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CR-LIGHTLIKE SUBMANIFOLDS OF INDEFINITE PARA-SASAKIAN MANIFOLDS

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Abstract. The purpose of this paper is to study a totally contact umbilical contact CR-lightlike submanifolds of an indefinite para-Sasakian manifold. In this paper, we prove that a totally contact umbilical CR-lightlike submanifold is totally contact geodesic. Further, we obtain a necessary and sufficient condition for a CR-lightlike submanifold to be anti-invariant submanifold. Finally, we obtain the integrability condition of distributions and also characterize a contact CR-lightlike submanifold of indefinite para-Sasakian manifold to be a contact CR-lightlike product.
Keywords: Lightlike submanifolds, contact CR-lightlike submanifolds, indefinite para-Sasakian manifolds, totally contact umbilical submanifolds.

1. Introduction

The geometry of submanifolds is very important topic to study in differential geometry. The geometry of submanifolds with positive definite metric may not be applicable to the other branches of mathematics and physics where the metric is not positive definite. Many geometrical properties of semi Riemannian submanifold has similarties with Riemannian case but the geometry in case of lightlike submanifold is so different and difficult. Cauchy-Riemann (CR)-submanifold of Kaehlerian manifolds with Riemannian metric were introduced by Bejancu [2] and further studied by [1],[3],[5],[6] and many more. Then contact CR-submanifolds of Sasakian manifolds with definite metric were introduced and studied by Yano and Kon [11],[12] and Kobayashi also studied CR-submanifolds of a Sasakian manifolds in [10]. Duggal and Sahin [9], who introduced the theory of contact CR-lightlike submanifolds of indefinite Sasakian manifolds also studied slant lightlike submnifolds of indefinite Hermitian manifolds [7] and slant lightlike submnifolds of indefinite Saskian mnifolds [8]. Since significant applications of the contact geometry (Maclane [14], Nazaikinskii et al. [15], Arnol'd [16]) and very limited information available on its lightlike case. Many interesting results for sectional curvature of indefinite Sasakian manifolds are obtained by R. Kumar in [13]. For application of semi-Riemannian

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manifolds see [4]. Moreover, the growing importance of lightlike submanifolds in mathematical physics and relativity theory. Motivated by above we extended this theory for totally contact umbilical contact CR-lightlike submanifolds of indefinite para-Sasakian manifolds.

The paper is organized as follows. In section two, we recall some basic definitions and fundamentals for further use. In section three, we give definition and example of contact umbilical CR-lightlike submanifold and obtain the necessary and sufficient condition of a totally contact umbilical proper CR-lightlike submanifold of an indefinite para-Sasakian manifold to be totally geodesic. In section four, we obtain the integrability condition of distributions $\mathcal{D}_1$ and $\mathcal{D}_2$. Further we obtain necessary and sufficient condition for a CR-lightlike submanifold to be an anti-invariant submanifold. Finally, we characterize a contact CR-lightlike submanifold of indefinite para-Sasakian manifold to be a contact CR-lightlike product.

2. Preliminaries

In this section, we recall some notations, definitions and fundamentals for lightlike submanifolds.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1, 1 \leq q \leq (m+n-1)$ and (M, g) be a $m-$dimensional submanifold of $\bar{M}$ and g be the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M,

$$TM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u, v) = 0,\ \forall v \in T_xM, x \in M\}, \tag{2.1}$$

is a degenerate $n-$dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exist a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ which is known as radical subspace. If the mapping

$$RadTM : x \in M \longrightarrow RadT_xM, \tag{2.2}$$

defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an $r-$lightlike submanifold, and $RadTM$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution $Rad(TM)$ in TM, that is

$$TM = Rad(TM) \perp S(TM), \tag{2.3}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}|_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, respectively. Then, we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.4}$$

$$(2.5) \qquad T\bar{M}|_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^\perp)$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as $\{\xi_1, ..., \xi_r, W_1, ..., W_n, N_1, ..., N_r, X_{r+1}, ..., X_m\}$, where $\{\xi_1, ..., \xi_r\}$ and $\{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(Rad(TM)|_u)$, $\Gamma(ltr(TM)|_u)$ and $\{W_1, ..., W_n\}$, $\{X_{r+1}, ..., X_m\}$ are local orthonormal bases of $\Gamma(S(TM)^\perp)|_u$ and $\Gamma(S(TM))|_u$, respectively. For the quasi-orthonormal basis of frames, we obtain

Theorem 2.1. *Let $M, g, S(TM), S(TM)^\perp$ be an $r-$lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^\perp)^\perp$ and a basis of $\Gamma(ltr(TM)|_u)$ consisting of smooth section N_i of $S(TM^\perp)^\perp|_u$, where u is a coordinate neighborhood of M, such that*

$$(2.6) \qquad \bar{g}(N_i, \xi_i) = \delta_{ij}, \; \bar{g}(N_i, N_i) = 0$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then according to decomposition of (2.5), the Gauss and Weingarten formulas are given by

$$(2.7) \qquad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \; \forall X, Y \in \Gamma(TM),$$

$$(2.8) \qquad \bar{\nabla}_X U = -A_U X + \nabla^\perp_X U, \; \forall X \in \Gamma(TM), U \in \Gamma(tr(TM)),$$

where $\{\nabla_X Y, A_X Y\}$ and $\{h(X, Y), \nabla^\perp_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$, called as a second fundamental form. A_V is a linear operator on M, called a shape operator.
According to (2.4), considering the projection morphism L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ respectively, (2.7) and (2.8) become

$$(2.9) \qquad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y) + h^s(X, Y),$$

$$(2.10) \qquad \bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U,$$

where, we put $h^l(X, Y) = L(h(X, Y))$,$h^s(X, Y) = S(h(X, Y))$, $D^l_X U = L(\nabla^\perp_X U)$, $D^s_X U = S(\nabla^\perp_X U)$.
As h^l and h^s are $\Gamma(ltr(TM))$-valued and $S(TM^\perp)$-valued respectively, therefore they are called the lightlike second fundamental forms on M. In particular

$$(2.11) \qquad \bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N),$$

$$(2.12) \qquad \bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W),$$

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))-$valued and $W \in \Gamma(S(TM^\perp))$.
Using (2.4)-(2.5) and (2.9)-(2.12), we get

$$(2.13) \qquad \bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^s(X, W)) = g(A_W X, Y),$$

$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y,h^l(X,\xi)) + g(Y,\nabla_X\xi) = 0, \tag{2.14}$$

$$\bar{g}(A_N X,N') + \bar{g}(N,A_{N'}X) = 0, \tag{2.15}$$

$$\bar{g}(N,\bar{\nabla}_X\bar{P}Y) = g(A_N X,\bar{P}Y), \tag{2.16}$$

for any $\xi \in \Gamma(Rad(TM)), W \in \Gamma(S(TM^{\perp}))$ and $N, N' \in \Gamma(ltr(TM))$. P is a projection of TM on $S(TM)$.
Now, we consider the decomposition (2.3), we can write

$$\nabla_X\bar{P}Y = \nabla^*_X\bar{P}Y + h(X,\bar{P}Y) \tag{2.17}$$

$$\nabla_X\xi = -A^*_\xi X + \nabla^{*\perp}_X\xi \tag{2.18}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X\bar{P}Y, A^*_\xi X\}$ and $\{h^*(X,\bar{P}Y), \nabla^{*\perp}_X\xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. Here ∇^* and $\nabla^{*\perp}$ are linear connections on $S(TM)$ and $Rad(TM)$, respectively.
By using (2.9) – (2.10) and (2.17) – (2.18), we have

$$\bar{g}(h^l(X,PY),\xi) = g(A^*_\xi X,PY), \tag{2.19}$$

$$\bar{g}(h^*(X,PY),N) = g(A_N X,PY). \tag{2.20}$$

2.1. Indefinite para-Sasakian manifold

An odd dimensional semi-Riemannian manifold $(\bar{M},g)$ is called a contact metric manifold if there are a $(1,1)$ tensor field ϕ, a vector field V and a one form η satisfying

$$g(\phi X,\phi Y) = g(X,Y) - \epsilon\eta(X)\eta(Y), \quad \bar{g}(V,V) = \epsilon \tag{2.21}$$

$$\phi^2 X = X - \eta(X)V \tag{2.22}$$

$$\bar{g}(X,V) = \eta(X), \quad \epsilon = \pm 1 \tag{2.23}$$

it follows that $\phi V = 0$, $\eta \circ \phi = 0$, $\eta(V) = \epsilon$, where V is called characteristic vector field and $(\phi, V, \eta, \bar{g})$ is called an indefinite contact metric structure of $\bar{M}$ and $\bar{M}$ is called an indefinite contact metric manifold.If $d\eta(X,Y) = g(\phi X,Y)$ then M is said to have contact metric structure $(\phi, V, \eta, \bar{g})$. If $N_\phi + d\eta \otimes V = 0$, where N_ϕ is the Nijenhuis tensor field then $\bar{M}$ is called an indefinite para-Sasakian manifold, for which we have.

$$\bar{\nabla}_X V = -\phi X, \tag{2.24}$$

$$(\bar{\nabla}_X\phi)Y = -\bar{g}(\phi X,\phi Y) - \epsilon\eta(Y)\phi^2 X, \tag{2.25}$$

3. Totally contact umbilical CR-lightlike submanifold

In this section, we give definitions and an example of contact umbilical CR-lightlike submanifold and obtain the necessary and sufficient condition of totally contact umbilical proper CR-lightlike submanifold of indefinite para-Sasakian manifold to be totally geodesic.

Definition 3.1. Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field V, immersed in an indefinite Para-Saskaian manifold $(\bar{M}, \bar{g})$. Then M is said to be a contact CR-lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:
$Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \phi(Rad(TM)) = \{0\}$ There exist vector bundles $\mathcal{D}_1$ and $\mathcal{D}_2$ over M such that

$$S(TM) = \{\phi(Rad(TM)) \oplus \mathcal{D}_2\} \perp \mathcal{D}_1 \perp \{V\}, \tag{3.1}$$

$$\phi\mathcal{D}_1 = \mathcal{D}_1, \ \phi\mathcal{D}_2 = L_1 \perp ltr(TM), \tag{3.2}$$

where $\mathcal{D}_1$ is non degenerate and L_1 is a vector subbundle of $S(TM^{\perp})$. Therefore

$$TM = \mathcal{D} \oplus \{V\} \oplus \mathcal{D}_2, \quad \mathcal{D} = Rad(TM) \perp \phi(Rad(TM)) \perp \mathcal{D}_1. \tag{3.3}$$

A contact CR-lightlike submanifold is proper if $\mathcal{D}_1 \neq \{0\}$ and $L_1 \neq \{0\}$. If $\mathcal{D}_1 = \{0\}$, then M is said to be anti-invariant lightlike submanifold.

Denote the orthogonal complement subbundle to $L_1 \in S(TM^{\perp})$ by $L_1^{\perp}$, therefore

$$\phi X = \nu X + \omega X, \quad \forall X \in \Gamma(TM), \tag{3.4}$$

where $\nu X \in \Gamma(\mathcal{D}), \omega X \in \Gamma(L_1 \perp ltr(TM))$ and

$$\phi W = \mathcal{B}W + \mathcal{C}W, \quad \forall W \in \Gamma(S(TM^{\perp})), \tag{3.5}$$

where $\mathcal{B}W \in \Gamma(\phi L_! \subset \Gamma(\mathcal{D}_2)), \quad \mathcal{C}W \in \Gamma(L_1^{\perp})$.
Using (2.9), (2.10), (3.4) and (3.5) in (2.5) and then comparing tangential and transversal components, we obtain

$$\begin{aligned} (\nabla_X \nu)Y &= A_{\omega Y}X + \mathcal{B}(h^s(X, Y)) + \phi(h^l(X, Y)) \\ &\quad -g(\phi X, \phi Y)V - \epsilon\eta(Y)\phi^2 X \end{aligned} \tag{3.6}$$

and

$$(\nabla^t_X \omega) = \mathcal{C}(h^s(X, Y)) - h(X, \nu Y), \tag{3.7}$$

where

$$(\nabla_X \nu)Y = \nabla_X \nu Y - \nu \nabla_X Y \tag{3.8}$$

$$(\nabla^t_X \omega)Y = \nabla^t_X \omega Y - \omega \nabla_X Y \tag{3.9}$$

Definition 3.2. If the second fundamental form h of a submanifold, tangent to structure vector field V, of an indefinite para-Sasakian manifold $\bar{M}$ is of the form

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha + \eta(X)h(Y,V) + \eta(Y)h(X,V), \quad (3.10)$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is called totally contact umbilical and totally contact geodesic if $\alpha = 0$.

For a totally contact umbilical M, we have

$$h^l(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_l + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V), \quad (3.11)$$

$$h^s(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_s + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V). \quad (3.12)$$

Theorem 3.1. *Let M be a totally contact umbilical proper contact CR-lightlike submanifold of an indefinite para-Sasakian manifold $\bar{M}$ and screen distribution be totally geodesic in M. Then*

$$\nabla_X \phi X = \phi \nabla_X X \quad (3.13)$$

for any $X \in \Gamma(\mathcal{D}_1)$.

Proof. From (3.7) and (3.9), we obtain

$$\nabla^t_X \omega Y - \omega \nabla_X Y = \mathcal{C}(h^s(X,Y)) - h(X, \nu Y). \quad (3.14)$$

Let $X \in \Gamma(\mathcal{D}_1)$, then $\phi X = \nu X$ and $\omega X = 0$, then

$$-\omega \nabla_X X = \mathcal{C}(h^s(X,X)) - h(X, \phi X). \quad (3.15)$$

Since M is totally contact umbilical then using (3.10), we obtain

$$-\omega \nabla_X X = \mathcal{C}(h^s(X,X)) - \{g(X, \phi X)\}\alpha, \quad (3.16)$$

or

$$\omega \nabla_X X + \mathcal{C}(h^s(X,X)) = 0. \quad (3.17)$$

Hence

$$\nabla_X X \in \Gamma(\mathcal{D}), \quad h^s(X,Y) \in \Gamma(L_1). \quad (3.18)$$

Since $\mathcal{D} = Rad(TM) \perp \phi(Rad(TM)) \perp \mathcal{D}_1$ therefore $\nabla_X X \in \mathcal{D}_1, \nabla_X X \in Rad(TM)$ or $\nabla_X X \in \phi(Rad(TM))$. Let $N \in Rad(TM)$ therefore using (2.11) and (2.20), we get

$$g(\nabla_X X, N) = \bar{g}(X, \bar{\nabla}_X N) = \bar{g}(h^*(\phi X, X), N). \quad (3.19)$$

Also, using 2.21 and (2.25), we obtain

$$g(\nabla_X X, \phi N) = \bar{g}(X, \phi \bar{\nabla}_X N) = \bar{g}(h^*(\phi X, X), N). \quad (3.20)$$

Since screen distribution is totally geodesic in M therefore $\nabla_X X \in \mathcal{D}_1$. Let $X, Y \in \Gamma(\mathcal{D}_1)$, then

$$\begin{aligned} g(\nabla_X \phi X, Y) &= \bar{g}(\bar{\nabla}_X \phi X, Y) \\ &= \bar{g}(\phi \bar{\nabla}_X X - g(\phi X, \phi Y) - \epsilon\eta(X)\phi^2 X, Y) \\ &= \bar{g}(\phi \bar{\nabla}_X \phi X, Y). \end{aligned}$$

Now, using (2.21) ,(2.22), (2.25) and $\eta(X) = 0$, we obtain

$$g(\nabla_X \phi X, Y) = \bar{g}(\bar{\nabla}_X X, \phi Y) = g(\nabla_X X, \phi Y) = g(\phi \nabla_X X, Y), \tag{3.21}$$

then non degeneracy of $\mathcal{D}_1$, gives the result.

Lemma 3.1. *Let M be a totally contact umbilical proper contact CR-lightlike submanifold of an indefinite para-Sasakian manifold $\bar{M}$. Then $\alpha_l = 0$*

Proof. Proof of this Lemaa is same as in [9].

Theorem 3.2. *Let M be a totally contact umbilical proper contact CR-lightlike submanifold of an indefinite para-Sasakian manifold $\bar{M}$ and screen distribution be totally geodesic in M. Then M is a totally contact geodesic.*

Proof. For $X \in \Gamma(\mathcal{D}_1)$ and $W \in \Gamma(S(TM^\perp))$, using (2.25), (3.13) and Theorem 3.1, we obtain

$$\begin{aligned} \bar{g}(\phi \bar{\nabla}_X X, \phi W) &= \bar{g}(\bar{\nabla}_X \phi X + g(\phi X, \phi X)V, \phi W) = \bar{g}(\bar{\nabla}_X \phi X, \phi W) \\ &= \bar{g}(\nabla_X \phi X + h^s(X, \phi X), \mathcal{B}W + \mathcal{C}W) \\ &= \bar{g}(\nabla_X \phi X, \phi W) + \bar{g}(h^s(X, \phi X), \mathcal{C}W) \\ &= \bar{g}(\nabla_X \phi X, \phi W) + g(X, \phi X)\bar{g}(\alpha_s, \mathcal{C}W) \\ &= \bar{g}(\nabla_X X, W) = 0. \end{aligned} \tag{3.22}$$

Now, using (2.21), (3.12) and (3.13), we obtain

$$\begin{aligned} \bar{g}(\phi \bar{\nabla}_X X, \phi W) &= \bar{g}(\bar{\nabla}_X \phi X, \phi W) \\ &= g(\nabla_X X, W) + g(h^s(X, X), W) \\ &= g(h^s(X, X), W) = g(X, X)g(\alpha_s, W), \end{aligned} \tag{3.23}$$

therefore from (3.14) and (3.15), we obtain

$$g(X, X)g(\alpha_s, W) = 0, \tag{3.24}$$

then non degeneracy of $\mathcal{D}_1$ and $S(TM^\perp)$ implies that $\alpha_s = 0$,
Using this with Lemma 3.1, we obtain our result.

Example 3.1. Let M be a lightlike hypersurface of $\bar{M}$. For $\xi \in \Gamma(Rad(TM))$, we get $g(\phi\xi, \xi) = 0$ this implies $\phi\xi \in \Gamma(TM)$ and we have a rank-1 distribution $\phi(TM^\perp)$ on M such that $\phi(TM^\perp \cap TM^\perp) = \{0\}$. This implies that $\phi(TM^\perp)$ is a vector subbundle of $S(TM)$. Since for any $N \in \Gamma(ltr(TM))$, $\bar{g}(\phi N, \xi) = \bar{g}(N, \phi\xi) = 0$ and $\bar{g}(\phi N, N) = 0$, therefore $\phi N \in \Gamma(S(TM))$. Taking $\mathcal{D}_2 = \phi(tr(TM))$, we obtain $S(TM) = \{\phi(TM^\perp) \oplus \mathcal{D}_2\} \perp \mathcal{D}_1$, where $\mathcal{D}_1$ is non degenerate distribution and distribution and $\phi(\mathcal{D}_2 = tr(TM))$. Hence, M is a contact CR-lightlike hypersurface.

4. Integrability of distributions of contact CR-lightlike submanifold of indefinite para-Sasakian manifold.

In this section, we obtain the integrability conditions of distributions. Further, we obtain necessary and sufficient condition for a CR-lightlike submanifold to be an anti-invariant submanifold. Also, we characterize a contact CR-lightlike submanifold of indefinite para-Sasakian manifold to be a contact CR-lightlike product.

Theorem 4.1. *Let M be a contact CR-lightlike submanifold of an indefinite para-Sasakian manifold $\bar{M}$. Then $\mathcal{D} \oplus \{V\}$ is integrable if and only if*

$$h(X, \phi Y) = h(\phi X, Y), \tag{4.1}$$

for any $X, Y \in \Gamma(\mathcal{D} \oplus \{V\})$.

Proof. Proof of this Theorem is same as in [9], so we omit it.

Theorem 4.2. *Let M be a contact CR-lightlike submanifold of an indefinite para-Sasakian manifold $\bar{M}$. Then $\mathcal{D}_2$ is integrable if and only if*

$$A_{\phi W} Z = A_{\phi Z} W, \tag{4.2}$$

for any $W, Z \in \Gamma(\mathcal{D}_2)$.

Proof. Using (3.6) and (3.8) for any $W, Z \in \Gamma(\mathcal{D}_2)$, we get

$$-\nu \nabla_Z W = A_{\omega W} Z + \mathcal{B} h^s(Z, W) + \phi h^l(Z, W) - g(\phi Z, \phi W) V. \tag{4.3}$$

Then, we have

$$-\nu [Z, W] = A_{\phi W} Z - A_{\phi Z} W, \tag{4.4}$$

Which completes the proof.

Lemma 4.1. *For $Y \in \Gamma(\mathcal{D}_2)$ and $Z \in \Gamma(\mathcal{D})$, we have*

$$g(\nabla_X Y, Z) = -g(\nu A_{\omega Y} X, Z),$$

Proof. From (3.6), we obtain

$$\begin{aligned} \nabla_X \nu Y - \nu \nabla_X Y &= A_{\omega Y} X + \mathcal{B}(h^s(X, Y)) + \phi(h^l(X, Y)) \\ &\quad - g(\phi X, \phi Y) - \epsilon \eta(Y) \phi^2 X. \end{aligned}$$

Let $Y \in \Gamma(\mathcal{D}_2)$ then $\phi Y = \omega Y,\ \nu Y = 0$, therefore, we get

$$\nu \nabla_X Y = -A_{\omega Y} X - \mathcal{B}(h^s(X, Y)) - \phi(h^l(X, Y)) + g(\phi X, \phi Y) V.$$

Let $Y \in \Gamma(\mathcal{D})$ then $\phi Z \in \mathcal{D}$, therefore

$$\begin{aligned} g(\nu \nabla_X Y, \phi Z) &= -g(A_{\omega Y} X, \phi Z) - g(\mathcal{B}(h^s(X, Y)), \phi Z) - g(\phi(h^l(X, Y)), \phi Z) \\ &+ g(\phi X, \phi Y) g(V, \phi Z) \\ &= g(A_{\omega Y} X, \phi Z). \end{aligned}$$

Now, using (2.21), we have

$$g(\nabla_X Y, Z) - \epsilon\eta(\nabla_X Y)\eta(Z) = g(A_{\omega Y}X, \phi Z).$$

This implies $g(\nabla_X Y, Z) = g(\nu A_{\omega Y}X, Z)$. Which completes the proof.
Next, for any $U \in \Gamma(tr(TM))$, we put

$$\phi U = \mathcal{P}U + \mathcal{Q}U, \tag{4.5}$$

where $\mathcal{P}U$ and $\mathcal{Q}U$ are tangential and transversal component of ϕU, respectively. Using (2.22), we have

$$\nu^2 = I - \mathcal{P}\omega - \eta \otimes V, \tag{4.6}$$

$$\omega\nu + \mathcal{Q}\omega = 0, \tag{4.7}$$

$$\mathcal{Q}^2 = I - \omega\mathcal{P}, \tag{4.8}$$

$$\nu\mathcal{P} + \mathcal{P}\mathcal{Q} = 0, \tag{4.9}$$

clearly, if M is tangent to structure vector field V, then it is a CR-submanifold if and only if one of the following condition is satisfied [9]:

$$\omega\mathcal{Q} = 0,\ \mathcal{Q}\omega = 0,\ \nu\mathcal{P} = 0,\ \mathcal{P}\mathcal{Q} = 0 \tag{4.10}$$

Theorem 4.3. *Let M be a contact CR-lightlike submanifold of an indefinite para-Sasakian manifold $\bar{M}$. Then ν is parallel if and only if M is an anti-invariant submanifold.*

Proof. Taking $Y = V$ in (3.6), we obtain

$$(\nabla_X \nu)V = A_{\omega V}X + \mathcal{P}(h(X, V)) - g(\phi X, \phi Y) - \epsilon\eta(V)\phi^2 X. \tag{4.11}$$

Since $\phi V = 0$ hence $\omega V = 0 = \nu V$. Therefore

$$(\nabla_X \nu)V = X - \epsilon\eta(X)V + \mathcal{P}(h(X, V)). \tag{4.12}$$

Let ν be parallel then we get

$$X - \epsilon\eta(X)V + \mathcal{P}(h(X, V)) = 0, \tag{4.13}$$

operating ν to this equation then using (4.8), we have $\nu X = 0$. Hence M is an anti-invariant submanifold. Converse is obvious from (3.8).

Theorem 4.4. *Let M be a contact CR-lightlike submanifold of an indefinite para-Sasakian manifold $\bar{M}$. Then $\mathcal{D} \oplus \{V\}$ defines totally geodesic foliation in M if and only if*

$$h^l(X, \phi Y) = 0,\ \mathcal{B}h^s(X, \phi Y) = 0, \tag{4.14}$$

for any $X, Y \in \mathcal{D} \oplus \{V\}$.

Proof. Since $\mathcal{D}\oplus\{V\}$ defines a totally geodesic foliation in M if and only if $\nabla_X Y \in \Gamma(\mathcal{D}\oplus\{V\})$, for any vector $X, Y \in \mathcal{D}\oplus\{V\}$. Also $\mathcal{D}_2 = \phi(L_1 \perp ltr(TM))$, therefore $\nabla_X Y \in \Gamma(\mathcal{D}\oplus\{V\})$ if and only if

$$\bar{g}(\nabla_X Y, \phi W_i) = 0, \quad i \in \{1, ..., s\}, \tag{4.15}$$

and

$$\bar{g}(\nabla_X Y, \phi \xi_i) = 0, \quad j \in \{1, ..., r\}, \tag{4.16}$$

where $\{N_1, ..., N_r\}$ is a basis of $\Gamma(ltr(TM))$ with respect to the basis $\{\xi_1, ..., \xi_r\}$ of $\Gamma(Rad(TM))$ and $\{W_1, ..., W_s\}$ is a basis of $\Gamma(L_1)$. Using (2.9), (2.21), (2.22) and (2.25), we obtain

$$\bar{g}(\nabla_X Y, \phi W_i) = \bar{g}(\bar{\nabla}_X \phi Y, W_i) = -\bar{g}(h^s(X, \phi Y), W_i), \tag{4.17}$$

and similarly, we also have

$$\bar{g}(\nabla_X Y, \phi \xi_i) = -\bar{g}(h^l(X, \phi Y), \xi_j), \tag{4.18}$$

By (4.15) and (4.16), we get our result.

Definition 4.1. A contact CR-lightlike submanifold M of an indefinite para-Sasakian manifold $\bar{M}$ is called contact CR-lightlike product if both the distribution $\mathcal{D}\oplus\{V\}$ and $\mathcal{D}_2$ define totally geodesic foliation in M.

Theorem 4.5. *A CR-submanifold of a Kaehler manifold $\bar{M}$ is a CR-product if and only if P is parallel, i.e. $\bar{\nabla}P = 0$, where $\bar{J}X = PX + FX$*

Now, we give the characterization of contact CR-lightlike product of an indefinite para-Sasakian manifold.

Theorem 4.6. *Let M be a contact CR-lightlike submanifold of an indefinite para-Sasakian manifold $\bar{M}$ if ν is parallel, i.e. $\nabla\nu = 0$, then M is a contact CR-lightlike product.*

Proof. Let $X, Y \in \Gamma(\mathcal{D}_2)$ then $(\nabla_X \nu)Y = 0$ implies that $(\nabla_X \nu)Y - \nu(\nabla_x Y) = 0$. Since $Y \in \Gamma(\mathcal{D}_2)$, therefore $\nu Y = 0$, hence $\nu \nabla_X Y = 0, \forall X, Y \in \Gamma(\mathcal{D}_2)$, this implies that the distribution $\mathcal{D}_2$ defines a totally geodesic foliation in M.
Let $X, Y \in \mathcal{D}\oplus\{V\}$, then $\omega Y = 0$ and using (3.6), we have

$$(\nabla_X \nu)Y = \mathcal{B}(h^s(X, Y)) + \phi(h^l(X, Y)) - g(\phi X, \phi Y)V - \epsilon\eta(Y)\phi^2 X. \tag{4.19}$$

Taking into account that ν is parallel, we obtain

$$\mathcal{B}(h^s(X, Y)) + \phi(h^l(X, Y)) - g(\phi X, \phi Y)V - \epsilon\eta(Y)\phi^2 X = 0. \tag{4.20}$$

Comparing the transversal components, we have

$$\mathcal{B}(h^s(X, Y)) = 0, \tag{4.21}$$

and

$$h^l(X, Y) = 0. \tag{4.22}$$

Hence, the distribution $\mathcal{D}\oplus\{V\}$ defines a totally geodesic foliation in M. Consequently, M is a contact CR-lightlike product of an indefinite para-Sasakian manifold.

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