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# SEMI GENERALIZED $\varphi$-RECURRENT TRANS-SASAKIAN MANIFOLDS

**Jagannath Chowdhury,* Rajesh Kumar and Jay P. Singh**

**Abstract.** In this paper we studied semi generalized $\varphi$-recurrent and concircular $\varphi$-recurrent Trans-Sasakian manifolds.
**keywords**: Trans-Sasakian manifold, semi generalized $\varphi$-recurrent, $\eta$-Einstein, $\eta$-parallel, constant curvature.

## 1. Introduction

In 1985, Oubina [3] introduced a class of almost contact metric manifolds known as trans-Sasakian manifolds.This class contains $\alpha$-Sasakian, $\beta$-Kenmotsu and co-symplectic manifolds. An almost contact metric structure on a manifold $M$ is called a trans-Sasakian structure if the product manifold $M \times R$ belongs to the class $W_4$, a class of Hermitian manifolds which are closely related to a locally conformal Kähler manifolds. Trans-Sasakian manifolds were studied extensively by Marrero [10], Tripathi [12], De et al.([5], [7], [8]) and others. Tripathi [12] proved that trans-Sasakian manifolds are always generalized quasi-Sasakian. De et al. and Bagewadi et al.([4], [1]) have obtained results on the conservativeness of Projective, Pseudo projective, Conformal concircular, Quasi conformal curvature tensors on $k$-contact,Kenmotsu and trans-Sasakian manifolds. De et al. [5] generalized the notion of locally $\varphi$-symmetric and introduced the notion of $\varphi$-recurrent Sasakian manifolds. The author Nagaraja [11] introduced the notion of $\varphi$-recurrent trans-Sasakian manifolds. In the present paper we study generalized $\varphi$-recurrent and concircular $\varphi$-recurrent Trans-Sasakian manifolds.

## 2. Preliminaries

Let $M$ be a $n$ dimensional almost contact metric manifold [6] with an almost contact metric structure $(\varphi, \xi, \eta, g)$ where $\varphi, \xi, \eta$ are tensor fields on $M$ of types (1,1),(1,0),

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*Corresponding author

(0,1) respectively and $g$ is the Riemannian metric on $M$ such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0. \tag{2.1}$$

The Riemannian metric $g$ on $M$ satisfies the condition

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \varphi Y) = -g(\varphi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in TM$. An almost contact metric structure $(\varphi, \xi, \eta, g)$ in $M$ is called a trans-Sasakian structure [2] if the product manifold $(M \times R, J, G)$ belongs to the class $W_4$ where $J$ is the complex structure on $(M \times R)$ defined by

$$J(X, \lambda \frac{d}{dt}) = (\varphi X - \lambda \xi, \eta(X) \frac{d}{dt}), \tag{2.4}$$

for all vector fields $X$ on $M$ and smooth functions $\lambda$ on $(M \times R)$ and $G$ is the product metric on $(M \times R)$. This may be expressed by the following condition [7]

$$(\nabla_X \varphi)(Y) = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{2.5}$$

where $\alpha$ and $\beta$ are smooth functions on $M$. From (2.5), we have

$$\nabla_X \xi = -\alpha(\varphi X) + \beta(X, \eta(X)\xi) \tag{2.6}$$

and

$$(\nabla_X \eta)(Y) = -\alpha g(\varphi X, Y) + \beta(\varphi X, \varphi Y). \tag{2.7}$$

In an $n$-dimensional trans-Sasakian manifold, from (2.5) (2.6) (2.7), we can derive [7]

$$\begin{aligned} R(X, Y)\xi &= (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) \\ &+ 2\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y) - (X\alpha)\varphi Y \\ &+ (Y\alpha)\varphi X - (X\beta)\varphi^2 Y + (Y\beta)\varphi^2 X, \end{aligned} \tag{2.8}$$

$$S(X, \xi) = \left\{(n-1)(\alpha^2 - \beta^2) - \xi\beta\right\} \eta(X) - (n-2)(X\beta) - (\varphi X)\alpha \tag{2.9}$$

and

$$Q\xi = \left\{(n-1)(\alpha^2 - \beta^2) - \xi\beta\right\} \xi - (n-2) grad\beta + \varphi(grad\alpha), \tag{2.10}$$

where $R$ is the curvature tensor, $S$ is the Ricci-tensor and $r$ is the scalar curvature. Also

$$g(QX, Y) = S(X, Y), \tag{2.11}$$

where $Q$ being the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor $S$.
When

$$\varphi(grad\alpha) = (n-2)grad\beta, \tag{2.12}$$

then (2.9) and (2.10) reduces to

$$S(X,\xi) = (n-1)(\alpha^2 - \beta^2)\eta(X), \tag{2.13}$$

and

$$Q\xi = (n-1)(\alpha^2 - \beta^2)\xi. \tag{2.14}$$

A Sasakian manifold is said to be a $\varphi$-recurrent manifold if there exists a non zero 1-form $A$ such that

$$\varphi^2((\nabla_W R)(X,Y)Z) = A(X)R(Y,Z)W \tag{2.15}$$

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.
Further we have

$$2\alpha\beta + \xi\alpha = 0. \tag{2.16}$$

## 3. Semi-generalized $\varphi$-recurrent trans Sasakian Manifolds

**Definition 3.1.** A trans-Sasakian manifold $(M^n, g)$ is called semi-generalized $\varphi$-recurrent if its curvature tensor $R$ satisfies the condition

$$\varphi^2\Big((\nabla_W R)(X,Y)Z\Big) = A(W)R(X,Y)Z + B(W)g(Y,Z)X, \tag{3.1}$$

for all $X, Y, Z, W \in TM$, where $A$ and $B$ are two 1-forms, $B$ is non zero and these are defined by

$$A(W) = g(W,\rho_1), \qquad B(W) = g(W,\rho_2), \tag{3.2}$$

where $\rho_1$, $\rho_2$ being the vector fields associated to the 1-form $A$, $B$ respectively.

**Theorem 3.1.** *A semi-generalized $\varphi$-recurrent trans-Sasakian manifold $(M^n, g)$ satisfying $\varphi(grad\alpha) = (n-2)grad\beta$, is an Einstein manifold and more over the 1-forms $A$ and $B$ are related as*

$$(n-1)(\alpha^2 - \beta^2)A(W) + nB(W) = 0.$$

**Proof**: Let us consider a semi-generalized $\varphi$-recurrent trans-Sasakian manifold. Then by virtue of (2.1) and (3.1) we have

$$\begin{aligned} &- (\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi \\ &= A(W)R(X,Y)Z + B(W)g(Y,Z)X, \end{aligned} \tag{3.3}$$

from which we get

$$\begin{aligned} &- g((\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) \\ &= A(W)R(X,Y,Z,U) + B(W)g(Y,Z)g(X,U). \end{aligned} \tag{3.4}$$

Let $\{e_i\}$, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.4) and taking summation over $i$, $1 \le i \le n$, we get

$$\begin{aligned} -(\nabla_W S)(Y,Z) &+ \sum_{i=1}^{n} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) \\ &= A(W)S(Y,Z) + nB(W)g(Y,Z). \end{aligned} \tag{3.5}$$

The second term of (3.5) by putting $Z = \xi$ takes the form $g((\nabla_W R)(e_i,Y)\xi, e_i)$.

Consider

$$\begin{aligned} g((\nabla_W R)(e_i,Y)\xi,\xi) &= g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi) \\ &- g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi) \end{aligned} \tag{3.6}$$

at $p \in M$. Since $\{e_i\}$ is an orthonormal basis, so $\nabla_X e_i = 0$ at $p$.
Using (2.1), (2.3) and (2.8), we have

$$\begin{aligned} g(R(e_i,\nabla_W Y)\xi,\xi) &= g\Big((\alpha^2 - \beta^2)(\eta(\nabla_W Y)e_i - \eta(e_i)(\nabla_W Y)) \\ &+ 2\alpha\beta(\eta(\nabla_W Y)\varphi e_i - \eta(e_i)\varphi(\nabla_W Y)) \\ &+ (\nabla_W Y\alpha)\varphi e_i - (e_i\alpha)\varphi(\nabla_W Y) \\ &+ (\nabla_W Y\beta)\varphi^2 e_i - (e_i\beta)\varphi^2(\nabla_W Y), \xi\Big) \\ &= 0. \end{aligned} \tag{3.7}$$

Using (3.7) in (3.6), we obtain

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi). \tag{3.8}$$

Since $(\nabla_W g) = 0$, we have

$$g(\nabla_W R(e_i,Y)\xi,\xi) + g(R(e_i,Y)\xi,\nabla_W \xi) = 0,$$

which implies

$$g(\nabla_W R(e_i,Y)\xi,\xi) = -g(R(e_i,Y)\xi,\nabla_W \xi) - g(R(e_i,Y)\nabla_W \xi,\xi). \tag{3.9}$$

Using (2.6) in (3.9), we get

$$\begin{aligned} g((\nabla_W R)(e_i,Y)\xi,\xi) &= -g\Big(R(e_i,Y)\xi, -\alpha\varphi W + \beta(W - \eta(W)\xi)\Big) \\ &- g\Big(R(e_i,Y)(-\alpha\varphi W + \beta(W - \eta(W)\xi)),\xi\Big) \\ &= \alpha g(R(e_i,Y)\xi,\varphi W) - \beta g(R(e_i,Y)\xi,W) \\ &+ \beta\eta(W)g(R(e_i,Y)\xi,\xi) + \alpha g(R(e_i,Y)\varphi W,\xi) \\ &- \beta g(R(e_i,Y)W,\xi) + \beta\eta(W)g(R(e_i,Y)\xi,\xi) \\ &= 0. \end{aligned} \tag{3.10}$$

Replacing $Z$ by $\xi$ in (3.5) and using (2.3),(2.13) and (3.10) we have

$$(\nabla_W S)(Y,\xi) = -\left[(n-1)(\alpha^2-\beta^2)A(W) + nB(W)\right]\eta(Y). \tag{3.11}$$

Now, we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi).$$

Using (2.6) and (2.13) in the above relation, it follows that

$$\begin{aligned} &(\nabla_W S)(Y,\xi) = (n-1)(\alpha^2-\beta^2)[-\alpha g(\varphi W,Y) + \beta g(\varphi Y,\varphi W)] \\ &+\alpha S(Y,\varphi W) - S(Y,\beta W) + (n-1)\beta(\alpha^2-\beta^2)\eta(Y)\eta(W). \end{aligned} \tag{3.12}$$

By virtue of (2.2), we obtain from (3.12) that

$$\begin{aligned} (\nabla_W S)(Y,\xi) &= (n-1)(\alpha^2-\beta^2)[-\alpha g(Y,\varphi W) + \beta g(Y,W)] \\ &+ \alpha S(Y,\varphi W) - \beta S(Y,W). \end{aligned} \tag{3.13}$$

From (3.11) and (3.13), we have

$$\begin{aligned} &(n-1)(\alpha^2-\beta^2)[-\alpha g(Y,\varphi W) + \beta g(Y,W)] + \alpha S(Y,\varphi W) \\ &- \beta S(Y,W) = -\left[(n-1)(\alpha^2-\beta^2)A(W) + nB(W)\right]\eta(Y). \end{aligned} \tag{3.14}$$

Replacing $Y = \xi$ in (3.14) then using (2.1), (2.3), (2.12) and (2.13) we get

$$(n-1)(\alpha^2-\beta^2)A(W) + nB(W) = 0. \tag{3.15}$$

Again replacing $Y$ and $W$ by $\varphi Y$ and $\varphi W$ respcetively in (3.14) and then using (2.1), (2.3), (2.11), (2.12) and (2.14), we obtain

$$S(Y,W) = (n-1)(\alpha^2-\beta^2)g(Y,W) \tag{3.16}$$

and

$$S(\varphi Y,W) = (n-1)(\alpha^2-\beta^2)g(\varphi Y,W).$$

Which proves the theorem.

## 4. Semi-generalized concircular $\varphi$-recurrent trans-Sasakian Manifolds

**Definition 4.1.** A trans-Sasakian manifold is said to be concircular $\varphi$-recurrent manifold if there exists a non zero 1-form $A$ such that

$$\varphi^2((\nabla_W C)(X,Y)Z) = A(W)C(X,Y)Z, \tag{4.1}$$

for arbitrary vector fields $X, Y, Z, W$ where $C$ is a concircular curvature tensor given by

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{4.2}$$

where $R$ is the curvature tensor and $r$ is the scalar curvature.

**Definition 4.2.** A trans-Sasakian manifold is called a semi-generalized concircular $\varphi$-recurrent if its concircular curvature tensor $C$ defined in (4.2) satisfies the condition

$$\varphi^2(\nabla_W C)(X,Y,Z) = A(W)C(X,Y,Z) + B(W)g(Y,Z)X, \tag{4.3}$$

where $A$ and $B$ are defined as (3.2).

**Theorem 4.1.** *Let $(M^n, g)$ be a semi-generalized concircular $\varphi$-recurrent trans-Sasakian manifold then*

$$\left[(n-1)(\alpha^2 - \beta^2) - \frac{r}{n}\right] A(W) + nB(W) = 0.$$

**Proof**:

Let $(M^n, g)$ be a semi-generalized concircular $\varphi$-recurrent trans-Sasakian manifold. Then by virtue of (2.1) and (4.3), we have

$$\begin{aligned} - \quad & (\nabla_W C)(X,Y,Z) + \eta(\nabla_W C)(X,Y,Z))\xi \\ & = A(W)C(X,Y,Z) + B(W)g(Y,Z)X, \end{aligned} \tag{4.4}$$

from which it follows that

$$\begin{aligned} - \quad & g((\nabla_W C)(X,Y,Z),U) + \eta((\nabla_W C)(X,Y,Z))\eta(U) \\ & = A(W)g(C(X,Y,Z),U) + B(W)g(Y,Z)g(X,U). \end{aligned} \tag{4.5}$$

Let $\{e_i\}, i = 1, 2, \ldots, n$ be orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (4.5) and taking summation over $i$, $1 \leq i \leq n$, we get

$$\begin{aligned} & -(\nabla_W S)(X,U) + \frac{W(r)}{n}g(X,U) + (\nabla_W S)(X,\xi)\eta(U) \\ & -\frac{W(r)}{n}\eta(X)\eta(U) = A(W)\left[S(X,U) - \frac{r}{n}g(X,U)\right] \\ & +nB(W)g(X,U). \end{aligned} \tag{4.6}$$

Replacing $U$ by $\xi$ in (4.6) and using (2.1) and (2.13), we get

$$A(W)\left[(n-1)(\alpha^2-\beta^2)-\frac{r}{n}\right]\eta(X)+nB(W)\eta(X)=0. \tag{4.7}$$

Putting $X=\xi$ in (4.7), we have

$$\left[(n-1)(\alpha^2-\beta^2)-\frac{r}{n}\right]A(W)+nB(W)=0. \tag{4.8}$$

This completes the proof.

**Theorem 4.2.** *A semi-generalized concircular $\varphi$-recurrent trans-Sasakian manifold is an Einstein manifold, provided $\alpha$ and $\beta$ are constants.*

**Proof**: Putting $X=U=e_i$ in (4.5) and taking summation over $i$, $1\le i\le n$, we get

$$\begin{aligned} - \quad & (\nabla_W S)(Y,Z) = -\sum_{i=1}^{n} g((\nabla_W R)(e_i,Y,Z),\xi)g(e_i,\xi) \\ - \quad & \frac{W(r)}{n}g(Y,Z)+\frac{W(r)}{n(n-1)}\left[g(Y,Z)-\eta(Y)\eta(Z)\right] \\ + \quad & A(W)\left[S(Y,Z)-\frac{r}{n}g(Y,Z)\right]+nB(W)g(Y,Z). \end{aligned} \tag{4.9}$$

Replacing $Z$ by $\xi$ in (4.9) and using (4.7), we have

$$(\nabla_W S)(Y,\xi)=\frac{W(r)}{n}\eta(Y). \tag{4.10}$$

Now, we have

$$(\nabla_W S)(Y,\xi)=\nabla_W S(Y,\xi)-S(\nabla_W Y,\xi)-S(Y,\nabla_W\xi). \tag{4.11}$$

Using (2.6) and (2.9) in the above relation, it follows that

$$\begin{aligned} (\nabla_W S)(Y,\xi) &= (n-1)(\alpha^2-\beta^2)[-\alpha g(\varphi W,Y)+\beta g(W,Y)] \\ &+ \alpha S(Y,\varphi W)-\beta S(Y,W). \end{aligned} \tag{4.12}$$

In view of (4.10) and (4.12)

$$\begin{aligned} & (n-1)(\alpha^2-\beta^2)[-\alpha g(\varphi W,Y)+\beta g(W,Y)] \\ + \quad & \alpha S(Y,\varphi W)-\beta S(Y,W)=\frac{W(r)}{n}\eta(Y). \end{aligned} \tag{4.13}$$

Replacing $Y$ by $\varphi Y$ in (4.13) and using (2.2), we get

$$\begin{aligned} - \quad & \alpha S(\varphi Y,\varphi W)+\beta S(\varphi Y,W) \\ = \quad & (n-1)(\alpha^2-\beta^2)[\beta g(W,\varphi Y)-\alpha g(\varphi W,\varphi Y)]. \end{aligned} \tag{4.14}$$

Interchanging $Y$ by $W$ in (4.14) and by using the skew symmetry of $\varphi$, we obtain

$$\alpha S(\varphi W, \varphi Y) = (n-1)\alpha(\alpha^2 - \beta^2)g(\varphi W, \varphi Y). \tag{4.15}$$

By skew symmetry of $\varphi$ and using (2.9), we obtain

$$\begin{aligned} S(\varphi W, \varphi Y) &= S(\varphi^2 W, Y) \\ &= S(W, Y) - (n-1)(\alpha^2 - \beta^2)\eta(W)\eta(Y). \end{aligned}$$

Substituting this in (4.15), we get

$$S(W, Y) = (n-1)(\alpha^2 - \beta^2)g(W, Y). \tag{4.16}$$

i.e $M$ is an Einstein manifold. Hence the theorem is verified.

## 5. Conclusion

This paper is all about the study of geometrical properties of a semi-generalized $\phi$-recurrent trans-Sasakian manifold. We prove that a semi-generalized $\phi$-recurrent trans-Sasakian manifold is an Einstein manifold. It is proved that a semi-generalized concircular $\phi$-recurrent trans-Sasakian manifold is also an Einstein manifold.

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Jagannath Choudhury
Department of Mathematics and Computer Science
Mizoram University
Tanhril-796004, Mizoram
jagai_76@yahoo.com

Rajesh Kumar
Department of Mathematics
Pachhunga University College, Aizawl,
Mizoram-796001, INDIA
Rajesh_mzu@yahoo.com

Jay P. Singh
Department of Mathematics and Computer Science
Mizoram University
Tanhril-796004, Mizoram
jpsmaths@gmail.com