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ON LACUNARY STATISTICAL BOUNDEDNESS OF ORDER α

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Abstract. The aim of this paper is to introduce and examine the concept of lacunary statistical boundedness of order α and give the relations between statistical boundedness and lacunary statistical boundedness of order α.

Keywords: Density, statistical convergence, statistical boundedness, lacunary sequence

1. Introduction and preliminaries

Throughout this paper, the w denotes the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences $x = (x_k)$, respectively, with the usual norm $\|x\|_\infty = \sup |x_k|$, where $k \in \mathbb{N}$, the set of natural numbers.

The concept of statistical convergence was introduced by Fast [11] and Steinhaus [32], independently, in the year 1951. Actually, the idea of statistical convergence was first presented under the name "almost convergence" by Zygmund in the first edition Zygmund [33] of the celebrated monograph [34]. Some basic properties of this notion has been studied by Schoenberg [29] and Šalát [31]. Over the years and under different names statistical convergence was discussed in the theory of Banach spaces, Trigonometric series, Turnpike theory, Number theory, Ergodic theory, Measure theory and Fourier analysis. It was also further investigated from the sequence space point of view and linked with summability theory by Bhardwaj et al. ([1,2]), Connor [7], Et [8], Fridy et al. [13–15], Güngör and Et [17], Işık [19], Mohiuddine et al. ([4,9,18,20]), Mursaleen et al. [25,27], Rath and Tripathy [28], and many others.

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$. The *density* of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by

$$\delta(\mathbb{E}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathbb{E}}(k),$$

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provided that the limit exists. A sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$,

$$\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0.$$

The concept of statistical boundedness was given by Fridy and Orhan as follows: The sequence $x = (x_k)$ of reals is said to be *statistically bounded* if there is a number B such that $\delta\left(\{k : |x_k| > B\}\right) = 0$. We shall write $S(b)$ for the set of all statistically bounded sequences.

It is well known that every bounded sequence is statistically bounded, but the converse is not true.

The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan [16] and after then statistical convergence of order α was studied by Çolak ([5, 6]).

By a lacunary sequence, we mean an increasing integer sequence $\theta = (k_r)$ such that $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. Throughout this paper, the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r. In recent years, lacunary sequences have been studied in ([10, 12, 14, 22–24, 26, 30]).

The concept of lacunary statistical convergence of order α was given by Şengül and Et [30] as follows: Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. The sequence $x = (x_k) \in w$ is said to be *lacunary statistically convergent of order* α, if there is a real number L such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right| = 0.$$

The set of all lacunary statistically convergent sequences of order α will be denoted by S_θ^α.

Let X be a sequence space. Then X is called

(i) *Solid* (or *normal*), if $(\alpha_k x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, whenever $(x_k) \in X$,

(ii) *Monotone* if it contains the canonical preimages of all its stepspaces,

(iii) *Symmetric*, if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where π is a permutation of $\mathbb{N}$,

(iv) *Sequence algebra* if $x.y \in X$, whenever $x, y \in X$.

2. Main results

Here, we give some new definitions and main results of the paper. In Theorem 10 we give the relations between lacunary statistically bounded sequences of order α and lacunary statistically bounded sequences of order β. In Theorem 15 we give the relationship between lacunary statistical boundedness of order α and lacunary statistical boundedness of order β for different θ's.

Definition 1. Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. We define *lacunary α-density* of a subset $\mathbb{E}$ of $\mathbb{N}$ by

$$\delta_\theta^\alpha(\mathbb{E}) = \lim_n \frac{1}{h_r^\alpha} |\{k_{r-1} < k \leq k_r : k \in \mathbb{E}\}|$$

provided the limit exists. We remark that lacunary α-density $\delta_\theta^\alpha(\mathbb{E})$ reduces to the natural density $\delta(\mathbb{E})$ in the special cases $\alpha = 1$ and $\theta = (2^r)$.

If $x = (x_k)$ is a sequence such that x_k satisfies the property $p(k)$ for all k except a set of *lacunary α-density* zero, then we say that x_k satisfies $p(k)$ for " lacunary almost all k according to α" and we abbreviate this by "$a.a.k_r(\alpha)$".

Proposition 2. Let $\theta = (k_r)$ be a lacunary sequence and $\alpha, \beta \in (0,1]$ such that $\alpha \leq \beta$, then $\delta_\theta^\beta(\mathbb{E}) \leq \delta_\theta^\alpha(\mathbb{E})$.

Proof. Proof follows from the following inequality

$$\frac{1}{h_r^\beta} |\{k_{r-1} < k \leq k_r : k \in \mathbb{E}\}| \leq \frac{1}{h_r^\alpha} |\{k_{r-1} < k \leq k_r : k \in \mathbb{E}\}|. \qquad \square$$

Definition 3. Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. The sequence $x = (x_k) \in w$ is said to be *lacunary statistically bounded of order α*, if there is a $M \geq 0$ such that

$$(1) \qquad \lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k| > M\}| = 0, \qquad i.e. \qquad |x_k| \leq M \qquad a.a.k_r(\alpha),$$

where $I_r = (k_{r-1}, k_r]$ and $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, ..., h_r^\alpha, ...)$. The set of all lacunary statistically bounded sequences of order α will be denoted by $S_\theta^\alpha(b)$. For $\theta = (2^r)$, lacunary statistically bounded sequences of order α reduces to statistically bounded sequences of order α; for $\alpha = 1$, lacunary statistically bounded sequences of order α reduces to lacunary statistically bounded sequences which were defined and studied by Bhardwaj et al. [3] and for $\alpha = 1$ and $\theta = (2^r)$ lacunary statistically bounded sequences of order α reduces to statistically bounded sequences. The set of all statistically bounded sequences of order α and the set of all lacunary statistically bounded sequences $S^\alpha(b)$ and $S_\theta(b)$, respectively.

The proof of each of the following results is fairly straightforward, so we choose to state these results *without* proof.

Proposition 4. Every convergent sequence is lacunary statistically bounded, but the converse is not true.

Proposition 5. Every lacunary statistically convergent sequence is lacunary statistically bounded, but the converse is not true.

Proposition 6. Every bounded sequence is lacunary statistically bounded, but the converse is not true.

Proposition 7. Every bounded sequence is lacunary statistically bounded of order α, but the converse is not true.

Proposition 8. Every lacunary statistically convergent sequence of order α is lacunary statistically bounded of order α, but the converse is not true.

Proof. Let $x \in S_\theta^\alpha$ and $\varepsilon > 0$ be given. Then there exists $L \in \mathbb{C}$ such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right| = 0.$$

The result follows from the following inequality

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : |x_k| > |L| + \varepsilon\}\right| \leq \lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right|.$$

To show the strictness of the inclusion, let $\theta = (2^r)$ be given and the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & k = 2n \\ -1, & k \neq 2n \end{cases} \quad k, n \in \mathbb{N}.$$

Then $x \in S_\theta^\alpha(b)$, but $x \notin S_\theta^\alpha$. $\square$

Theorem 9. We have the following:

(i) $S_\theta^\alpha(b)$ is not symmetric,

(ii) $S_\theta^\alpha(b)$ is normal and hence monotone,

(iii) $S_\theta^\alpha(b)$ is a sequence algebra.

Proof. (i) Let $x = (x_k) = (1, 0, 0, 2, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 4, ...) \in S_\theta^\alpha(b)$. Let $y = (y_k)$ be a rearrangement of the sequence (x_k), which is defined as follows:

$$\begin{aligned}(y_k) &= (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, ...) \\ &= (1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, 7, 0, ...).\end{aligned}$$

Clearly for any $M > 0$, $\delta_\theta^\alpha(\{k : |y_k| > M\}) \neq 0$, in the special case $\alpha = 1$ and $\theta = (2^r)$.

(ii) Let $x=(x_k)\in S_\theta^\alpha(b)$ and $y=(y_k)$ be a sequence such that $|y_k|\leq|x_k|$ for all $k\in\mathbb{N}$. Since $x\in S_\theta^\alpha(b)$ there exists a number M such that $\delta_\theta^\alpha(\{k:|x_k|>M\})=0$. Clearly $y\in S_\theta^\alpha(b)$ as $\{k:|y_k|>M\}\subset\{k:|x_k|>M\}$. So $S_\theta^\alpha(b)$ is normal. It is well known that every normal space is monotone, so $S_\theta^\alpha(b)$ is monotone.

(iii) Let $x,y\in S_\theta^\alpha(b)$. Then there exists $K,M>0$ such that

$$\delta_\theta^\alpha(\{k:|x_k|\geq K\})=0 \quad\text{and}\quad \delta_\theta^\alpha(\{k:|y_k|\geq M\})=0.$$

The proof follows from the following inclusion

$$\{k:|x_ky_k|\geq KM\}\subset\{k:|x_k|>K\}\cup\{k:|y_k|>M\}. \qquad \square$$

Theorem 10. If $0<\alpha\leq\beta\leq1$ then $S_\theta^\alpha(b)\subseteq S_\theta^\beta(b)$ and the inclusion is strict.

Proof. The inclusion part of proof is easy. To show the strictness of the inclusion, let θ be given and the sequence $x=(x_k)$ defined by

$$x_k=\begin{cases}\left[\sqrt{h_r}\right], & k=1,2,3,...,\left[\sqrt{h_r}\right]\\ 0, & \text{otherwise}\end{cases}.$$

Then $x\in S_\theta^\beta(b)$ for $\frac{1}{2}<\beta\leq1$ but $x\notin S_\theta^\alpha(b)$ for $0<\alpha\leq\frac{1}{2}$. $\square$

The above Theorem 10 yields the following corollary.

Corollary 11. If a sequence is lacunary statistically bounded of order α, then it is lacunary statistically bounded.

Theorem 12. Let $0<\alpha\leq1$ and $\theta=(k_r)$ be a lacunary sequence. If $\liminf_r q_r>1$, then $S^\alpha(b)\subset S_\theta^\alpha(b)$.

Proof. Suppose that $\liminf_r q_r>1$, then there exists a $\delta>0$ such that $q_r\geq1+\delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r}\geq\frac{\delta}{1+\delta}\Longrightarrow\left(\frac{h_r}{k_r}\right)^\alpha\geq\left(\frac{\delta}{1+\delta}\right)^\alpha\Longrightarrow\frac{1}{k_r^\alpha}\geq\frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}.$$

If $(x_k)\in S^\alpha(b)$, then for $M>0$ and for sufficiently large r, we have

$$\frac{1}{k_r^\alpha}\left|\{k\leq k_r:|x_k|>M\}\right|\geq\frac{1}{k_r^\alpha}\left|\{k\in I_r:|x_k|>M\}\right|\geq\frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}\left|\{k\in I_r:|x_k|>M\}\right|;$$

this proves the proof. $\square$

Theorem 13. Let $0 < \alpha \leq 1$ and $\theta = (k_r)$ be a lacunary sequence. If $\limsup_r q_r < \infty$, then $S_\theta^\alpha(b) \subset S(b)$.

Proof of the above theorem is straightforward and therefore omitted. $\square$

Theorem 14. If

$$\lim_{r\to\infty} \inf \frac{h_r^\alpha}{k_r} > 0 \tag{2}$$

then $S(b) \subset S_\theta^\alpha(b)$.

Proof. For $M > 0$, we have

$$\{k \leq k_r : |x_k| > M\} \supset \{k \in I_r : |x_k| > M\}.$$

Therefore,

$$\frac{1}{k_r} |\{k \leq k_r : |x_k| > M\}| \geq \frac{1}{k_r} \{k \in I_r : |x_k| > M\} = \frac{h_r^\alpha}{k_r} \frac{1}{h_r^\alpha} \{k \in I_r : |x_k| > M\}.$$

Taking limit as $r \to \infty$ and using (2), we get

$$x_k \to S(b) \Longrightarrow x_k \to S_\theta^\alpha(b).$$

This completes the proof of the theorem. $\square$

Let $\theta = (k_r)_{r\in\mathbb{N}}$ and $\theta' = (s_r)_{r\in\mathbb{N}}$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$. Suppose also that the parameters α and β are fixed real numbers such that $0 < \alpha \leq \beta \leq 1$. We shall now give general inclusion relation between lacunary statistically bounded sequences of order α and lacunary statistically bounded sequences of order β for different choices for θ. Our results stated as Theorem 15 below would yield Theorem 10 as its special cases.

Theorem 15. Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$. Let α and β be such that $0 < \alpha \leq \beta \leq 1$. We have the following:
(i) If

$$\lim_{r\to\infty} \inf \frac{h_r^\alpha}{\ell_r^\beta} > 0 \tag{3}$$

then $S_{\theta'}^\beta(b) \subseteq S_\theta^\alpha(b)$,

(ii) If

$$\lim_{r\to\infty} \frac{\ell_r}{h_r^\beta} = 1 \tag{4}$$

then $S_{\theta}^{\alpha}(b) \subseteq S_{\theta'}^{\beta}(b)$.

Proof. (i) Suppose that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let (3) be satisfied. For $M > 0$ we have

$$\{k \in J_r : |x_k| > M\} \supseteq \{k \in I_r : |x_k| > M\}$$

and so

$$\frac{1}{\ell_r^{\beta}} |\{k \in J_r : |x_k| > M\}| \geq \frac{h_r^{\alpha}}{\ell_r^{\beta}} \frac{1}{h_r^{\alpha}} |\{k \in I_r : |x_k| > M\}|$$

for all $r \in \mathbb{N}$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$, $\ell_r = s_r - s_{r-1}$. Now taking the limit as $r \to \infty$ in the last inequality and using (3) we get $S_{\theta'}^{\beta}(b) \subseteq S_{\theta}^{\alpha}(b)$.

(ii) Let $x = (x_k) \in S_{\theta}^{\alpha}(b)$ and (4) be satisfied. Since $I_r \subset J_r$, for $M > 0$ we may write

$$\begin{aligned}
\frac{1}{\ell_r^{\beta}} |\{k \in J_r : |x_k| > M\}| &= \frac{1}{\ell_r^{\beta}} |\{s_{r-1} < k \leq k_{r-1} : |x_k| > M\}| \\
&\quad + \frac{1}{\ell_r^{\beta}} |\{k_r < k \leq s_r : |x_k| > M\}| \\
&\quad + \frac{1}{\ell_r^{\beta}} |\{k_{r-1} < k \leq k_r : |x_k| > M\}| \\
&\leq \frac{k_{r-1} - s_{r-1}}{\ell_r^{\beta}} + \frac{s_r - k_r}{\ell_r^{\beta}} + \frac{1}{\ell_r^{\beta}} |\{k \in I_r : |x_k| > M\}| \\
&= \frac{\ell_r - h_r}{\ell_r^{\beta}} + \frac{1}{\ell_r^{\beta}} |\{k \in I_r : |x_k| > M\}| \\
&\leq \frac{\ell_r - h_r^{\beta}}{h_r^{\beta}} + \frac{1}{h_r^{\beta}} |\{k \in I_r : |x_k| > M\}| \\
&\leq \left(\frac{\ell_r}{h_r^{\beta}} - 1\right) + \frac{1}{h_r^{\alpha}} |\{k \in I_r : |x_k| > M\}|
\end{aligned}$$

for all $r \in \mathbb{N}$. Since $\lim_{r\to\infty} \frac{\ell_r}{h_r^{\beta}} = 1$ by (4) the first term and since $x \in S_{\theta}^{\alpha}(b)$ the second term of right hand side of above inequality tend to 0 as $r \to \infty$. This implies that $S_{\theta}^{\alpha}(b) \subseteq S_{\theta'}^{\alpha}(b)$. $\square$

The following results are derivable easily from the above Theorem 15.

Corollary 16. Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$. If the condition (3) is satisfied, then

(i) $S_{\theta'}^{\alpha}(b) \subseteq S_{\theta}^{\alpha}(b)$ for each $\alpha \in (0, 1]$,

(ii) $S_{\theta'}(b) \subseteq S_{\theta}^{\alpha}(b)$ for each $\alpha \in (0, 1]$,

(iii) $S_{\theta'}(b) \subseteq S_{\theta}(b)$.

Furthermore, if the condition (4) is satisfied, then

(i) $S_{\theta}^{\alpha}(b) \subseteq S_{\theta'}^{\alpha}(b)$ for each $\alpha \in (0,1]$,

(ii) $S_{\theta}^{\alpha}(b) \subseteq S_{\theta'}(b)$ for each $\alpha \in (0,1]$,

(iii) $S_{\theta}(b) \subseteq S_{\theta'}(b)$.

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