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GROWTH OF MEROMORPHIC FUNCTIONS DEPENDING ON (p, q)-th RELATIVE ORDER

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Abstract. In this paper, for any two positive integers p and q, we wish to introduce an alternative definition of relative (p, q)-th order of a meromorphic function with respect to another entire function which improves the earlier definition of relative (p, q)-th order of meromorphic function introduced by Banerjee and Jana [5]. Also in this paper we discuss some growth rates of composite entire and meromorphic functions on the basis of the improved definition of relative (p, q)-th order of meromorphic function.

Keywords: Meromorphic function, entire function, index-pair, (p, q)-th order, relative (p, q)-th order, composition, growth

1. Introduction

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f is defined on $|z| = r$ as follows:

$$M_f(r) = \max_{|z|=r} |f(z)| \ .$$

When f is meromorphic, $M_f(r)$ cannot be defined as f is not analytic throughout the complex plane. In this situation, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f, playing the same role as $M_f(r)$.

The integrated counting function $N_f(r, a)$ $\left(\overline{N}_f(r, a)\right)$ of a-points (distinct a-points) of f is defined as

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} dt + n_f(0, a) \log r$$

$$\left(\overline{N}_f(r, a) = \int_0^r \frac{\overline{n}_f(t, a) - \overline{n}_f(r, a)}{t} dt + \overline{n}_f(0, a) \log r\right) \ ,$$

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where we denote by $n_f(t,a)$ $(\overline{n}_f(t,a))$ the number of a-points (distinct a-points) of f in $|z| \leq t$ and an ∞ -point is a pole of f . In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\overline{N}_f(r)$ respectively. The function $N_f(r,a)$ is called the enumerative function.

On the other hand, the function $m_f(r) \equiv m_f(r,\infty)$ known as the proximity function is defined as

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta,$$

where $\log^+ x = \max(\log x, 0)$ for all $x \geqslant 0$ and an ∞ -point is a pole of f .

Analogously, $m_{\frac{1}{f-a}}(r) \equiv m_f(r,a)$ is defined when a is not an ∞-point of f.

Thus the Nevanlinna's characteristic function $T_f(r)$ corresponding to f is defined as

$$T_f(r) = N_f(r) + m_f(r) \ .$$

When f is entire, $T_f(r)$ coincides with $m_f(r)$ as $N_f(r) = 0$.

Further for given any two meromorphic functions f and g the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their Nevanlinna's Characteristic function.

The *order* of a meromorphic function f which is generally used in computational purpose is defined in terms of the growth of f respect to the exponential function as

$$\rho_f = \limsup_{r\to\infty}\frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty}\frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty}\frac{\log T_f(r)}{\log(r) + O(1)} \ .$$

Lahiri and Banerjee [4] introduced the relative order of a meromorphic function with respect to an entire function to avoid comparing growth just with $\exp z$. Extending the notion of relative order as cited in the reference, in this paper we extend some results related to the growth rates of entire and meromorphic functions on the basis of avoiding some restriction, introducing a new type of relative order (p,q), and revisiting ideas developed by a number of authors including Banerjee and Jana [5].

2. Notation and Preliminary Remarks

We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [2] and [9]. Hence we do not explain those in details. Now we state the following notation which will be needed in the sequel:

$$\begin{aligned}
\log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1,2,3,\cdots ; \\
\log^{[0]} x &= x
\end{aligned}$$

and

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots ;$$
$$\exp^{[0]} x = x.$$

Taking this into account the definitions of order and lower order of entirer and meromorphic functions are as follows:

Definition 2.1. The order ρ_f and the lower order λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} .$$

If f is a meromorphic function, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} .$$

Definition 2.2. [7] Let l be an integer ≥ 2. The generalized order $\rho_f^{[l]}$ and generalized lower order $\lambda_f^{[l]}$ of an entire function f are defined as

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r} \text{ and } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r} .$$

If f is meromorphic, one can easily verify that

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} T_f(r)}{\log r} \text{ and } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} T_f(r)}{\log r} .$$

When $l = 2$, Definition 2.2 coincides with Definition 2.1.

Juneja, Kapoor and Bajpai [3] defined the (p,q)-th order and (p,q)-th lower order of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

where p, q are any two positive integers with $p \geq q$.
When f is meromorphic one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r},$$

where p, q are any two positive integers with $p \geq q$.

If $p = l$ and $q = 1$ then we write $\rho_f(l,1) = \rho_f^{[l]}$ and $\lambda_f(l,1) = \lambda_f^{[l]}$.

Also for $p = 2$ and $q = 1$ we respectively denote $\rho_f(2,1)$ and $\lambda_f(2,1)$ by ρ_f and λ_f.

In this connection, we just recall the following definition :

Definition 2.3. [3] An entire function f is said to have index-pair (p,q), $p \geq q \geq 1$ if $b < \rho_f(p,q) < \infty$ and $\rho_f(p-1,q-1)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ if $p > q$. Moreover if $0 < \rho_f(p,q) < \infty$, then

$$\rho_f(p-n,q) = \infty \text{ for } n < p, \ \rho_f(p,q-n) = 0 \text{ for } n < q \text{ and}$$

$$\rho_f(p+n,q+n) = 1 \text{ for } n = 1,2,.... .$$

Similarly for $0 < \lambda_f(p,q) < \infty$, one can easily verify that

$$\lambda_f(p-n,q) = \infty \text{ for } n < p, \ \lambda_f(p,q-n) = 0 \text{ for } n < q \text{ and}$$

$$\lambda_f(p+n,q+n) = 1 \text{ for } n = 1,2,.... .$$

An entire function for which (p,q)-th order and (p,q)-th lower order are the same is said to be of regular (p,q)-growth. Functions which are not of regular (p,q)-growth are said to be of irregular (p,q)-growth.

Analogously, one can easily verify that the Definition 2.3 of index-pair can also be applicable for a meromorphic function f.

Given a non-constant entire function f defined in the open complex plane $\mathbb{C}$ its maxmimum modulus function and Nevanlinna's characteristic function are strictly increasing and continuous. Hence there exists its inverse functions $M_f^{-1}(r)$: $(|f(0)|,\infty) \to (0,\infty)$ with $\lim_{s\to\infty} M_f^{-1}(s) = \infty$ and $T_f^{-1}(r) : (|f(0)|,\infty) \to (0,\infty)$ with $\lim_{s\to\infty} T_f^{-1}(s) = \infty$.

In this connection, Bernal [1] introduced the definition of relative order of an entire function f with respect to another entire function g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0.\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r} .\end{aligned}$$

The definition coincides with the classical one [8] if $g(z) = \exp z$.

Similarly one can define the relative lower order of an entire function f with respect to another entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r} .$$

Extending this notion, Lahiri and Banerjee **[4]** introduced the definition of relative order of a meromorphic function with respect to an entire function in the following way :

Definition 2.4. **[4]** Let f be any meromorphic function and g be any entire function. The relative order of f with respect to g is defined as

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} . \end{aligned}$$

It is known {cf. [4] } that if $g(z) = \exp z$ then Definition 4 coincides with the classical definition of order of a meromorphic function f .

In the case of relative order, it therefore seems reasonable to define suitably the relative (p, q) th order of meromorphic functions. Banerjee and Jana [5] also introduced such definition in the following manner:

Definition 2.5. [5] Let p and q be any two positive integers with $p > q$. The relative (p, q) th order of a non-constant meromorphic function f with respect to another non-constant entire function g is defined by

$$\begin{aligned} \rho_g^{(p,q)}(f) &= \inf\left\{ \begin{array}{c} \mu > 0 : T_f(r) < T_g\left(\exp^{[p-1]}\left(\mu \log^{[q]} r\right)\right) \\ \text{for all } r > r_0(\mu) > 0 \end{array} \right\} \\ &= \limsup_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\log^{[q]} r}. \end{aligned}$$

If $p = 2, q = 1$ then $\rho_g^{(p,q)}(f) = \rho_g(f)$. If $g = \exp z$ then $\rho_g^{(p,q)}(f) = \rho_f(p, q)$.

Now we intend to give an alternative definition of relative (p, q) th order of a meromorphic function with respect to an entire function in the light of index-pair which is as follows:

Definition 2.6. Let f be any meromorphic function and g be any entire function with index-pairs (m_1, q) and (m_2, p) respectively where $m_1 = m_2 = m$ and p, q, m are all positive integers such that $m \geq p$ and $m \geq q$. Then the relative (p, q) th order of f with respect to g is defined as

$$\rho_g^{(p,q)}(f) = \inf\left\{ \begin{array}{c} \mu > 0 : T_f(r) < T_g\left[\exp^{[p]}\left\{\log^{[m_2-1]} \exp^{[m_1-1]}\left(\mu \log^{[q]} r\right)\right\}\right] \\ \text{for all } r > r_0(\mu) > 0 \end{array} \right\}$$

$$= \inf\left\{ \begin{array}{c} \mu > 0 : T_f(r) < T_g\left(\exp^{[p]}\left(\mu \log^{[q]} r\right)\right) \\ \text{for all } r > r_0(\mu) > 0 \end{array} \right\}$$
$$= \limsup_{r\to\infty} \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r}.$$

Similarly, one can define the relative (p,q) th lower order of a meromorphic function f with respect to an entire function g denoted by $\lambda_g^{(p,q)}(f)$ where p and q are any two positive integers in the following way:

$$\lambda_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r}.$$

In fact, Definition 2.6 improves Definition 2.5 ignoring the restriction $p \geq q$.

If the meromorphic functions f and entire function g have the same index-pair $(p,1)$ where p is any positive integer, we may get the definition of relative order of meromorphic function introduced by Lahiri and Banerjee [4] and if $g = \exp^{[m-1]} z$, then $\rho_g(f) = \rho_f^{[m]}$ and $\rho_g^{(p,q)}(f) = \rho_f(m,q)$. Also Definition 2.6 coincides with the classical one if f is a meromorphic function with index-pair $(2,1)$ and $g = \exp z$.

In this paper we wish to prove some results related to the growth rates of composite entire and meromorphic functions on the basis of relative (p,q) th order and relative (p,q) th lower order of a meromorphic function with respect to an entire function for any two positive integers p and q. In this connection we would also like to mention that the improvement of the results of Banerjee and Jana [5] which are solely based on the assumption $p \geq q$ can be carried out in view of Definition 2.6 ignoring the restriction $p \geq q$.

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let f a meromorphic function and g be an entire function with index-pairs (m,q) and (m,p) respectively where p,q,m are all positive integers such that $m \geq p$ and $m \geq q$. Then*

$$\frac{\lambda_f(m,q)}{\rho_g(m,p)} \leq \lambda_g^{(p,q)}(f) \leq \min\left\{\frac{\lambda_f(m,q)}{\lambda_g(m,p)}, \frac{\rho_f(m,q)}{\rho_g(m,p)}\right\}$$
$$\leq \max\left\{\frac{\lambda_f(m,q)}{\lambda_g(m,p)}, \frac{\rho_f(m,q)}{\rho_g(m,p)}\right\} \leq \rho_g^{(p,q)}(f) \leq \frac{\rho_f(m,q)}{\lambda_g(m,p)}.$$

Proof. From the definitions of $\rho_f(m,q)$ and $\lambda_f(m,q)$, we have for all sufficiently large values of r that

$$T_f(r) \leq \exp^{[m-1]}\left\{(\rho_f(m,q)+\varepsilon)\log^{[q]} r\right\}, \tag{3.1}$$

$$T_f(r) \geq \exp^{[m-1]}\left\{(\lambda_f(m,q)-\varepsilon)\log^{[q]} r\right\} \tag{3.2}$$

and also for a sequence of values of r tending to infinity we get that

$$T_f(r) \geq \exp^{[m-1]}\left\{(\rho_f(m,q)-\varepsilon)\log^{[q]} r\right\}, \tag{3.3}$$

$$T_f(r) \leq \exp^{[m-1]}\left\{(\lambda_f(m,q)+\varepsilon)\log^{[q]} r\right\}. \tag{3.4}$$

Similarly from the definitions of $\rho_g(m,p)$ and $\lambda_g(m,q)$, it follows for all sufficiently large values of r that

$$\begin{aligned} T_g(r) &\leq \exp^{[m-1]}\left\{(\rho_g(m,p)+\varepsilon)\log^{[p]} r\right\} \\ \textit{i.e., } r &\leq T_g^{-1}\left[\exp^{[m-1]}\left\{(\rho_g(m,p)+\varepsilon)\log^{[p]} r\right\}\right] \\ \textit{i.e., } T_g^{-1}(r) &\geq \exp^{[p]}\left[\frac{\log^{[m-1]} r}{(\rho_g(m,p)+\varepsilon)}\right], \end{aligned} \tag{3.5}$$

$$\begin{aligned} T_g(r) &\geq \exp^{[m-1]}\left\{(\lambda_g(m,p)-\varepsilon)\log^{[p]} r\right\} \\ \textit{i.e., } r &\geq T_g^{-1}\left[\exp^{[m-1]}\left\{(\lambda_g(m,p)-\varepsilon)\log^{[p]} r\right\}\right] \\ \textit{i.e., } T_g^{-1}(r) &\leq \exp^{[p]}\left[\frac{\log^{[m-1]} r}{(\lambda_g(m,p)-\varepsilon)}\right] \end{aligned} \tag{3.6}$$

and for a sequence of values of r tending to infinity we obtain that

$$\begin{aligned} T_g(r) &\geq \exp^{[m-1]}\left\{(\rho_g(m,p)-\varepsilon)\log^{[p]} r\right\} \\ \textit{i.e., } r &\geq T_g^{-1}\left[\exp^{[m-1]}\left\{(\rho_g(m,p)-\varepsilon)\log^{[p]} r\right\}\right] \\ \textit{i.e., } T_g^{-1}(r) &\leq \exp^{[p]}\left[\frac{\log^{[m-1]} r}{(\rho_g(m,p)-\varepsilon)}\right], \end{aligned} \tag{3.7}$$

$$\begin{aligned} T_g(r) &\leq \exp^{[m-1]}\left\{(\lambda_g(m,p)+\varepsilon)\log^{[p]} r\right\} \\ \textit{i.e., } r &\leq T_g^{-1}\left[\exp^{[m-1]}\left\{(\lambda_g(m,p)+\varepsilon)\log^{[p]} r\right\}\right] \\ \textit{i.e., } T_g^{-1}(r) &\geq \exp^{[p]}\left[\frac{\log^{[m-1]} r}{(\lambda_g(m,p)+\varepsilon)}\right]. \end{aligned} \tag{3.8}$$

Now from (3.3) and in view of (3.5), we get for a sequence of values of r tending to infinity that

$$\log^{[p]} T_g^{-1} T_f(r) \geq \log^{[p]} T_g^{-1}\left[\exp^{[m-1]}\left\{(\rho_f(m,q)-\varepsilon)\log^{[q]} r\right\}\right]$$

$$i.e.,\ \log^{[p]} T_g^{-1} T_f(r) \geq \log^{[p]} \exp^{[p]}\left[\frac{\log^{[m-1]} \exp^{[m-1]}\left\{(\rho_f(m,q)-\varepsilon)\log^{[q]} r\right\}}{(\rho_g(m,p)+\varepsilon)}\right]$$

$$\begin{aligned} i.e.,\ \log^{[p]} T_g^{-1} T_f(r) &\geq \frac{(\rho_f(m,q)-\varepsilon)}{(\rho_g(m,p)+\varepsilon)}\log^{[q]} r \\ i.e.,\ \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r} &\geq \frac{(\rho_f(m,q)-\varepsilon)}{(\rho_g(m,p)+\varepsilon)}. \end{aligned}$$

As $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\begin{aligned} \limsup_{r\to\infty}\frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r} &\geq \frac{\rho_f(m,q)}{\rho_g(m,p)} \\ i.e.,\ \rho_g^{(p,q)}(f) &\geq \frac{\rho_f(m,q)}{\rho_g(m,p)}. \end{aligned} \tag{3.9}$$

Analogously from (3.2) and in view of (3.8), it follows for a sequence of values of r tending to infinity that

$$\log^{[p]} T_g^{-1} T_f(r) \geq \log^{[p]} T_g^{-1}\left[\exp^{[m-1]}\left\{(\lambda_f(m,q)-\varepsilon)\log^{[q]} r\right\}\right]$$

$$i.e.,\ \log^{[p]} T_g^{-1} T_f(r) \geq \log^{[p]} \exp^{[p]}\left[\frac{\log^{[m-1]} \exp^{[m-1]}\left\{(\lambda_f(m,q)-\varepsilon)\log^{[q]} r\right\}}{(\lambda_g(m,p)+\varepsilon)}\right]$$

$$\begin{aligned} i.e.,\ \log^{[p]} T_g^{-1} T_f(r) &\geq \frac{(\lambda_f(m,q)-\varepsilon)}{(\lambda_g(m,p)+\varepsilon)}\log^{[q]} r \\ i.e.,\ \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r} &\geq \frac{(\lambda_f(m,q)-\varepsilon)}{(\lambda_g(m,p)+\varepsilon)}. \end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\begin{aligned} \limsup_{r\to\infty}\frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r} &\geq \frac{\lambda_f(m,q)}{\lambda_g(m,p)} \\ i.e.,\ \rho_g^{(p,q)}(f) &\geq \frac{\lambda_f(m,q)}{\lambda_g(m,p)}. \end{aligned} \tag{3.10}$$

Again in view of (3.6) we have from (3.1), for all sufficiently large values of r that

$$\log^{[p]} T_g^{-1} T_f(r) \le \log^{[p]} T_g^{-1}\left[\exp^{[m-1]}\left\{(\rho_f(m,q)+\varepsilon)\log^{[q]} r\right\}\right]$$

$$i.e.,\ \log^{[p]} T_g^{-1} T_f(r) \le \log^{[p]} \exp^{[p]}\left[\frac{\log^{[m-1]}\exp^{[m-1]}\left\{(\rho_f(m,q)+\varepsilon)\log^{[q]} r\right\}}{(\lambda_g(m,p)-\varepsilon)}\right]$$

$$\begin{aligned} i.e.,\ \log^{[p]} T_g^{-1} T_f(r) &\le \frac{(\rho_f(m,q)+\varepsilon)}{(\lambda_g(m,p)-\varepsilon)}\log^{[q]} r \\ i.e.,\ \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r} &\le \frac{(\rho_f(m,q)+\varepsilon)}{(\lambda_g(m,p)-\varepsilon)}. \end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\begin{aligned} \limsup_{r\to\infty}\frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r} &\le \frac{\rho_f(m,q)}{\lambda_g(m,p)} \\ i.e.,\ \rho_g^{(p,q)}(f) &\le \frac{\rho_f(m,q)}{\lambda_g(m,p)}. \end{aligned} \tag{3.11}$$

Again from (3.2) and in view of (3.5) we get for all sufficiently large values of r that

$$\log^{[p]} T_g^{-1} T_f(r) \ge \log^{[p]} T_g^{-1}\left[\exp^{[m-1]}\left\{(\lambda_f(m,q)-\varepsilon)\log^{[q]} r\right\}\right]$$

$$i.e.,\ \log^{[p]} T_g^{-1} T_f(r) \ge \log^{[p]} \exp^{[p]}\left[\frac{\log^{[m-1]}\exp^{[m-1]}\left\{(\lambda_f(m,q)-\varepsilon)\log^{[q]} r\right\}}{(\rho_g(m,p)+\varepsilon)}\right]$$

$$\begin{aligned} i.e.,\ \log^{[p]} T_g^{-1} T_f(r) &\ge \frac{(\lambda_f(m,q)-\varepsilon)}{(\rho_g(m,p)+\varepsilon)}\log^{[q]} r \\ i.e.,\ \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r} &\ge \frac{(\lambda_f(m,q)-\varepsilon)}{(\rho_g(m,p)+\varepsilon)}. \end{aligned}$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\begin{aligned} \liminf_{r\to\infty}\frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r} &\ge \frac{\lambda_f(m,q)}{\rho_g(m,p)} \\ i.e.,\ \lambda_g^{(p,q)}(f) &\ge \frac{\lambda_f(m,q)}{\rho_g(m,p)}. \end{aligned} \tag{3.12}$$

Also in view of (3.7), we get from (3.1) for a sequence of values of r tending to infinity that

$$\log^{[p]} T_g^{-1} T_f(r) \le \log^{[p]} T_g^{-1}\left[\exp^{[m-1]}\left\{(\rho_f(m,q)+\varepsilon)\log^{[q]} r\right\}\right]$$

$$i.e.,\ \log^{[p]} T_g^{-1}T_f(r) \le \log^{[p]} \exp^{[p]} \left[\frac{\log^{[m-1]} \exp^{[m-1]} \left\{ (\rho_f(m,q) + \varepsilon) \log^{[q]} r \right\}}{(\rho_g(m,p) - \varepsilon)} \right]$$

$$i.e.,\ \log^{[p]} T_g^{-1}T_f(r) \le \frac{(\rho_f(m,q) + \varepsilon)}{(\rho_g(m,p) - \varepsilon)} \log^{[q]} r$$

$$i.e.,\ \frac{\log^{[p]} T_g^{-1}T_f(r)}{\log^{[q]} r} \le \frac{(\rho_f(m,q) + \varepsilon)}{(\rho_g(m,p) - \varepsilon)}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, we get from above that

$$\liminf_{r\to\infty} \frac{\log^{[p]} T_g^{-1}T_f(r)}{\log^{[q]} r} \le \frac{\rho_f(m,q)}{\rho_g(m,p)}$$

$$i.e.,\ \lambda_g^{(p,q)}(f) \le \frac{\rho_f(m,q)}{\rho_g(m,p)}. \tag{3.13}$$

Similarly from (3.4) and in view of (3.6), it follows for a sequence of values of r tending to infinity that

$$\log^{[p]} T_g^{-1}T_f(r) \le \log^{[p]} T_g^{-1} \left[\exp^{[m-1]} \left\{ (\lambda_f(m,q) + \varepsilon) \log^{[q]} r \right\} \right]$$

$$i.e.,\ \log^{[p]} T_g^{-1}T_f(r) \le \log^{[p]} \exp^{[p]} \left[\frac{\log^{[m-1]} \exp^{[m-1]} \left\{ (\lambda_f(m,q) + \varepsilon) \log^{[q]} r \right\}}{(\lambda_g(m,p) - \varepsilon)} \right]$$

$$i.e.,\ \log^{[p]} T_g^{-1}T_f(r) \le \frac{(\lambda_f(m,q) + \varepsilon)}{(\lambda_g(m,p) - \varepsilon)} \log^{[q]} r$$

$$i.e.,\ \frac{\log^{[p]} T_g^{-1}T_f(r)}{\log^{[q]} r} \le \frac{(\lambda_f(m,q) + \varepsilon)}{(\lambda_g(m,p) - \varepsilon)}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to\infty} \frac{\log^{[p]} T_g^{-1}T_f(r)}{\log^{[q]} r} \le \frac{\lambda_f(m,q)}{\lambda_g(m,p)}$$

$$i.e.,\ \lambda_g^{(p,q)}(f) \le \frac{\lambda_f(m,q)}{\lambda_g(m,p)}. \tag{3.14}$$

Thus the theorem follows from (3.9), (3.10), (3.11), (3.12), (3.13) and (3.14). □

In view of Theorem 3.1 one can easily verify the following corollaries:

Corollary 3.1. *Let f be a meromorphic function with regular (m,q) growth and g be an entire function having index-pair (m,q) where p,q,m are all positive integers such that $m \geq p$ and $m \geq q$. Then*

$$\lambda_g^{(p,q)}(f) = \frac{\rho_f(m,q)}{\rho_g(m,p)} \quad \text{and} \quad \rho_g^{(p,q)}(f) = \frac{\rho_f(m,q)}{\lambda_g(m,p)} \ .$$

Corollary 3.2. *Let f be a meromorphic function with index-pair (m,q) and g be an entire function of regular (m,p)-growth where p,q,m are all positive integers such that $m \geq p$ and $m \geq q$. Then*

$$\lambda_g^{(p,q)}(f) = \frac{\lambda_f(m,q)}{\rho_g(m,p)} \quad \text{and} \quad \rho_g^{(p,q)}(f) = \frac{\rho_f(m,q)}{\rho_g(m,p)} \ .$$

Corollary 3.3. *Let f be a meromorphic function and g be an entire function with regular (m,q) growth and regular (m,p) growth respectively where p,q,m are all positive integers with $m \geq \max\{p,q\}$. Then*

$$\lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) = \frac{\rho_f(m,q)}{\rho_g(m,p)} \ .$$

Corollary 3.4. *Let f be a meromorphic function with index-pair (m,q) where m,q are positive integers with $m \geq q$. Then for any entire function g,*

$$\begin{aligned} (i) \ \lambda_g^{(p,q)}(f) &= \infty \text{ when } \rho_g(m,p) = 0 \ , \\ (ii) \ \rho_g^{(p,q)}(f) &= \infty \text{ when } \lambda_g(m,p) = 0 \ , \\ (iii) \ \lambda_g^{(p,q)}(f) &= 0 \text{ when } \rho_g(m,p) = \infty \end{aligned}$$

and

$$(iv) \ \rho_g^{(p,q)}(f) = \infty \text{ when } \lambda_g(m,p) = \infty,$$

where p is any positive integer with $m \geq p$.

Corollary 3.5. *Let g be an entire function with index-pair (m,p) where m,p are positive integers with $m \geq p$. Then for any meromorphic function f,*

$$\begin{aligned} (i) \ \rho_g^{(p,q)}(f) &= 0 \text{ when } \rho_f(m,q) = 0 \ , \\ (ii) \ \lambda_g^{(p,q)}(f) &= 0 \text{ when } \lambda_f(m,q) = 0 \ , \\ (iii) \ \rho_g^{(p,q)}(f) &= \infty \text{ when } \rho_f(m,q) = \infty \end{aligned}$$

and

$$(iv) \ \lambda_g^{(p,q)}(f) = \infty \text{ when } \lambda_f(m,q) = \infty \ ,$$

where q is any positive integer such that $m \geq q$.

Now a question may arise about the index-pair and existence of the relative order of f with respect to g when f and g are any two entire functions with index-pairs (p,q) and (m,n) respectively where p,q,m,n are all positive integer such that $p\geq q$ and $m\geq n$. The next theorem may provide this answer.

Theorem 3.2. *Let f be a meromorphic function and g be an entire function with index-pairs (m,q) and (n,p) respectively where p,q,m,n are all positive integers such that $m\geq q$ and $n\geq p$. Then*

$$(i)\ \limsup_{r\to\infty}\frac{\log^{[p+m-n]}T_g^{-1}T_f(r)}{\log^{[q]}r}=\limsup_{r\to\infty}\frac{\log^{[m-1]}T_f(r)}{\log^{[q]}r}\ \textit{and}$$
$$\liminf_{r\to\infty}\frac{\log^{[p+m-n]}T_g^{-1}T_f(r)}{\log^{[q]}r}=\liminf_{r\to\infty}\frac{\log^{[m-1]}T_f(r)}{\log^{[q]}r}\ \textit{for } m>n$$

and

$$(ii)\ \limsup_{r\to\infty}\frac{\log^{[p]}T_g^{-1}T_f(r)}{\log^{[q+n-m]}r}=\limsup_{r\to\infty}\frac{\log^{[p]}r}{\log^{[n-1]}T_g(r)}\ \textit{and}$$
$$\liminf_{r\to\infty}\frac{\log^{[p]}T_g^{-1}T_f(r)}{\log^{[q+n-m]}r}=\liminf_{r\to\infty}\frac{\log^{[p]}r}{\log^{[n-1]}T_g(r)}\ \textit{for } m<n.$$

Proof. In view of Definition 2.3, we obtain that

$$\rho_g(m,p+m-n)=1\ \text{ for } m>n \tag{3.15}$$

and

$$\rho_f(n,q+n-m)=1\ \text{ for } m<n. \tag{3.16}$$

Thus the first part of the theorem follows from (3.15) and in view of Theorem 3.1. Now, in the line of Theorem 3.1 and using (3.16), one may easily prove the second part of Theorem 3.2. □

Theorem 3.3. *Let f be any meromorphic function and g, h be any two entire functions such that $\rho_h^{(p,q)}(f)<\infty$ and $\lambda_h^{(p,q)}(f\circ g)=\infty$ where p and q are any positive integers. Then for every $\mu\,(>0)$,*

$$\lim_{r\to\infty}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\log^{[p]}T_h^{-1}T_f(r^{\mu})}=\infty\ .$$

Proof. If possible, let there exist a constant β such that for a sequence of values of r tending to infinity that

$$\log^{[p]}T_h^{-1}T_{f\circ g}(r)\leq\beta.\log^{[p]}T_h^{-1}T_f(r^{\mu})\ . \tag{3.17}$$

Again from the definition of $\rho_h^{(p,q)}(f)$, it follows for all sufficiently large values of r that

$$\log^{[p]} T_h^{-1} T_f(r^\mu) \le \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]}(r^\mu) \ . \tag{3.18}$$

If $q = 1$, then from (3.18) we get for all sufficiently large values of r that

$$\log^{[p]} T_h^{-1} T_f(r^\mu) \le \left(\rho_h^{(p)}(f) + \varepsilon\right) \mu \log r \ . \tag{3.19}$$

Also for $q > 1$, we obtain from (3.18) for all sufficiently large values of r that

$$\log^{[p]} T_h^{-1} T_f(r^\mu) \le \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} r + O(1) \ . \tag{3.20}$$

Now if $q = 1$, from (3.17) and (3.19), we have for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[p]} T_h^{-1} T_{f\circ g}(r) &\le \beta.\left(\rho_h^{(p)}(f) + \varepsilon\right) \mu. \log r \\ i.e.,\ \lambda_h^{(p)}(f\circ g) &\le \beta.\mu\left(\rho_h^{(p)}(f) + \varepsilon\right), \end{aligned}$$

which contradicts the condition $\lambda_h^{(p)}(f\circ g) = \infty$.
Again when $q > 1$, combining (3.17) and (3.20) we obtain for a sequence of values of r tending to infinity,

$$\begin{aligned} \log^{[p]} T_h^{-1} T_{f\circ g}(r) &\le \beta.\left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} r + O(1) \\ i.e.,\ \lambda_h^{(p,q)}(f\circ g) &\le \beta.\left(\rho_h^{(p,q)}(f) + \varepsilon\right), \end{aligned}$$

which also contradicts the condition $\lambda_h^{(p,q)}(f\circ g) = \infty$.
So for any positive integer q and for all sufficiently large values of r we get that

$$\log^{[p]} T_h^{-1} T_{f\circ g}(r) \ge \beta. \log^{[p]} T_h^{-1} T_f(r^\mu),$$

from which the theorem follows. □

Remark 3.1. Theorem 3.3 is also valid with "limit superior" instead of "limit" if $\lambda_h^{(p,q)}(f\circ g) = \infty$ is replaced by $\rho_h^{(p,q)}(f\circ g) = \infty$ and the other conditions remain the same.

Corollary 3.6. *Under the assumptions of Theorem 3.3 and Remark 3.1,*

$$\lim_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p-1]} T_h^{-1} T_f(r^\mu)} = \infty \ \textit{and} \ \limsup_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p-1]} T_h^{-1} T_f(r^\mu)} = \infty$$

respectively.

Proof. By Theorem 3.3 we obtain for all sufficiently large values of r and for $K > 1$,

$$\begin{aligned} \log^{[p]} T_h^{-1} T_{f \circ g}(r) &\geq K \log^{[p]} T_h^{-1} T_f(r^{\mu}) \\ i.e., \ \log^{[p-1]} T_h^{-1} T_{f \circ g}(r) &\geq \left\{ \log^{[p-1]} T_h^{-1} T_f(r^{\mu}) \right\}^K , \end{aligned}$$

from which the first part of the corollary follows.
Similarly, using Remark 3.1, we obtain the second part of the corollary. □

Analogously one may also state the following theorem and corollaries without proofs as they may be carried out in the line of Remark 3.1, Theorem 3.3 and Corollary 3.6, respectively.

Theorem 3.4. *Let f be any meromorphic function and g, h be any two entire functions such that $\rho_h^{(p,q)}(g) < \infty$ and $\rho_h^{(p,q)}(f \circ g) = \infty$ where p and q are any two positive integers. Then for every $\mu\,(> 0)$,*

$$\limsup_{r \to \infty} \frac{\log^{[p]} T_h^{-1} T_{f \circ g}(r)}{\log^{[p]} T_h^{-1} T_g(r^{\mu})} = \infty \ .$$

Corollary 3.7. *Theorem 3.4 is also valid with "limit " instead of "limit superior" if $\rho_h(f \circ g) = \infty$ is replaced by $\lambda_h(f \circ g) = \infty$ and the other conditions remain the same.*

Corollary 3.8. *Under the assumptions of Theorem 3.4 and Corollary 3.7,*

$$\limsup_{r \to \infty} \frac{\log^{[p-1]} T_h^{-1} T_{f \circ g}(r)}{\log^{[p-1]} T_h^{-1} T_g(r^{\mu})} = \infty \quad \text{and} \quad \lim_{r \to \infty} \frac{\log^{[p-1]} T_h^{-1} T_{f \circ g}(r)}{\log^{[p-1]} T_h^{-1} T_g(r^{\mu})} = \infty$$

respectively holds.

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