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RICCI SOLITONS AND GRADIENT RICCI SOLITONS IN AN $LP-$SASAKIAN MANIFOLD *

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Abstract. The objective of the present paper is to study an LP-Sasakian manifold admitting Ricci solitons and gradient Ricci solitons.

1. Introduction

An $n-$dimensional Lorentzian manifold M is a smooth connected para-contact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric second order tensor field g such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-, +, +, ..., +)$, where T_pM denotes the tangent vector space of M at p. The study of Lorentzian almost paracontact manifold was initiated by Matsumoto [16]. Later on several authors studied Lorentzian almost paracontact manifolds and their different classes, viz. LP-Sasakian and LSP-Sasakian manifolds(cf. [10], [11], [17], [18], [21]).

Ricci solitons are natural generalization of Einstein metrics and have been a branch of study in mathematics, as they correspond to special solutions of Ricci flow[2]. In a Riemannian manifold (M, g), g is called a Ricci soliton if [14]

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0, \tag{1.1}$$

where $\pounds$ is the Lie derivative, S is the Ricci tensor, V is a vector field on M and λ is a constant. Metrics satisfying (1.1) are interesting and useful in physics and are often referred as quasi-Einstein (e.g. [3],[4],[12]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up

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limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan who discusses some aspects of it [12].

The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (1.1) assumes the form

$$\nabla\nabla f = S + \lambda g. \tag{1.2}$$

The obvious examples of Ricci solitons are Einstein solitons, where g is an Einstein metric and X is a Killing vector field. A Ricci soliton on a compact manifold has constant curvature in dimension 2 [14] and also in dimension three [15]. For details we refer to Chow and Knopf [5] and Derdzinski [6]. We also recall the following significant result of Perelman [19]: A Ricci soliton on a compact manifold is a gradient Ricci soliton.

In [20], Sharma has started the study of Ricci solitons in K-contact manifolds. Also, in a subsequent paper [13] Ghosh, Sharma and Cho studied gradient Ricci soliton of a non-Sasakian (k,μ)-contact manifold. In a K-contact manifold the structure vector field ξ is Killing, that is, $\pounds_\xi g = 0$, which is not in general, in a $P-$Sasakian manifold. In [7], U. C. De have studied Ricci solitons in $P-$Sasakian manifolds. Recently in [1], B. Barua and U. C. De have studied Ricci solitons in Riemannian manifolds.

Motivated by these circumstances, in this paper we study Ricci solitons and gradient Ricci solitons in an $LP-$Sasakian manifold.

The paper is organized as follows. After preliminaries in section 2 among others we prove that in an LSP-Sasakian manifold if g admits a Ricci soliton (g,V,λ) and V is point-wise colinear with ξ, then the manifold is an $\eta-$Einstein manifold and also we show that if an LSP-Sasakian manifold admits a compact Ricci soliton, then the manifold is Einstein. Finally we prove that if an $\eta-$Einstein LP-Sasakian manifold admits a gradient Ricci soliton, then the manifold reduces to an Einstein manifold under certain condition.

2. $LP-$Sasakian Manifold

Let M be an n-dimensional Lorentzian para Sasakian (LP-Sasakian) manifold with structure $\sum = (\phi,\xi,\eta,g)$, where ϕ is a (1,1)-tensor field, ξ is a contravariant vector field, η is a 1-form and g is a Lorentzian metric, then by definition, they satisfies [22]

$$\eta(\xi) = -1, \quad \phi^2 = I + \eta\otimes\xi, \tag{2.1}$$

$$\phi\xi = 0, \quad \eta.\phi = 0, \quad \nabla_X\xi = \phi X, \quad rank(\phi) = n-1, \tag{2.2}$$

$$\eta(X) = g(\xi, X), \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X \eta)(Y) = \Omega(Y,X), \quad \Omega(X,Y) = \Omega(Y,X) \quad (\Omega(Y,X) = g(\phi Y, X)), \tag{2.4}$$

$$\begin{aligned}(\nabla_Z \Omega)(X,Y) &= \{g(X,Z) + \eta(X)\eta(Z)\}\eta(Y) \\ &\quad + \{g(Y,Z) + \eta(Y)\eta(Z)\}\eta(X)\end{aligned} \tag{2.5}$$

for any vector fields X, Y and Z on M, where I denotes the identity map on T_pM (the tangent vector space at p of M) and the symbol $\otimes$ is the tensor product.

An n-dimensional Lorentzian manifold (M, g) is said to be Lorentzian special para Sasakian (LSP-Sasakian) if M admits a timelike unit vector field ξ with its associated 1-form η satisfies

$$\Omega(X,Y) = (\nabla_X \eta)(Y) = \epsilon\{g(X,Y) + \eta(X)\eta(Y)\}, \quad \epsilon^2 = 1. \tag{2.6}$$

Of course, an LSP-Sasakian manifold is LP-Sasakian.

On the other hand, the eigenvalues of ϕ are -1, 0 and 1. And the multiplicity of 0 is 1 by (2.2). Let K and l be the multiplicities of -1 and 1 respectively. Then $tr\phi = l - K$. So, if $(tr\phi)^2 = (n-1)^2$, then $l = 0$ or $K = 0$. In this case, we call our structure is a trivial LP-Sasakian structure.

In an n-dimensional LP-Sasakian manifold with structure $\sum$, we know the following relations

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \quad S(X,\xi) = (n-1)\eta(X), \tag{2.7}$$

$$R(\xi,Y)X = g(Y,X)\xi - \eta(X)Y, \quad R(Y,X)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$\begin{aligned}\phi(R(X,\phi Y)Z) &= R(X,Y)Z + 2\{\eta(Y)X - \eta(X)Y\}\eta(Z) \\ &\quad + 2\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi \\ &\quad + \Omega(X,Z)\phi Y - \Omega(Y,Z)\phi X \\ &\quad + g(Y,Z)X - g(X,Z)Y,\end{aligned} \tag{2.9}$$

where R and S are respectively the curvature tensor and the Ricci tensor with respect to g.

An n-dimensional LP-Sasakian manifold is said to be η-Einstein if the Ricci tensor S satisfies

$$S = ag + b\eta \otimes \eta, \tag{2.10}$$

where a and b are smooth functions on the manifold. In[9] an $\eta-$einstein LP-Sasakian manifold the Ricci tensor S is of the form

$$S(X,Y) = [\frac{r}{n-1} - 1]g(X,Y) + [\frac{r}{n-1} - n]\eta(X)\eta(Y) \tag{2.11}$$

and the Ricci operator is of the form

$$QX = [\frac{r}{n-1} - 1]X + [\frac{r}{n-1} - n]\eta(X)\xi. \tag{2.12}$$

3. Ricci Solitons

Suppose an LP-Sasakian manifold admits a Ricci soliton defined by (1.1). It is well known that $\nabla g = 0$. Since λ in the Ricci soliton equation (1.1) is a constant, so $\nabla \lambda g = 0$. Thus $\pounds_V g + 2S$ is parallel. In [8] the author prove that if an LP-Sasakian manifold admits a symmetric parallel $(0,2)$ tensor, then the tensor is a constant multiple of the metric tensor. Hence $\pounds_V g + 2S$ is a constant multiple of metric tensors g, i.e., $\pounds_V g + 2S = ag$, where a is constant. Hence $\pounds_V g + 2S + 2\lambda g$ reduces to $(a + 2\lambda)g$. Using (1.1) we get $\lambda = -a/2$. So we have the following:

Proposition 3.1. *In an LP-Sasakian manifold the Ricci soliton (g, λ, V) is shrinking or expanding according as a is positive or negative.*

In particular, let V be point-wise collinear with ξ i.e. $V = b\xi$, where b is a function on the $LP-$ manifold. Then

$$(\pounds_V g + 2S + 2\lambda g)(X,Y) = 0, \tag{3.1}$$

which implies that

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X,Y) + 2\lambda g(X,Y) = 0,$$

or,

$$\begin{aligned} bg((\nabla_X \xi, Y) &+ (Xb)\eta(Y) + bg(\nabla_Y \xi, X) \\ &+ (Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \end{aligned}$$

Using (2.2), we obtain

$$\begin{aligned} 2bg(\phi X, Y) &+ (Xb)\eta(Y) + (Yb)\eta(X) \\ &+ 2S(X,Y) + 2\lambda g(X,Y) = 0. \end{aligned} \tag{3.2}$$

Putting $Y = \xi$ in (3.2) we get

$$(Xb) + \eta(X)\xi b + 2(n-1)\eta(X) + 2\lambda\eta(X)$$

or,

$$(Xb) = (1 - \lambda - n)\eta(X). \tag{3.3}$$

Since $d\eta = 0$ in $LP-$Sasakian manifold, from (3.3) we obtain

$$Xb = 0$$

provided $\lambda = 1 - n$.

Theorem 3.1. *If in an $LP-$Sasakian manifold the metric g is a Ricci soliton and V is point-wise collinear with ξ, then V is a constant multiple of ξ provided $\lambda = 1 - n$.*

In particular, let $V = \xi$. Then

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0,$$

implies that

$$2g(\phi X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{3.4}$$

Substituting $X = \xi$ we get $\lambda = -(n - 1)$. Thus the Ricci soliton is shrinking.

If, in particular, the manifold is an $LSP-$Sasakian manifold, then

$$(\nabla_X \eta)(Y) = g(\phi X, Y) = \epsilon\{g(X, Y) + \eta(X)\eta(Y)\} \quad \epsilon^2 = 1. \tag{3.5}$$

Hence using (3.3),(3.5) equation (3.2)takes the form

$$S(X, Y) = (b\epsilon - \lambda)g(X, Y) + (\lambda + n - 1 - b\epsilon)\eta(X)\eta(Y), \tag{3.6}$$

that is, an $\eta-$Einstein manifold.

So we have the following:

Theorem 3.2. *If in an LSP-Sasakian manifold the metric g is a Ricci soliton and V is point-wise colinear with ξ, then the manifold is an $\eta-$Einstein manifold.*

Conversely, let M be an LSP-Sasakian $\eta-$Einstein manifold of the form

$$S(X, Y) = \delta g(X, Y) + \gamma\eta(X)\eta(Y), \tag{3.7}$$

where γ and δ are constants.

Now taking $V = \xi$ in (3.1)and using (3.7) we obtain

$$\begin{aligned}(\pounds_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) &= 2(\epsilon + \lambda + \delta)g(X, Y) \\ &\quad -2(\epsilon + \gamma)\eta(X)\eta(Y).\end{aligned} \tag{3.8}$$

From equation (3.8) it follows that M admits a Ricci soliton (g, ξ, λ) if $\epsilon + \lambda + \delta = 0$ and $\epsilon + \gamma = 0$. these implies that $\gamma = -\epsilon = constant$.

Also from (3.7) we have $\delta = n - 1 + \gamma = constant$. Therefore $\lambda = -\delta - \epsilon$ which is a constant. So we have the following:

Theorem 3.3. *If an LSP-Sasakian manifold is $\eta-$Einstein of the form $S = \delta g + \gamma\eta\otimes\eta$ with $\gamma, \delta = constant$, then the manifold admits a Ricci soliton $(g, \xi, -(\delta+\epsilon))$.*

Again on contraction we get from (3.6)

$$r = (n-1)(b\epsilon - \lambda - 1) = constant.$$

Therefore the scalar curvature is constant.

In [20] Sharma proved that a compact Ricci soliton of constant scalar curvature is Einstein. Hence from Theorem 3.1. we state the following:

Corollary 3.1. *If an LSP-Sasakian manifold admits a compact Ricci soliton, then the manifold is Einstein.*

4. Gradient Ricci Solitons

If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and (1.1) assume the form

$$\nabla\nabla f = S + \lambda g. \tag{4.1}$$

This reduces to

$$\nabla_Y Df = QY + \lambda Y, \tag{4.2}$$

where D denotes the gradient operator of g. From (4.2) it is clear that

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \tag{4.3}$$

This implies

$$g(R(\xi,Y)Df, \xi) = g((\nabla_\xi Q)Y, \xi) - g((\nabla_Y Q)\xi, \xi). \tag{4.4}$$

Now using (2.12) and (2.2) we have

$$(\nabla_Y Q)(X) = \frac{1}{n-1}(X + \eta(X)\xi) + [\frac{r}{n-1} - n](g(Y,\phi X)\xi + \eta(X)\phi Y). \tag{4.5}$$

Then clearly

$$g((\nabla_X Q)\xi - (\nabla_\xi Q)X, \xi) = 0. \tag{4.6}$$

Then we have from (4.4)

$$g(R(\xi,X)Df, \xi) = 0. \tag{4.7}$$

From (2.8) and (4.7)we get

$$g(R(\xi,Y)Df,\xi) = -g(Y,Df) - \eta(Df)\eta(Y) = 0.$$

Hence

$$Df = -\eta(Df)\xi = -g(Df,\xi)\xi = -(\xi f)\xi. \tag{4.8}$$

Using (4.8) in (4.2) we get

$$\begin{aligned} S(X,Y) + \lambda g(X,Y) &= -g(\nabla_Y((\xi f)\xi), X) \\ &= -Y(\xi f)\eta(X) - \xi f g(X,\phi Y). \end{aligned} \tag{4.9}$$

Putting $X = \xi$ in (4.9) and using (3.3) we get

$$Y(\xi f) = (n + \lambda - 1)\eta(Y). \tag{4.10}$$

From this it is clear that if $\lambda = 1 - n$, then $\xi f = constant$. Therefore from (4.8) we have

$$Df = -(\xi f)\xi = c\xi.$$

In particular taking a frame field $\xi f = 0$ we get from (4.8) $f = constant$.

Therefore equation (4.1) reduces to

$$S(X,Y) = (n-1)g(X,Y),$$

that is, M is an Einstein manifold.

Theorem 4.1. *If an $\eta-$Einstein LP-Sasakian manifold admits a gradient Ricci soliton then the manifold reduces to an Einstein manifold provided $\lambda = 1 - n$ within the frame field $\xi f = 0$.*

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