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ASYMPTOTIC STABILITY PROPERTIES OF SOLUTIONS TO A BRESSE SYSTEM WITH A WEAK VISCOELASTIC TERM *

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Abstract. We consider the Bresse system in bounded domain with a weak viscoelastic terms acting in the one equation of the system under some conditions imposed into the relaxation functions. We study the asymptotic behavior of solutions using suitable energy and Lyapunov functionals.
Keywords: Bresse system, relaxation function, Lyapunov functional, energy decay.

1. Introduction

In this paper, we investigate the decay properties of solutions for the initial boundary value problem of the linear Bresse system of the type

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + l\omega)_x - lk_3(\omega_x - l\varphi) = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + l\omega) + \sigma(t)\displaystyle\int_0^t g(s)\psi_{xx}(s)ds = 0, \\ \rho_1\omega_{tt} - k_3(\omega_x - l\varphi)_x + lk_1(\varphi_x + \psi + l\omega) = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in (0,L) \times (0,+\infty)$. This system is subject to the Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = \omega(0,t) = \omega(L,t) = 0, \quad t > 0$$

and to the initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad x \in (0,L) \\ \psi_t(x,0) = \psi_1(x), \quad \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \quad x \in (0,L) \end{cases}$$

where the initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1)$ belong to a suitable Sobolev space. By

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ω, ψ and φ we are denoting the longitudinal, vertical and shear angle displacements. The original Bresse system is given by the following equations (see [**1**]) :

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1, \\ \rho_2\psi_{tt} = M_x - Q + F_2, \\ \rho_1\omega_{tt} = N_x - lQ + F_3, \end{cases}$$

where we use N, Q and M to denote the axial force, the shear force and the bending moment respectively. These forces are stress-strain relations for elastic behavior and given by

$$N = Eh(\omega_x - l\varphi), \quad Q = Gh(\varphi_x + \psi + l\omega), \quad \text{and } M = EI\psi_x,$$

where G, E, I and h are positive constants. Finally, by the terms F_i we are denoting external forces.
The Bresse system is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered ($l = 0$). There are a number of publications concerning the stabilization of Timoshenko system with different kinds of damping (see [**2**], [**3**], [**4**] and [**5**]). Raposo et al. [**6**] proved the exponential decay of the solution for the following linear system of Timoshenko-type beam equations with linear frictional dissipative terms:

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1\varphi_t = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + \widetilde{\mu_1}\psi_t = 0. \end{cases}$$

Messaoudi and Mustafa [**3**] (see also [**5**]) considered the stabilization for the following Timoshenko system with nonlinear internal feedbacks:

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + g_1(\psi_t) = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + g_2(\psi_t) = 0. \end{cases}$$

Recently, Park and Kang [**5**] considered the stabilization of the Timoshenko system with weakly nonlinear internal feedbacks.
For the Timoshenko system, along with the new theory og Green and Naghdi [**14**], Messaoudi and Said-Houari [**15**] considered a Timoshenko system of thermoelasticity of type III of the form

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, &]0, L[\times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_x = 0, &]0, L[\times \mathbb{R}_+, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0, &]0, L[\times \mathbb{R}_+, \end{cases}$$

where φ, ψ and θ are function of (x,t), which model the transverse displacement of the beam, the rotation angle of the filament and the difference temperature, respectively. They proved an exponential decay in the case of equal speeds $\left(\frac{k}{\rho_1} = \frac{b}{\rho_2}\right)$. This result was later established by Messaoudi and Said-Houari [20] for above system in the presence of a viscoelastic damping of the form

$$\int_0^\infty g(s)\psi_{xx}(x,t-s)ds$$

acting in the second equation. Moreover, the case of nonequal speeds $\left(\frac{k}{\rho_1} \neq \frac{b}{\rho_2}\right)$ was studied and a polynomial decay result was proved for solutions with smooth initial data. A more general decay result, from which the exponential and polynomial rates of decay are only special cases, was also established by Kafini **[21]**. Raposo et al. **[6]** proved the exponential decay of the solution for the following linear system of Timoshenko-type beam equations with linear frictional dissipative terms:

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1\varphi_t = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + \widetilde{\mu_1}\psi_t = 0. \end{cases}$$

Recently F. A Boussouira and J. Munoz Rivera **[22]** studied the following problem:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + \sigma\psi_t = 0, \\ \rho_1\omega_{tt} - k_3\omega_{xx} = 0, \end{cases}$$

and proved that this dissipative mechanism is enough to stabilize the whole system provided the velocities of waves propagations are the same.
Motivated by the previous works, in this paper it is interesting to give a more general decay estimates of the solutions to the problem (1.1) for a weak viscoelastic term. To the best of our knowledge there is no result of decay estimate of the Bresse system in the presence of a weak viscoelastic term. Under suitable assumptions on both functions $g(t)$ and $\sigma(t)$ that will be specified later, the initial data and the parameters in the equations, we establish general decay estimates by using suitable energy and Lyapunov functionals.

2. Preliminary Results

In this section, we present some material for the proof of our result. For the relaxation function g and σ we assume

$(\mathbf{A_0})$ $g, \sigma : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing differentiable functions satisfying

$$g(0) > 0,\ l_0 = \int_0^\infty g(s)ds < \infty,$$
$$\sigma(t) > 0,\ 1 - \sigma(t)\int_0^t g(s)ds > l > 0\ for\ t > 0, \tag{2.1}$$

there exists a nonincreasing differentiable function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\eta(t) > 0,\ g'(t) \leq -\eta(t)g(t) \quad for\ t > 0, \quad \lim_{t\to\infty} \frac{-\sigma^{'}(t)}{\eta(t)\sigma(t)} = 0. \tag{2.2}$$

The following inequality will be proved in Lemma 3.1 by contradiction arguments. It is easy to see that there exists a positive constant $\widetilde{k}_0$ such that, for $(\varphi, \psi, \omega) \in (H_0^1(]0, L[))^3$, we have

$$\widetilde{k}_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + \omega_x^2\right) dx \leq$$
$$\int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + l\omega)^2 + k_3(w_x - l\varphi)^2\right) dx. \tag{2.3}$$

On the other hand, thanks to Poincare's inequality, there exists a positive constant $\widetilde{k}_0$ such that, for $(\varphi, \psi, \omega) \in (H_0^1(]0, L[))^3$, we get

$$\int_0^L (k_2\psi_x^2 + k_1\left(\varphi_x + \psi + l\omega\right)^2 + k_3(\omega_x - l\varphi)^2\Big) dx \leq$$
$$\widetilde{k}_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + \omega_x^2\right) dx. \tag{2.4}$$

We first state some Lemmas which will be needed later.

Lemma 2.1. *(Sobolev-Poincaré's inequality). Let q be a number with $2 \leq q < +\infty$. Then there is a constant $c_* = c_*((0,1), q)$ such that*

$$\|\psi\|_q \leq c_*\|\psi_x\|_2 \quad for \quad \psi \in H_0^1((0,1)).$$

Now we give some estimates related to the convolution operator. By direct calculations, as in **[16-17]** we find

$$\begin{aligned} \sigma(t)(g*\psi, \psi_t) &= -\frac{d}{dt}\left[\frac{\sigma(t)}{2}(g\ o\ \psi)(t) - \frac{\sigma(t)}{2}\left(\int_0^\infty g(s)ds\right)\|\psi(t)\|_2^2\right] \\ &+ \tfrac{\sigma(t)}{2}(g^{'}\ o\ \psi)(t) + \tfrac{\sigma^{'}(t)}{2}(g\ o\ \psi)(t) \\ &- \tfrac{\sigma^{'}(t)}{2}\left(\int_0^\infty g(s)ds\right)\|\psi(t)\|_2^2 - \tfrac{\sigma(t)}{2}g(t)\|\psi(t)\|_2^2, \end{aligned} \tag{2.5}$$

where

$$(2.6)\qquad \begin{aligned}(g*\psi)(t) &= \int_0^\infty g(t-s)\psi(s)ds,\\ (g\ o\ \psi)(t) &= \int_0^\infty g(t-s)\psi(s)\|\psi(t)-\psi(s)\|_2^2 ds,\end{aligned}$$

and

$$(2.7)\qquad (g*\psi,\psi) \le 2\left(\int_0^t g(s)ds\right)\|\psi(t)\|_2^2 + \frac{1}{4}(g\ o\ \psi)(t).$$

The above system subjected to the following initial and boundary conditions

$$(2.8)\qquad \begin{cases} \varphi(0,t)=\varphi(L,t)=\psi(0,t)=\psi(L,t)=\omega(0,t)=\omega(L,t),\ t>0\\ \varphi(x,0)=\varphi_0, \varphi_t(x,0)=\varphi_1, \psi(x,0)=\psi_0, \psi_t(x,0)=\psi_1,\\ \omega(x,0)=\omega_0, \omega_t(x,0)=\omega_1, \quad x\in(0,L).\end{cases}$$

We define the energy associated to the solution of the problem (1.1) by the following formula:

$$(2.9)\qquad \begin{aligned} E(t) &= \frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|\omega_t\|_2^2\\ &+ \frac{k_1}{2}\|\varphi_x+\psi+l\omega\|_2^2 + \frac{k_3}{2}\|\omega_x - l\varphi\|_2^2\\ &+ \sigma(t)(g\ o\ \psi_x)(t) + \frac{1}{2}\left(k_2 - \sigma(t)\int_0^t g(s)ds\right)\|\psi_x(t)\|_2^2.\end{aligned}$$

Now we give an explicit upper bound for the derivative of the energy.

Lemma 2.2. *Let (φ,ψ,ω) be a solution of the problem (1.1). Then, the energy functional defined by (2.9) satisfies*

$$(2.10)\qquad E'(t) \le -\frac{\sigma(t)}{2}(g' o\ \psi_x)(t) - \frac{\sigma'(t)}{2}\int_0^t g(s)ds\|\psi_x(t)\|_2^2.$$

Proof. Multiplying the first equation in (1.1) by φ_t, the second by ω_t and the third equation by ψ_t, integrating the result over $(0,L)$ and using integration by parts, we get

$$\frac{1}{2}\rho_1\frac{d}{dt}\|\varphi_t\|_2^2 - k_1\int_0^L(\varphi_x+\psi+l\omega)_x\varphi_t dx - lk_3\int_0^L(\omega_x - l\varphi)\varphi_t dx = 0,$$

$$\frac{1}{2}\rho_2\frac{d}{dt}\|\psi_t\|_2^2+\frac{k_2}{2}\|\psi_x\|_2^2+k_1\int_0^L(\varphi_x+\psi+l\omega)\psi_t dx+\sigma(t)\int_0^L\int_0^t g(s)\psi_{xx}(s)dsdx=0,$$

$$\frac{1}{2}\rho_1\frac{d}{dt}\|\omega_t\|_2^2-k_3\int_0^L(\omega_x-l\varphi)_x\omega_t dx+lk_1\int_0^L(\varphi_x+\psi+l\omega)\omega_t dx=0.$$

Then

$$\begin{aligned} &+ \frac{d}{dt}\left(\frac{\rho_1}{2}\|\varphi_t\|_2^2+\frac{\rho_2}{2}\|\psi_t\|_2^2+\frac{\rho_1}{2}\|\omega_t\|_2^2+\frac{k_1}{2}\|\psi_x\|_2^2\right)\\ &+ \frac{d}{dt}\left(\frac{k_2}{2}\|\varphi_x+\psi+l\omega\|_2^2+\frac{k_3}{2}\|\omega_x-l\varphi\|_2^2\right)\\ &+ \sigma(t)(g^{'}o\ \psi_x)(t)-\sigma^{'}(t)\left(\int_0^t g(s)ds\right)\|\psi_x(t)\|_2^2\\ &- \frac{\sigma(t)}{2}g(t)\|\psi_x(t)\|_2^2+\sigma^{'}(t)(g\ o\ \psi_x)(t)=0. \end{aligned} \tag{2.11}$$

Integrating the above system over $[0,L]\times(0,1)$, we get

$$E(t)+\int_0^t\sigma^{'}(s)(go\psi_x)(s)ds-\frac{\sigma^{'}(t)}{2}\int_0^t g(s)\|\psi_x(t)\|_2^2 ds=E(0). \tag{2.12}$$

After deriving the last equality, we deduce the desired result. □

3. Asymptotic Stability

In this section, we prove the asymptotic stability result by constructing a suitable Lyapunov functional. Now, let us introduce the following functional

$$\mathcal{L}(t)=ME(t)+\sigma(t)I_4(t), \tag{3.1}$$

such that

$$I_4(t)=-\int_0^L(\rho_1\varphi\varphi_t+\rho_2\psi\psi_t+\rho_1\omega\omega_t)\,dx. \tag{3.2}$$

Then the following result holds.

Lemma 3.1. *There exists a positive constant C such that the following inequality holds for every $(\varphi,\psi,\omega)\in(H_0^1(0,L))^3$*

$$\begin{aligned}\int_0^L\left(|\varphi_x|^2+|\psi_x|^2+|\omega_x|^2\right)dx &\le C\int_0^L(k_2|\psi_x|^2+k_1|\varphi_x+\psi+l\omega|^2)dx\\ &+\int_0^L k_3|\omega_x-l\varphi|^2\,dx.\end{aligned} \tag{3.3}$$

Proof. We will argue by contradiction. Indeed, let us suppose that is not true. So, we can find a sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in N}$ in $(H_0^1(0,L))^3$ satisfying

$$\int_0^L (k_2|\psi_{\nu x}|^2 + k_1|\varphi_{\nu x} + \psi + l\omega_\nu|^2 + k_3|\omega_{\nu x} - l\varphi_\nu|^2)\,dx \le \frac{1}{\nu}, \tag{3.4}$$

and

$$\int_0^L (|\varphi_{\nu x}|^2 + |\psi_{\nu x}|^2 + |\omega_{\nu x}|^2)\,dx = 1. \tag{3.5}$$

From (3.5), the sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in N}$ is bounded in $(H_0^1(0,L))^3$. Since the embedding $H_0^1(0,L) \hookrightarrow \mathbb{L}^2(0,L)$ is compact, then the sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in N}$ converge strongly in $(\mathbb{L}^2(0,L))^3$. From (3.5), we get

$$\psi_{\nu x} \to 0 \text{ strongly in } \mathbb{L}^2(0,L). \tag{3.6}$$

Using Poincaré's inequality, we can conclude that

$$\psi_\nu \to 0 \text{ strongly in } \mathbb{L}^2(0,L). \tag{3.7}$$

Now, setting $\varphi_\nu \to \varphi$ and $\omega_\nu \to \omega$ strongly in $\mathbb{L}^2(0,L)$. From (3.6), we have

$$\varphi_{\nu x} + \psi_\nu + l\omega_\nu \to 0 \text{ strongly in } \mathbb{L}^2(0,L). \tag{3.8}$$

Then

$$\varphi_{\nu x} + \psi_\nu + l\omega_\nu = \varphi_{\nu x} + \psi_\nu + l(\omega_\nu - \omega) + l\omega \to 0 \text{ strongly in } \mathbb{L}^2(0,L). \tag{3.9}$$

which implies that

$$\varphi_{\nu x} \to -l\omega \text{ strongly in } \mathbb{L}^2(0,L). \tag{3.10}$$

Then, $\{\varphi_\nu\}_n$ is a Cauchy sequence in $H^1(0,L)$. Therefore $\{\varphi_\nu\}_n$ converge to a function φ_1 in $H^1(0,L)$. Consequently $\{\varphi_\nu\}_n$ converge to φ_1 in $\mathbb{L}^2(0,L)$. Thus by the uniqueness of the limit $\varphi_1 = \varphi$. Moreover $\varphi \in H_0^1(0,L)$. From (3.10), we deduce that

$$\varphi_x + l\omega = 0 \text{ a.e } x \in (0,L). \tag{3.11}$$

Similarly, we have

$$\omega_x - l\varphi = 0 \text{ a.e } x \in (0,L), \tag{3.12}$$

and $\omega \in H_0^1(0,L)$. The equations (3.9) and (3.11) provides us $\varphi = \omega = 0$, contradicting (3.4). The proof is hence complete. ☐

Lemma 3.2. *The functional defined in (3.2) satisfies for any $c_1 > 0$*

$$\begin{aligned} I_4'(t) \le & -\int_0^L \left\{(\rho_1+\epsilon)\varphi_t^2 + (\rho_2+\epsilon)\psi_t^2 + (\rho_1+\epsilon)\omega_t^2\right\} dx \\ & - c_1 \int_0^L \left\{\psi_x^2 + (\varphi_x + \psi + l\omega)^2 + (\omega_x - l\varphi)^2\right\} dx \\ & +2\ \sigma(t)\left(\int_0^t g(s)ds\right) \|\psi_x\|_2^2 + \tfrac{\sigma(t)}{4}(g\ o\ \psi_x)(t). \end{aligned} \tag{3.13}$$

Proof. By taking the derivative of (3.2) and using the system (1.1), we get

$$\begin{aligned} I_4'(t) &= -\int_0^L (\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1\omega_t^2)dx + k_1\int_0^L(\varphi_x+\psi+l\omega)^2dx \\ &- k_3\int_0^L(\omega_x-l\varphi)^2dx+\sigma(t)(g*\psi_x,\psi_x)-k_2\int_0^L\psi_x^2dx. \end{aligned} \tag{3.14}$$

Using Poincare and Hölder inequalities, we find

$$\begin{aligned} I_4'(t) &\le -\int_0^L \left\{(\rho_1+\epsilon)\varphi_t^2+(\rho_2+\epsilon)\psi_t^2+(\rho_1+\epsilon)\omega_t^2\right\}dx \\ &- (k_0-2\epsilon)\int_0^L(\varphi_x^2+\psi_x^2+\omega_x^2)dx+\sigma(t)(g*\psi_x,\psi_x). \end{aligned} \tag{3.15}$$

Inserting (2.7) into (3.15), we get

$$\begin{aligned} I_4'(t) &\le -\int_0^L \left\{(\rho_1+\epsilon)\varphi_t^2+(\rho_2+\epsilon)\psi_t^2+(\rho_1+\epsilon)\omega_t^2\right\}dx \\ &- (k_0-2\epsilon)\int_0^L(\varphi_x^2+\psi_x^2+\omega_x^2)dx+\sigma(t)(g*\psi_x,\psi_x) \\ &+ 2\sigma(t)\left(\int_0^t g(s)ds\right)\|\psi_x\|_2^2+\tfrac{\sigma(t)}{4}(g\ o\ \psi_x)(t). \end{aligned} \tag{3.16}$$

Finally using (2.7) in the last inequality, we get the desired result. □

Lemma 3.3. *Let $\mathcal{L}(t)$ the functional defined in (3.1), then $\mathcal{L}(t)$ satisfies*

$$\frac{d}{dt}\mathcal{L}(t)\le -C_1\sigma(t)E(t)+C_2\sigma(t)(g\ o\ \psi_x)(t),\ \forall t\ge 0. \tag{3.17}$$

Proof. We take the derivative of (3.1), we get

$$\frac{d}{dt}\mathcal{L}(t)=M\frac{d}{dt}E(t)+\sigma(t)\frac{d}{dt}I_4(t)+\sigma'(t)I_4(t), \tag{3.18}$$

making use of the inequalities

$$\sigma'(t)\left|\int_0^L\varphi\varphi_tdx\right|\le\sigma'(t)\frac{c_s^2}{\alpha_1}\|\varphi_x\|_2^2+\sigma'(t)\alpha_1\|\varphi_t\|_2^2, \tag{3.19}$$

$$\sigma'(t)\left|\int_0^L\psi\psi_tdx\right|\le\sigma'(t)\frac{c_s^2}{\alpha_1}\|\psi_x\|_2^2+\sigma'(t)\alpha_1\|\psi_t\|_2^2, \tag{3.20}$$

$$\sigma'(t)\left|\int_0^L\omega\omega_tdx\right|\le\sigma'(t)\frac{c_s^2}{\alpha_1}\|\omega_x\|_2^2+\sigma'(t)\alpha_1\|\omega_t\|_2^2. \tag{3.21}$$

Combining (2.10), (3.13) and (3.19)-(3.21), we have

$$
\begin{aligned}
\mathcal{L}'(t) \leq\ & -\sigma(t)\left\{2(\rho_1+\epsilon)+\tfrac{\sigma'(t)}{\sigma(t)}\alpha_1\rho_1\right\}\|\varphi_t\|_2^2 \\
& - \sigma(t)\left\{2(\rho_2+\epsilon)+\tfrac{\sigma'(t)}{\sigma(t)}\alpha_1\rho_2\right\}\|\psi_t\|_2^2 \\
& - \sigma(t)\left\{2(\rho_1+\epsilon)+\tfrac{\sigma'(t)}{\sigma(t)}\alpha_1\rho_2\right\}\|\omega_t\|_2^2 \\
& - \sigma(t)\left\{c_1-2\int_0^t g(s)ds+(k_0-2\epsilon)+\frac{\sigma'(t)}{\sigma(t)}\frac{c_s^2\rho_2}{\alpha_1}\right\}\|\psi_x\|_2^2 \\
& - \sigma(t)\left\{\tfrac{\sigma'(t)}{\sigma(t)}\tfrac{c_s^2\rho_1}{\alpha_1}\right\}\|\varphi_x\|_2^2-\sigma(t)\left\{\tfrac{\sigma'(t)}{\sigma(t)}\tfrac{c_s^2\rho_1}{\alpha_1}\right\}\|\omega_x\|_2^2 \\
& - \sigma(t)\left\{c_1-(k_0-2\epsilon)\right\}\|\varphi_x+\psi+L\omega\|_2^2 \\
& - \sigma(t)\{c_1-1\}\|\omega_x-L\varphi\|_2^2+\tfrac{\sigma^2(t)}{4}(g\circ\psi_x)(t).
\end{aligned} \tag{3.22}
$$

Since $\lim_{t\to\infty}\frac{\sigma'(t)}{\sigma(t)}=0$, we can choose $t_0>0$ sufficiently large so that

$$
\left\{c_1-2\int_0^t g(s)ds+(k_0-2\epsilon)+\frac{\sigma'(t_0)}{\sigma(t_0)}\frac{c_s^2\rho_2}{\alpha_1}\right\}>0,\quad \left\{\frac{\sigma'(t_0)}{\sigma(t_0)}\frac{c_s^2\rho_1}{\alpha_1}\right\}>0.
$$

Using (2.7), we get

$$
\begin{aligned}
\mathcal{L}'(t) \leq\ & -\sigma(t)\left\{2(\rho_1+\epsilon)+\tfrac{\sigma'(t)}{\sigma(t)}\alpha_1\rho_1\right\}\|\varphi_t\|_2^2 \\
& - \sigma(t)\left\{2(\rho_2+\epsilon)+\tfrac{\sigma'(t)}{\sigma(t)}\alpha_1\rho_2\right\}\|\psi_t\|_2^2 \\
& - \sigma(t)\left\{2(\rho_1+\epsilon)+\tfrac{\sigma'(t)}{\sigma(t)}\alpha_1\rho_2\right\}\|\omega_t\|_2^2 \\
& - \sigma(t)\left\{c_1-1-\tfrac{\widetilde{k_0}}{k_1}\right\}\|\omega_x-L\varphi\|_2^2+\tfrac{\sigma(t)^2}{4}(g\circ\psi_x)(t) \\
& - \sigma(t)\left\{c_1-(k_0-2\epsilon)-\tfrac{\widetilde{k_0}}{k_1}\right\}\|\varphi_x+\psi+L\omega\|_2^2.
\end{aligned} \tag{3.23}
$$

We finally obtain

$$
\frac{d}{dt}\mathcal{L}(t)\leq -C_1\sigma(t)E(t)+C_2\sigma(t)(g\ o\ \psi_x)(t),\quad \forall t\geq 0.
$$

This completes the proof. □

Lemma 3.4. *There exists two positive constants λ_1, λ_2 such that*

$$
\lambda_1 E(t)\leq \mathcal{L}(t)\leq \lambda_2 E(t),\quad t\geq 0, \tag{3.24}
$$

for M sufficiently large.

Proof. By making use of the inequalities

$$\left|\int_0^L \varphi\varphi_t dx\right| \leq \frac{c_s^2}{\alpha_1}\|\varphi_x\|_2^2 + \alpha_1\|\varphi_t\|_2^2, \tag{3.25}$$

$$\left|\int_0^L \psi\psi_t dx\right| \leq \frac{c_s^2}{\alpha_1}\|\psi_x\|_2^2 + \alpha_1\|\psi_t\|_2^2, \tag{3.26}$$

$$\left|\int_0^L \omega\omega_t dx\right| \leq \frac{c_s^2}{\alpha_1}\|\omega_x\|_2^2 + \alpha_1\|\omega_t\|_2^2. \tag{3.27}$$

Combining (3.25)-(3.27), we have

$$\begin{aligned} |\mathcal{L}(t) - ME(t)| &\leq \sigma(t)\alpha_1\|\varphi_t\|_2^2 + \sigma(t)\alpha_1\|\psi_t\|_2^2 + \sigma(t)\alpha_1\|\omega_t\|_2^2 \\ &+ \tfrac{\sigma(t)c_s^2}{\alpha_1}\|\varphi_x\|_2^2 + \tfrac{\sigma(t)c_s^2}{\alpha_1}\|\psi_x\|_2^2 + \tfrac{\sigma(t)c_s^2}{\alpha_1}\|\omega_x\|_2^2, \end{aligned} \tag{3.28}$$

using the fact that $\frac{\sigma(t)}{\sigma(0)} \leq 1$ and the inequality (2.3), to get

$$|\mathcal{L}(t) - ME(t)| \leq \sigma(0)\alpha_1 E(t) + \frac{\sigma(0)c_s^2}{\alpha_1}E(t), \tag{3.29}$$

finally

$$|\mathcal{L}(t) - ME(t)| \leq c_6 E(t), \tag{3.30}$$

where $c_6 = max\left\{\sigma(0)\alpha_1, \quad \frac{\sigma(0)c_s^2}{\alpha_1}\right\}$. Thus, from the definition of $E(t)$ and selecting M sufficiently large, we can easily find

$$\lambda_2 E(t) \leq \mathcal{L}(t) \leq \lambda_1 E(t). \tag{3.31}$$

Where $\lambda_1 = (M - c_6), \quad \lambda_2 = (M + c_6)$. This ends the proof. □

Theorem 3.1. *We suppose that the following equalities are satisfied*

$$\frac{\rho_1}{\rho_1} = \frac{k_1}{k_2}, \quad k_1 = lk_3.$$

Then, there exist positive constants C_0, θ and t_1 such that

$$E(t) \leq C_0 e^{-\theta \int_{t_1}^{t} \sigma(s)\eta(s)ds} \tag{3.32}$$

Proof. Multiplying (3.17) by $\eta(t)$ and using the Lemma 2.2, we get

$$
\begin{aligned}
\eta(t)\tfrac{d}{dt}\mathcal{L}(t) &\le -C_1\sigma(t)\eta(t)E(t) + C_2\sigma(t)\eta(t)(go\psi_x)(t) \\
&\le -C_1\sigma(t)\eta(t)E(t) - C_2\sigma(t)\eta(t)(g^{'}o\psi_x)(t) \qquad (3.33)\\
&\le -C_1\sigma(t)\eta(t)E(t) + C_2\left(-2\tfrac{d}{dt}E(t) - \sigma^{'}(t)\int_0^t g(s)ds\|\psi_x\|_2^2\right).
\end{aligned}
$$

Since η is nonincreasing, from the definition of $E(t)$ and assumption (2.2), we have

$$\frac{d}{dt}\left(\eta(t)\mathcal{L}(t) + 2C_2E(t)\right) \le -\sigma(t)\eta(t)\left(C_1 + \frac{2C_2l_0\sigma^{'}(t)}{\lambda l\sigma(t)\eta(t)}\right)E(t) \quad for\ t > t_0,$$

as we have $\lim\limits_{t\to\infty}\dfrac{2C_2l_0\sigma^{'}(t)}{\lambda l\sigma(t)\eta(t)} = 0$, we can choose $t_1 > t_0$ such that

$$C_3 = C_1 + \frac{2C_2l_0\sigma^{'}(t)}{\lambda l\sigma(t)\eta(t)} > 0 \quad \text{for} \quad t > t_1.$$

Now let

$$\chi(t) = \eta(t)\mathcal{L}(t) + 2C_2E(t).$$

Then we can verify that

$$\theta_1E(t) \le \chi(t) \le \theta_2E(t). \qquad (3.34)$$

Where θ_1, θ_2 are two positive constants, thus we arrive at

$$\frac{d}{dt}\chi(t) \le -C_4\sigma(t)\eta(t)\chi(t) \quad for\ t > t_1.$$

Integrating the previous differential inequality between t_1 and t gives the following estimate for the function χ

$$\chi(t) \le \chi(t_1)e^{-C_4\int_{t_1}^t\sigma(s)\eta(s)ds}, \quad \forall t \ge t_1.$$

Consequently, by using (3.33), we conclude

$$E(t) \le \hat{C}e^{-C_4\int_{t_1}^t\sigma(s)\eta(s)ds}, \quad \forall t \ge t_1.$$

This completes the proof. □

Remark 3.1. We illustrate the energy decay rate given by Theorem 3.1 through the following examples.

1. If $g(t) = ae^{-b(1+t)^{\nu}}$, $\sigma(t) = \frac{1}{1+t}$ for $a, b > 0$ and $0 < \nu \leq 1$, then $\eta(t) = b\nu(1+t)^{\nu-1}$ satisfies the conditions (2.1) and (2.2). Thus (3.32) gives the estimate
$$E(t) \leq C_0 e^{-\theta(1+t)^{\nu-1}}.$$
2. If $g(t) = ae^{-b\ln^{\nu}(1+t)}$, $\sigma(t) = \frac{1}{\ln(1+t)}$ for $a, b > 0$ and $1 < \nu$, then $\eta(t) = \frac{b\nu \ln^{\nu-1}(1+t)}{(1+t)}$ satisfies the conditions (2.1) and (2.2). Thus (3.32) gives the estimate
$$E(t) \leq C_0 e^{-\theta \ln^{\nu}(1+t)}.$$
3. If $g(t) = e^{-at}$, $\sigma(t) = \frac{b}{(1+t)}$ for $a, b > 0$ then $\eta(t) \equiv a$ satisfies the conditions (2.1) and (2.2). Thus (3.32) gives the estimate
$$E(t) \leq C_0(1+t)^{-\theta ab}.$$
4. If $g(t) = e^{-at}$, $\sigma(t) \equiv b$. Note that in this case (3.32) reduces to one of [23].

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