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A NEW APPROACH TOWARDS GEODESIC CURVATURE AND GEODESIC TORSION OF TRANSVERSAL INTERSECTION IN $\mathbb{R}^3$

Mohamd Saleem Lone, Mehraj Ahmad Lone and Mohammad Hasan Shahid

Abstract. In this paper, we formulate new method to obtain the geodesic curvature and geodesic torsion of two regular parametric surfaces in $\mathbb{R}^3$. The new method will be different from the older ones in the sense that we will be making use of Rodrigues rotation formula and a operator $\mathscr{D}$ [2].

Keywords: Geodesic curvature, geodesic torsion, curvature, tangent vector.

1. Introduction

The geometry of intersection problems of surfaces is a fundamental process needed in formualting complex shapes in CAD/CAM system. Intersections are useful in the representation of the design of complex objects, in computer animations and in NC machining for trimming off the region bounded by self intersection curves of offset surfaces. The two types of surfaces commonly used in geometric designing systems are parametric and implicit surfaces, that results in three types of surface-surface intersection (SSI) problems i.e., parametric-parametric(P-P), implicit-implicit(I-I), implicit-parametric(I-P). The SSI is called transversal or tangential if the normal vectors of the surfaces are linearly independent or linearly dependent, respectively at the intersection point. In transversal intersection problems, the tangent vectors of the intersection curve can be found by the vector product of the normal vectors of the surfaces as in [8, 9, 12, 20], but in our paper, we shall not rely on this technique of cross product of normal vectors. We will develop a whole new method to obtain the tangent and curvature vector.
The geometric properties of the parametrically defined curves can be found in the classical literature on differential geometry in [14, 18] and in the contemporary literature of geometric modelling [4, 6]. Also the higher curvatures of curves in $\mathbb{R}^n$ can be found in textbooks such as [19] and papers such as[5]. On the other hand, for the differential geometry of the intersection curves, there exists little literature. Willmore [18] obtained the unit tangent, the unit principal normal and unit binormal, as well as the curvature and the torsion of the intersection curve of two implicit surfaces in $\mathbb{R}^3$. Ye and Maekawa [20] in 1999 provide

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algorithms for obtaining the Frenet apparatus of the intersection curves of two parametric surfaces in $\mathbb{R}^3$, they also provide algorithms for the evaluation of higher order derivatives for transversal as well as tangential intersections. Then Hartmann [3] provides formulae for computing the curvature and geodesic curvature of the intersection curves of all the three types of intersection problems in $\mathbb{R}^3$ using the implicit function theorem. Similarly Abdel-All et al.[11] provide an algorithm for the evaluation of the Frenet apparatus of the intersection curves of two implicit surfaces using implicit function theorem. Soliman et al.[7] obtained an algorithm for the Frenet apparatus of the intersection curves of two surfaces (implicit-parametric) in $\mathbb{R}^3$. Goldmann [15] derived closed formulae for computing the curvature and the torsion of the intersection curve of two implicit surfaces in $\mathbb{R}^3$ and the curvature of the intersection curves in $\mathbb{R}^{n+1}$. For non-transversal intersection we refer [13, 17].

In this study, we give new method to calculate the geodesic curvature and geodesic torsion of the transversal intersection curve of two regular parametric surface in $\mathbb{R}^3$. Althought, the same things are discussed in [1], but we try to obtain them using a new technique[2]. In section 2 basic definitions and other required geometric terms are compiled. In section 3, we show how we make use of Rodrigues' rotation formula to obtain the geodesic curvature and geodesic torsion. Finally to be more constructive, we present an example in section 4.

2. Preliminaries

Definition 2.1. Let e_1, e_2, e_3 be the standard basis of three dimensional Euclidean space $\mathbb{R}^3$. The vector product of vectors $x = \sum_{i=1}^{3} x_i e_i$, $y = \sum_{i=1}^{3} y_i e_i$ is defined as

$$x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}. \tag{2.1}$$

The vector product yields a vector that is orthogonal to x and y.

Definition 2.2. Let $M \subset E^3$ be a regular surface given by $R = R(u,v)$ and $\alpha : I \subset \mathbb{R} \to E^3$ be an arbitrary curve with arc length parametrisation. If $\{T, V, N\}$ is the darboux frame of α, where T is the unit tangent, N is the unit normal restricted to α and $V = T \times N$, then we have

$$\begin{cases} T' = \kappa_g V + \kappa_n N \\ V' = -\kappa_g T + \tau_g N \\ N' = -\kappa_n T - \tau_g V \end{cases} \tag{2.2}$$

where κ_n is the normal curvature of the surface in T direction, κ_g is the geodesic curvature and τ_g is the geodesic torsion of α, respectively. From (2.2), we obtain

$$\kappa_g = \langle T', V \rangle, \quad \tau_g = \langle V', N \rangle. \tag{2.3}$$

where $\langle , \rangle$ denotes the scalar product.

Definition 2.3. Suppose the curve $\alpha(s)$ lies on M, we can write $\alpha(s) = R(u(s), v(s))$, then we have

$$\begin{aligned}
\alpha'(s) &= R_1 u' + R_2 v', && (2.4)\\
\alpha''(s) &= R_{11}(u')^2 + 2R_{12}u'v' + R_{22}(v')^2 + R_1 u'' + R_2 v'', && (2.5)\\
\alpha'''(s) &= R_{111}(u')^3 + 3R_{112}(u')^2 v' + 3R_{122}u'(v')^2 + R_{222}(v')^3 + 3[R_{11}u'u'' \\
&\quad + R_{12}(u''v' + u'v'') + R_{22}v'v''] + R_1 u''' + R_2 v''', && (2.6)
\end{aligned}$$

where $R_1 = \frac{\partial R}{\partial u}$ and $R_2 = \frac{\partial R}{\partial v}$.

Definition 2.4. $\mathscr{D}$ operator: Let u be a non-zero vector in $\mathbb{R}^3$, then for any arbitrary choice of linearly independent vector(υ) with u, we have[2]

$$\mathscr{D}(u) = \upsilon \times u.$$

Hence $\mathscr{D}$ gives a nonzero vector orthogonal to u.

Definition 2.5. For any choice of a unit vector $Q = (q_1, q_2, q_3)$ in $\mathbb{R}^3$, a vector can be rotated about the axis in the direction of Q with a rotation angle θ by Rodrigues' rotation formula given as[16]

$$\mathscr{R} = I_3 + (\sin\theta)P + (1 - \cos\theta)P^2, \tag{2.7}$$

where I_3 is a 3×3 identity matrix and

$$P = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix},$$

or, in matrix form as

(2.8)
$$\mathscr{R} = \begin{bmatrix} q_1^2 + (1-q_1^2)\cos\theta & q_1q_2(1-\cos\theta) - q_3\sin\theta & q_1q_3(1-\cos\theta) + q_2\sin\theta \\ q_1q_2(1-\cos\theta) + q_3\sin\theta & q_2^2 + (1-q_2^2)\cos\theta & q_2q_3(1-\cos\theta) - q_1\sin\theta \\ q_1q_3(1-\cos\theta) - q_2\sin\theta & q_2q_3(1-\cos\theta) + q_1\sin\theta & q_3^2 + (1-q_3^2)\cos\theta \end{bmatrix}$$

Remark 2.1. We assume that the intersection curve has a tangential direction at each point. The method do not work in at least second order contact.

3. Parametric-parametric surface intersection

3.1. Geodesic torsion

Let X and Y be two regular surfaces given by their parametric representation $X = X(u, v)$ and $Y = Y(p, q)$, respectively and $\alpha(s)$ be their intersecting curve of unit speed with arc length parametrisation. Then , their unit normals are defined as

$$N^X = \frac{X_u \times X_v}{\|X_u \times X_v\|}, \quad N^Y = \frac{Y_p \times Y_q}{\|Y_p \times Y_q\|}.$$

Denoting the darboux frame of α with respect to surface X and Y by $\{T^X, V^X, N^X\}$ and $\{T^Y, V^Y, N^Y\}$, respectively, then from (2.3), we obtain the geodesic torsion of the intersecting curve α with respect to surface X and Y as

$$\tau_g^X = \langle (V^X)', N^X \rangle \quad \tau_g^Y = \langle (V^Y)', N^Y \rangle \tag{3.1}$$

or, we can write (3.1) as [1]

$$\begin{cases} \tau_g^X = \frac{1}{\sqrt{E^X G^X - (F^X)^2}} \{(E^X M^X - F^X L^X)(u')^2 + (E^X N^X - G^X L^X) u'v' \\ \quad + (F^X N^X - G^X M^X)(v')^2\}, \\ \tau_g^Y = \frac{1}{\sqrt{E^Y G^Y - (F^Y)^2}} \{(E^Y M^Y - F^Y L^Y)(p')^2 + (E^Y N^Y - G^Y L^Y) p'q' \\ \quad + (F^Y N^Y - G^Y M^Y)(q')^2\}. \end{cases} \tag{3.2}$$

where (E, G, F) and (L, M, N) are the first and second fundamental coefficients respectively. From (3.2), we see that, to obtain τ_g^X and τ_g^Y, we need to find u', v', p' and q'. For that, we have the following method.

3.2. New Method:

By the definition of $\mathscr{D}$, the vectors $\mathscr{D}(N^X)$ and $\mathscr{D}(N^Y)$ lies in the tangent plane of X and Y, respectively. Then after a suitable choice of rotation angles θ and ϕ, $(0 \le \theta, \phi < \pi)$ around the axis of rotations in the direction of N^X and N^Y, respectively, the normalized vectors $\mathscr{D}(N^X)$ and $\mathscr{D}(N^Y)$ coincides with the unit tangent vector T of the intersection curve at point c, then we have

$$T = X_u u' + X_v v' = \mathscr{R}(\theta, N^X) \frac{\mathscr{D}(N^X)}{\|\mathscr{D}(N^X\|}, \tag{3.3}$$

$$T = Y_p p' + Y_q q' = \mathscr{R}(\phi, N^Y) \frac{\mathscr{D}(N^Y)}{\|\mathscr{D}(N^Y\|}. \tag{3.4}$$

Since the T is the common tangent, rotation angles can be find by equating

$$\mathscr{R}(\theta, N^X) \frac{\mathscr{D}(N^X)}{\|\mathscr{D}(N^X\|} = \mathscr{R}(\phi, N^Y) \frac{\mathscr{D}(N^Y)}{\|\mathscr{D}(N^Y\|} = f(\theta, \phi)(say).$$

Rewriting (3.3)and (3.4) in matrix form as

$$\begin{bmatrix} \\ X_u \\ \\ \end{bmatrix}_{3\times 1} u' + \begin{bmatrix} \\ X_v \\ \\ \end{bmatrix}_{3\times 1} v' = \begin{bmatrix} \\ f(\theta, \phi) \\ \\ \end{bmatrix}_{3\times 1} \tag{3.5}$$

and

$$\begin{bmatrix} \\ X_p \\ \\ \end{bmatrix}_{3\times 1} p' + \begin{bmatrix} \\ Y_q \\ \\ \end{bmatrix}_{3\times 1} q' = \begin{bmatrix} \\ f(\theta, \phi) \\ \\ \end{bmatrix}_{3\times 1}. \tag{3.6}$$

From (3.5) and (3.6), $(u',v'),(p',q')$ can be easily found respectively. Finally substituting their values in (3.2), we get the geodesic torsion of the intersection curve with respect to X and Y, respectively. Consequently, the unit tangent vector follows from (3.3) or (3.4).

3.3. Geodesic curvature

In this subsection, we evaluate the geodesic curvature of the intersection curve α with respect to X and Y, respectively.
From (2.3) the geodesic curvature can be written as

$$\begin{aligned}\kappa_g^X = & \frac{1}{\sqrt{E^X G^X-(F^X)^2}}\left(\left[\left(F_u^X-\frac{E_v^X}{2}\right)\langle X_u,T\rangle-\frac{E_u^X}{2}\langle X_v,T\rangle\right](u')^2+(G_u^X\langle X_u,T\rangle\right. \\ & \left.-E_v^X\langle X_v,T\rangle)u'v'+\left[\frac{G_v^X}{2}\langle X_u,T\rangle-\left(F_v^X-\frac{G_u^X}{2}\right)\langle X_v,T\rangle\right](v')^2\right) \\ & +\sqrt{E^X G^X-(F^X)^2}(u'v''-v'u'')\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\kappa_g^Y = & \frac{1}{\sqrt{E^Y G^Y-(F^Y)^2}}\left(\left[\left(F_p^Y-\frac{E_q^Y}{2}\right)\langle Y_p,T\rangle-\frac{E_p^Y}{2}\langle Y_q,T\rangle\right](p')^2+(G_p^Y\langle Y_p,T\rangle\right. \\ & \left.-E_q^Y\langle Y_q,T\rangle)p'q'+\left[\frac{G_q^Y}{2}\langle Y_p,T\rangle-\left(F_q^Y-\frac{G_p^Y}{2}\right)\langle Y_q,T\rangle\right](q')^2\right) \\ & +\sqrt{E^Y G^Y-(F^Y)^2}(p'q''-q'p'').\end{aligned} \tag{3.8}$$

The formula (3.7,3.8) can be found in any classical book of differential geometry. Since (u',v') and (p',q') are known from (3.5) and (3.6), respectively, we need to evaluate (u'',v'') and (p'',q'') to find the geodesic curvature of the intersection curve with respect to surfaces X and Y, respectively. Applying operator $\mathscr{D}$ to unit tangent vector T, $\mathscr{D}(T)$ yields a vector lying in the normal plane of the intersection curve. After a suitable desired rotation around the axis in the direction T with the rotation angle ϑ, the normalised vector $\mathscr{D}(T)$ will coincide with the principal normal vector n. Since, we know that $\alpha''=\kappa n$, where κ is the curvature. We may write

$$\alpha''=X_u u''+X_v v''+X_{uu}(u')^2+2X_{uv}u'v'+X_{vv}(v')^2=\kappa\mathscr{R}(\vartheta,T)\frac{\mathscr{D}(T)}{\|\mathscr{D}(T)\|}, \tag{3.9}$$

$$\alpha''=Y_p p''+Y_q q''+Y_{pp}(p')^2+2Y_{pq}p'q'+Y_{qq}(q')^2=\kappa\mathscr{R}(\vartheta,T)\frac{\mathscr{D}(T)}{\|\mathscr{D}(T)\|}. \tag{3.10}$$

(3.9) and (3.10) yields a linear equation depending on ϑ for (u'',v'') and (p'',q''), respectively. Considering α on X and Y and substituting the results in $\langle\alpha'',T\rangle=0$, gives the second equation for each pair, respectively. Consequently (u'',v'') and (p'',q'') can be found from (3.9) and (3.10), respectively. Substituting (u'', v'') and (p'', q'') in (3.7) and (3.8) gives the geodesic curvature of the transversal intersection curve α with respect to surface X and Y, respectively.

Theorem 3.1. *Let α be a unit speed intersection curve of two regular surfaces X and Y. Then, we have*

$$\kappa = \frac{1}{\sin\theta}\sqrt{(\kappa_g^X)^2 + (\kappa_g^Y)^2 - 2\kappa_g^X\kappa_g^Y\cos\theta}$$

where θ is the angle between the normal vectors[1].

4. Example:

Example 4.1. Consider the surfaces

$$X = (\sin u, v\cos u, v-1); \quad 0 \le u, v \le \pi$$

and

$$Y = (u-1, \cos v, \sin v); \quad 0 \le u, v \le \pi.$$

We find the geodesic curvature and the geodesic torsion at the intersection point

$$c = X(0,1) = Y(1,0) = (0,1,0).$$

Since the unit normal vectors of X and Y are

$$N^X = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad N^Y = (0,-1,0),$$

respectively, we see that X and Y intersect transversally at c.
Choosing $\upsilon = (-1,0,0)$ which is infact linearly independent to N^X, we have

$$\frac{\mathscr{D}(N^X)}{\|\mathscr{D}(N^X)\|} = \frac{\upsilon \times N^X}{\|\upsilon \times N^X\|} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$

Now

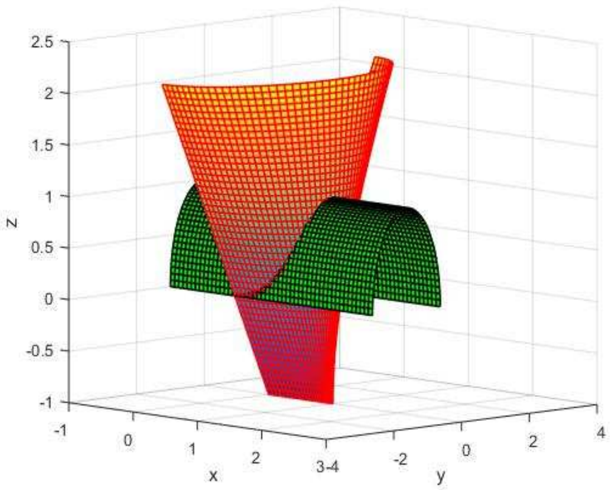
$$\begin{aligned}\mathscr{R}(N^X,\theta)\frac{\mathscr{D}(N^X)}{\|\mathscr{D}(N^X)\|} &= \begin{pmatrix} \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta & -\frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1+\cos\theta) & -\frac{1}{2}(1-\cos\theta) \\ \frac{1}{\sqrt{2}}\sin\theta & -\frac{1}{2}(1-\cos\theta) & \frac{1}{2}(1+\cos\theta) \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \left(-\sin\theta, \frac{1}{\sqrt{2}}\cos\theta, \frac{1}{\sqrt{2}}\cos\theta\right). \end{aligned} \tag{4.1}$$

Choosing $\upsilon = (-1,0,0)$ being linearly independent with N^Y, we have

$$\frac{\mathscr{D}(N^Y)}{\|\mathscr{D}(N^Y)\|} = \frac{\upsilon \times N^Y}{\|\upsilon \times N^Y\|} = (0,0,1).$$

Then

$$\begin{aligned}\mathscr{R}(N^Y,\phi)\frac{\mathscr{D}(N^Y)}{\|\mathscr{D}(N^Y)\|} &= \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0) \\ \sin\phi & 0 & \cos\phi \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= (-\sin\phi, 0, \cos\phi). \end{aligned} \tag{4.2}$$



On comparing (4.1) and (4.2), we obtain $\theta = \phi = \frac{\pi}{2}$.
Hence from (3.5) and (3.6), we obtain $(u', v') = (-1, 0)$ and $(p', q') = (-1, 0)$, respectively.
Now from (3.5) or (3.6), we obtain

$$t = (-1, 0, 0).$$

Thus from (3.2), we obtain $\tau_g^X = 0$ and $\tau_g^Y = 0$, respectively.
Now for $\vartheta = (0, 0, 1)$, we have

$$\kappa \mathscr{R}(\vartheta, T) \frac{\mathscr{D}(T)}{\|\mathscr{D}(T)\|} = (0, \kappa \cos\theta, -\kappa \sin\theta).$$

Therefore, from (3.9) and (3.10), we obtain

$$(u'', v'' - 1, v'') = (p'' + 1, 1, q'' + 1) = (0, \kappa \cos\theta, -\kappa \sin\theta).$$

Using $\langle \alpha', \alpha'' \rangle = 0$, we obtain

$$(u'', v'') = (0, 2) \quad \textit{and} \quad (p'', q'') = (-1, 1).$$

Hence from (3.7) and (3.8), we obtain $\kappa_g^X = -\frac{3}{\sqrt{2}}$ and $\kappa_g^Y = -1$, respectively.

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Mohamd Saleem Lone
Faculty of Basic and Applied Science
Department of Mathematics
Central University of Jammu
180011 Jammu, India
saleemraja2008@gmail.com

Mehraj Ahmad Lone
Faculty of Basic and Applied Science
Department of Mathematics
Central University of Jammu
180011 Jammu, India
mehraj.jmi@gmail.com

Mohammad Hasan Shahid
Faculty of Natural Sciences
Department of Mathematics
Jamia Millia Islamia
New Delhi-110 025, India
hasan_jmi@yahoo.com