n^2-4}(\delta^l_i \bar P^*_{jk} - \delta^l_j \bar P^*_{ik} + g_{ik}\bar P^{*l}_j - g_{jk}\bar P^{*l}_i - n\delta^l_k \bar P^*_{ij}) \\ & + \frac{1}{n(n^2-4)}[\delta^l_i(\bar R^*_{jk} - \bar R^*_{kj}) - \delta^l_j(\bar R^*_{ik} - \bar R^*_{ki}) + g_{ik}(\bar R^{*l}_j - \bar R^{*l}_{.j}) \\ & - g_{jk}(\bar R^{*l}_i - \bar R^{*l}_{.i}) - n\delta^l_k(\bar R^*_{ij} - \bar R^*_{ji})]. \end{aligned}$$

*is an invariant under the connection transformation of the mutual connection $\bar\nabla \to \bar\nabla^*$, where $\bar P^*_{ij} = \bar R^{*k}_{ijk}$.*

*Proof.* From (5.5) and (5.6) we have

$$\bar{R}^{*l}_{ijk} = \bar{R}^l_{ijk} + \delta^l_i \bar{\tau}_{jk} - \delta^l_j \bar{\tau}_{ik} + g_{ik}\bar{\tau}^l_j - g_{jk}\bar{\tau}^l_i - 2\delta^l_k \beta_{ij}, \tag{5.11}$$

where $\bar{\tau}_{ik} = \alpha_{ik} + \omega_{ik}$. By using contraction of the indices $i$ and $l$ of (4.9), we find

$$\bar{R}^*_{jk} = \bar{R}_{jk} + n\bar{\tau}_{jk} - g_{jk}\tau^h_h - 2\beta_{kj}, \tag{5.12}$$

Alternating the indices $j$ and $k$ of (5.12), we have

$$\bar{R}^*_{jk} - \bar{R}^*_{kj} = \bar{R}_{jk} - \bar{R}_{kj} + n(\bar{\tau}_{jk} - \bar{\tau}_{kj}) + 4\beta_{jk},$$

By using contraction of the indices $k$ and $l$ of (5.11), we obtained

$$\bar{P}^*_{ij} = P_{ij} - 2(\bar{\tau}_{ij} - \bar{\tau}_{ji}) - 2n\beta_{ij},$$

From the above two expressions, we obtained

$$\begin{aligned} &n(\bar{\tau}_{jk} - \bar{\tau}_{kj}) + 4\beta_{jk} = (\bar{R}^*_{jk} - \bar{R}^*_{kj}) - (\bar{R}_{jk} - \bar{R}_{kj}), \\ &2(\bar{\tau}_{jk} - \bar{\tau}_{kj}) + 2n\beta_{jk} = \bar{P}_{jk} - \bar{P}^*_{jk}, \end{aligned}$$

From these two expressions we find

$$\beta_{jk} = \frac{1}{2(n^2-4)}\{2[(\bar{R}_{jk} - \bar{R}_{kj}) - (\bar{R}^*_{jk} - \bar{R}^*_{kj})] + n(\bar{P}_{jk} - \bar{P}^*_{jk})\}, \tag{5.13}$$

Substituting (5.13) into (5.12),

$$\bar{\tau}_{jk} = \frac{1}{n}(\bar{R}^*_{jk} - \bar{R}_{jk} + g_{jk}\bar{\tau}^h_h + \frac{1}{n^2-4}\{2[(\bar{R}_{jk} - \bar{R}_{kj}) - (\bar{R}^*_{jk} - \bar{R}^*_{kj})] + n(\bar{P}_{jk} - \bar{P}^*_{jk})\}), \tag{5.14}$$

Substituting (5.13) into (5.14) into (5.11), by a direct computation, we have $V^{*l}_{ijk} = V^l_{ijk}$, where the tensor $V^l_{ijk}$ is

$$\begin{aligned} \bar{V}^l_{ijk} \;=\; & \bar{R}^l_{ijk} + \frac{1}{n}(\delta^l_j \bar{R}_{ik} - \delta^l_i \bar{R}_{jk} + g_{jk}\bar{R}^l_i - g_{ik}\bar{R}^l_j) \\ & + \frac{1}{n^2-4}(\delta^l_i \bar{P}_{jk} - \delta^l_j \bar{P}_{ik} + g_{ik}\bar{P}^l_j - g_{jk}\bar{P}^l_i - n\delta^l_k \bar{P}_{ij}) \\ & + \frac{1}{n(n^2-4)}[\delta^l_i(\bar{R}_{jk} - \bar{R}_{kj}) - \delta^l_j(\bar{R}_{ik} - \bar{R}_{ki}) + g_{ik}(\bar{R}^l_j - \bar{R}^l_{.j}) \\ & - g_{jk}(\bar{R}^l_i - \bar{R}^l_{.i}) - n\delta^l_k(\bar{R}_{ij} - \bar{R}_{ji})]. \end{aligned}$$

□

Using Theorem 5.3, the following theorem is proved without difficulty.

**Theorem 5.4.** *In order for the mutual connection of an $\alpha$-type $(\varphi, \omega)$ semi-symmetric non-metric connection to be conjugate symmetry, it is necessary and sufficient that corresponding Ricci tensor and volume curvature tensor be equal.*

## 6. Acknowledgments

We would like to thank Professor Peibiao Zhao for his encouragement and help.

## REFERENCES

1. N. S. Agache and M. R. Chafle: *A semi-symmetric non-metric connection on a Riemannian manifold*. Indian J. Pure Apple. Math. **23**(1992), 399-469.
2. S. K. Chaubeg and R. H. Ojha: *On semi-symmetric non-metric connection*. Filomat. **26**(2)(2012), 269-275.
3. P. Erdős: *On the distribution of the roots of orthogonal polynomials*. In: Proceedings of a Conference on Constructive Theory of Functions (G. Alexits, S. B. Steckhin, eds.), Akademiai Kiado, Budapest, 1972, pp. 145–150.
4. A. Ostrowski: *Solution of Equations and Systems of Equations*. Academic Press, New York, 1966.
5. E. B. Saff and R. S. Varga: *On incomplete polynomials II*. Pacific J. Math. **92** (1981), 161–172.
6. U. C. De and D. Kamila: *On a type of semi-symmetric non-metric connection on a Riemannian manifold*. J. Indian Inst. Sci. **75**(1995), 707-710.
7. U. C. De and S. C. Biswas: *On a type of semi-symmetric non-metric connection on a Riemannian manifold*. Istanbul Univ. Mat.Derg. **55/56**(1996/1997), 237-243.
8. U. C. De and J. Sengupta: *On a type of semi-symmetric metric connection on an almost contact metric manifold*. Filomat. **14**(2000), 33-42.
9. K. A. Dunn: *A Geometric model for scalar-tensor theories of gravitation*. Tensor. N. S. 29(1975), 214-216.
10. G. S. Hall: *Weyl manifolds and connections*. Journal of Mathematical Physics. **33**(7)(1992), 2633-2638.
11. G. S. Hall and B. M. Haddow: *Some remarks on metric and Weyl connections*. Journal of Mathematical Physics. **36**(10)(1995), 5938-5948.
12. A. Hayden: *Subspaces of a space with torsion*. Proc. London. Math. Soc, **34**(1932), 27-50.
13. T. Higa: *Weyl manifolds and Einstein-Weyl manifolds*. Comm. Math. Univ. Sancti Pauli, **12**(2)(1993), 143-159.
14. T. Imai: *Notes on semi-symmetric metric connection*. Tensor. N. S, **24**(1972), 293-296.
15. J. P. Jaiswal and R. H. Ojha: *Some properties of K-contact Riemannian manifolds admitting a semi-symmetric non-metric connection*. Filomat. **24**(1) (2010), 9-16.
16. H. V. Le: *Statistical manifolds are statistical models*. J. Geom, **84**(2005), 83-93.
17. G. Lyra: *Über eine Modifikation der Riemannschen Geo-metrie*. Mathematische Zeitschrift, **54**(1)(1951), 52-64.

18. E. S. Stepanova: *Dual symmetric statistical manifolds.* J. of Mathematical Sciences. **147**(1)(2007), 6507-6509.
19. I. Suhendro: *A new semi-symmetric unified field theory of the classical fields of gravity and electromagnetism.* Progress in physics, **4**(2007), 47-62.
20. D. K. Sen and J. R. Vanstone: *On Weyl and Lyra manifolds.* Journal of Mathematical Physics, **13**(7)(1972), 990-993.
21. F. Unal and A. Uysal: *Weyl manifolds with semi-symmetric connection.* Mathematical and Computational Applications, **10**(3)(2005), 351-358.
22. H. Weyl: *Gravitation und Elektrizität.* Sitzungsber. Preuss. Akad. Wiss., **26**(1918), 465-478.
23. M. P. Wojtkowski: *On some Weyl manifolds with nonpositive sectional curvature.* Proc. Amer. Math. Soc., **133**(11)(2005), 3395-3402.
24. K. Yano: *On semi-symmetric metric connection.* Rev. Roumanie. Math. Pure Apple.**15**(1970), 1579-1586.
25. P. B. Zhao: *Some properties of Projective semi-symmetric connections.* International Mathematical Forum, **3**(7)(2008), 341-347.
26. P. B. Zhao and L. Jiao: *Conformal transformations on Carnot Caratheodory spaces.* Nihonkai Mathematical Journal, **17**(2)(2006), 181-209.
27. P. B. Zhao and H. Z. Song: *An invariant of the projective Semi-symmetric connection.* Chin. Quart. J. of Math., **16**(4)(2001), 49-54.
28. P. B. Zhao, H. Z. Song and X. P. Yang: *Some invariant properties of semi-symmetric metric recurrent connections and curvature tensor expressions.* Chin. Quart. J. of Math., **19**(4)(2004), 355-361.

Yanling Han
Nanjing University of Science and Technology
Dept. of Applied Mathematics
Qilu University of Technology
School of Science
Nanjing 210094, P. R. China

Tal Yun Ho
Kim I1 Sung University
Faculty of Mathematics
Pyongyang, DPRK

Fengyun Fu
Guangdong University of Finance and Economics
School of Mathematics and Statistics
Guangzhou 510320, P.R.China
fengyunfu@aliyun.com