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AW(k)-TYPE CURVES ACCORDING TO PARALLEL TRANSPORT FRAME IN EUCLIDEAN SPACE $\mathbb{E}^4$

İlim Kişi, Sezgin Büyükkütük, Deepmala[1] and Günay Öztürk

Abstract. In this paper, we study AW(k)-type ($k = 1, 2, ..., 7$) curves according to the parallel transport frame in Euclidean space $\mathbb{E}^4$. We give the classification of these types curves with the parallel transport curvatures (Bishop curvatures). Finally, we consider the curvatures k_1, k_2, k_3 as constants respectively and give the relations between the parallel transport curvatures of AW(k)-type ($k = 1, 2, ..., 7$) curves.

Keywords: AW(k)-type curves, Parallel transport frame

1. Introduction

The Frenet frame can be constructed for a 3−time continuously differentiable curve. But, for some points the second derivative may vanish. Namely, the curvature may be zero. In this case, we must use a new frame in $\mathbb{E}^3$. Because of that, in [6], Bishop established a new frame named 'Bishop frame'. This frame is well defined even though the curve's second derivative vanishes. In [6, 8], the authors gave the positive features of the Bishop frame and the comparison of Frenet frame with the Bishop frame in Euclidean 3−space. In Euclidean 4−space $\mathbb{E}^4$, we have the same problem for a curve like being in Euclidean 3−space. That is, one of the $i - th$ ($1 < i < 4$) derivative of the curve may vanish. In this case, we must use a new frame.

In [7], using the similar idea authors considered such curves and constructed an alternative frame. They gave parallel transport frame of a curve, and they introduced the relationship between the Frenet frame and the parallel transport frame of the curve in $\mathbb{E}^4$. They generalized the relation to Euclidean space $\mathbb{E}^4$.

In [2], Arslan and West defined submanifolds of $AW(k)$-type. Especially, several authors have done many works related the curves of $AW(k)$-type. For instance, in [3], the authors obtained curvature conditions and characterized these curves in $\mathbb{E}^n$. In [9], $AW(k)$ ($k = 1, 2$ or 3)-type curves and surfaces were discussed. Further,

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[1]Corresponding author.

associated examples about curves and surfaces which satisfy $AW(k)$-type conditions were given.

In [10], the authors considered $AW(k)$-type curves in accordance with Bishop Frame in $\mathbb{E}^3$. Also, they mentioned the relationship between Bishop curvatures (k_1, k_2) for these type curves in $\mathbb{E}^3$.

Furthermore, in [1], the authors considered a generalization of $AW(k)$-type ($k = 1, 2, ..., 7$) curves in Euclidean n-space $\mathbb{E}^n$. They obtained the curvature conditions for these curves. In [4], the authors characterized curves in Galilean 3-space. In [5], the authors gave a short and understandable exposition on differential operators over modules and rings as a path to the generalized differential geometry. Also in [11], the authors studied projectively flatness of a new class of (α, β)-metrics.

In this paper, we study AW(k)-type ($k = 1, 2, ..., 7$) curves according to the parallel transport frame in Euclidean space $\mathbb{E}^4$. We give the classification of these types curves with the parallel transport curvatures (Bishop curvatures). Finally, we consider the curvatures k_1, k_2, k_3 as constants respectively and give the relations between the parallel transport curvatures of AW(k)-type ($k = 1, 2, ..., 7$) curves.

2. Basic Concepts

Let $\gamma = \gamma(s) : I \to \mathbb{E}^4$ be a curve in the Euclidean 4−space $\mathbb{E}^4$, where I is interval in $\mathbb{R}$. γ is called unit speed if $\|\gamma\,'(s)\| = 1$ (parametrized by arclength functions). Then the Frenet formulae of γ are:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where $\{T, N, B_1, B_2\}$ is the Frenet frame of γ and κ, τ, and σ are the curvature functions.

In [7], authors used $T(s)$ which is tangent to γ and the parallel transport vector fields $M_1(s)$, $M_2(s)$, and $M_3(s)$ to construct an alternative frame.

Theorem 2.1. *[7] Let $\{T, N, B_1, B_2\}$ be a Frenet frame along a unit speed curve $\gamma = \gamma(s) : I \to \mathbb{E}^4$ and $\{T, M_1, M_2, M_3\}$ denotes the parallel transport frame of the curve γ. The relationship may be given with*

$$\begin{aligned}
T &= T(s) \\
N &= \cos\theta(s)\cos\psi(s)M_1 + (-\cos\phi(s)\sin\psi(s) + \sin\phi(s)\sin\theta(s)\cos\psi(s))M_2 \\
&\quad +(\sin\phi(s)\sin\psi(s) + \cos\phi(s)\sin\theta(s)\cos\psi(s))M_3 \\
B_1 &= \cos\theta(s)\sin\psi(s)M_1 + (\cos\phi(s)\cos\psi(s) + \sin\phi(s)\sin\theta(s)\sin\psi(s))M_2 \\
&\quad +(-\sin\phi(s)\cos\psi(s) + \cos\phi(s)\sin\theta(s)\sin\psi(s))M_3 \\
B_2 &= -\sin\theta(s)M_1 + \sin\phi(s)\cos\theta(s)M_2 + \cos\phi(s)\cos\theta(s)M_3
\end{aligned}$$

where θ, ψ and ϕ are the Euler angles. Then the alternative parallel frame equations are

$$\begin{bmatrix} T\,' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \tag{2.1}$$

where k_1, k_2 and k_3 are principal curvature functions according to parallel transport frame of the curve γ and their expressions are as follows:

$$\begin{aligned} k_1 &= \kappa \cos\theta \cos\psi, \\ k_2 &= \kappa(-\cos\phi \sin\psi + \sin\phi \sin\theta \cos\psi), \\ k_3 &= \kappa(\sin\phi \sin\psi + \cos\phi \sin\theta \cos,\psi), \end{aligned}$$

where $\theta' = \frac{\sigma}{\sqrt{\kappa^2+\tau^2}}$, $\psi' = -\tau - \sigma\frac{\sqrt{\sigma^2-\theta'^2}}{\sqrt{\kappa^2+\tau^2}}$, $\phi' = -\frac{\sqrt{\sigma^2-\theta'^2}}{\cos\theta}$ and the following equalities

$$\begin{aligned} \kappa &= \sqrt{k_1^2+k_2^2+k_3^2}, \\ \tau &= -\psi' + \phi' \sin\theta, \\ \sigma &= \frac{\theta'}{\sin\psi}, \\ \phi' \cos\theta + \theta' \cot\psi &= 0 \end{aligned}$$

hold.

In [1], the authors obtained the higher order derivatives of γ in $\mathbb{E}^4$ as follows:

$$\begin{aligned} \gamma''(s) &= \kappa N, \\ \gamma'''(s) &= -\kappa^2 T + \kappa' N + \kappa\tau B_1, \\ \gamma^{(iv)}(s) &= -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B_1 + \kappa\tau\sigma B_2, \\ \gamma^{(v)}(s) &= (-3\kappa'^2 - 4\kappa\kappa'' + \kappa^4 + \kappa^2\tau^2)T + (\kappa''' - 6\kappa^2\kappa' - 3\kappa'\tau^2 - 3\kappa\tau\tau')N \\ &\quad +(3\kappa''\tau + 3\kappa'\tau' - \kappa^3\tau - \kappa\tau^3 + \kappa\tau'' - \kappa\tau\sigma^2)B_1 \\ &\quad +(3\kappa'\tau\sigma + 2\kappa\tau'\sigma + \kappa\tau\sigma')B_2. \end{aligned}$$

And also they gave the following notation and definition:

Notation:

$$\begin{aligned} N_1 &= \kappa N, \\ N_2 &= \kappa' N + \kappa\tau B_1, \\ N_3 &= \lambda N + \lambda_1 B_1 + \lambda_2 B_2, \\ N_4 &= \mu N + \mu_1 B_1 + \mu_2 B_2, \end{aligned}$$

where

$$\begin{aligned} \lambda &= \kappa'' - \kappa^3 - \kappa\tau^2, \\ \lambda_1 &= 2\kappa'\tau + \kappa\tau', \\ \lambda_2 &= \kappa\tau\sigma, \end{aligned}$$

and

$$\begin{aligned} \mu &= \kappa''' - 6\kappa^2\kappa' - 3\kappa'\tau^2 - 3\kappa\tau\tau', \\ \mu_1 &= 3\kappa''\tau + 3\kappa'\tau' - \kappa^3\tau - \kappa\tau^3 + \kappa\tau'' - \kappa\tau\sigma^2, \\ \mu_2 &= 3\kappa'\tau\sigma + 2\kappa\tau'\sigma + \kappa\tau\sigma' \end{aligned}$$

are differentiable functions.

Definition 2.1. [1] Frenet curves are

i) of $GAW\,(1)$ type if they satisfy

$$N_4 = 0,$$

ii) of $GAW\,(2)$ type if they satisfy

$$\|N_2\|^2\, N_4 = \langle N_2, N_4\rangle\, N_2,$$

iii) of $GAW(3)$ type if they satisfy

$$\|N_1\|^2\, N_4 = \langle N_1, N_4\rangle\, N_1,$$

iv) of $GAW\,(4)$ type if they satisfy

$$\|N_3\|^2\, N_4 = \langle N_3, N_4\rangle\, N_3,$$

v) of $GAW(5)$ type if they satisfy

$$N_4 = a_1 N_1 + b_1 N_2,$$

vi) of $GAW(6)$ type if they satisfy

$$N_4 = a_2 N_1 + b_2 N_3,$$

vii) of $GAW\,(7)$ type if they satisfy

$$N_4 = a_3 N_2 + b_3 N_3,$$

where a_i, b_i $(1 \leq i \leq 3)$ are non-zero real valued differentiable functions.

3. AW(k)-Type Curves with Parallel Transport Frame in $\mathbb{E}^4$

In this section, we consider $GAW(k)$-type curves according to the parallel transport frame in Euclidean space $\mathbb{E}^4$.

Let $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. By the use of parallel transport frame formulas (2.1), we obtain the higher order derivatives of γ as follows:

$$\begin{aligned}
\gamma''(s) &= T'(s) = k_1M_1 + k_2M_2 + k_3M_3,\\
\gamma'''(s) &= \left\{-k_1^2 - k_2^2 - k_3^2\right\}T + k_1'M_1 + k_2'M_2 + k_3'M_3,\\
\gamma^{(iv)}(s) &= \{-3k_1k_1' - 3k_2k_2' - 3k_3k_3'\}T\\
&\quad + \left\{k_1'' - k_1^3 - k_1k_2^2 - k_1k_3^2\right\}M_1\\
&\quad + \left\{k_2'' - k_2^3 - k_1^2k_2 - k_3^2k_2\right\}M_2\\
&\quad + \left\{k_3'' - k_3^3 - k_1^2k_3 - k_2^2k_3\right\}M_3,\\
\gamma^{(v)}(s) &= \left\{\begin{array}{c} -3k_1'^2 - 3k_2'^2 - 3k_3'^2 - 4k_1k_1'' - 4k_2k_2'' - 4k_3k_3'' \\ +k_1^4 + k_2^4 + k_3^4 + 2k_1^2k_2^2 + 2k_1^2k_3^2 + 2k_2^2k_3^2 \end{array}\right\}T\\
&\quad + \left\{-6k_1^2k_1' - 5k_1k_2k_2' - 5k_1k_3k_3' + k_1''' - k_1'k_2^2 - k_1'k_3^2\right\}M_1\\
&\quad + \left\{-6k_2^2k_2' - 5k_1k_2k_1' - 5k_3k_2k_3' + k_2''' - k_1^2k_2' - k_3^2k_2'\right\}M_2\\
&\quad + \left\{-6k_3^2k_3' - 5k_1k_3k_1' - 5k_2k_3k_2' + k_3''' - k_1^2k_3' - k_2^2k_3'\right\}M_3.
\end{aligned}$$

Then we give the following notation:

Notation:

$$\begin{aligned}
\overline{N_1} &= k_1M_1 + k_2M_2 + k_3M_3,\\
\overline{N_2} &= k_1'M_1 + k_2'M_2 + k_3'M_3,\\
\overline{N_3} &= \phi_1M_1 + \phi_2M_2 + \phi_3M_3,\\
\overline{N_4} &= \psi_1M_1 + \psi_2M_2 + \psi_3M_3,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\phi_1 &= k_1'' - k_1^3 - k_1k_2^2 - k_1k_3^2,\\
\phi_2 &= k_2'' - k_2^3 - k_1^2k_2 - k_3^2k_2,\\
\phi_3 &= k_3'' - k_3^3 - k_1^2k_3 - k_2^2k_3,
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\psi_1 &= -6k_1^2k_1' - 5k_1k_2k_2' - 5k_1k_3k_3' + k_1''' - k_1'k_2^2 - k_1'k_3^2,\\
\psi_2 &= -6k_2^2k_2' - 5k_1k_2k_1' - 5k_2k_3k_3' + k_2''' - k_1^2k_2' - k_3^2k_2',\\
\psi_3 &= -6k_3^2k_3' - 5k_1k_3k_1' - 5k_2k_3k_2' + k_3''' - k_1^2k_3' - k_2^2k_3'
\end{aligned} \tag{3.3}$$

are differentiable functions.

Definition 3.1. Let γ be a unit speed curve *in* $\mathbb{E}^4$. According to its *parallel transport frame,* γ is

$i)$ of $PAW\,(1)$ type if it satisfies

$$\overline{N_4} = 0, \tag{3.4}$$

$ii)$ of $PAW\,(2)$ type if it satisfies

$$\left\|\overline{N_2}\right\|^2 \overline{N_4} = \left\langle \overline{N_2}, \overline{N_4} \right\rangle \overline{N_2}, \tag{3.5}$$

$iii)$ of $PAW(3)$ type if it satisfies

$$\left\|\overline{N_1}\right\|^2 \overline{N_4} = \left\langle \overline{N_1}, \overline{N_4} \right\rangle \overline{N_1}, \tag{3.6}$$

$iv)$ of $PAW\,(4)$ type if it satisfies

$$\left\|\overline{N_3}\right\|^2 \overline{N_4} = \left\langle \overline{N_3}, \overline{N_4} \right\rangle \overline{N_3}, \tag{3.7}$$

$v)$ of $PAW(5)$ type if it satisfies

$$\overline{N_4} \ = a_1\overline{N_1} + b_1\overline{N_2}, \tag{3.8}$$

$vi)$ of $PAW(6)$ type if it satisfies

$$\overline{N_4} = a_2\overline{N_1} + b_2\overline{N_3}, \tag{3.9}$$

$vii)$ of $PAW\,(7)$ type if it satisfies

$$\overline{N_4} = a_3\overline{N_2} + b_3\overline{N_3}, \tag{3.10}$$

where a_i, b_i $(1 \leq i \leq 3)$ are non-zero real valued differentiable functions.

Theorem 3.1. *Let γ be a unit speed curve in $\mathbb{E}^4$. According to its parallel transport frame, γ is*

$i)$ *of $PAW(1)$ type if and only if*

$$\begin{aligned} -6k_1^2k_1' - 5k_1k_2k_2' - 5k_1k_3k_3' + k_1''' - k_1'k_2^2 - k_1'k_3^2 &= 0, \\ -6k_2^2k_2' - 5k_1k_2k_1' - 5k_3k_2k_3' + k_2''' - k_1^2k_2' - k_3^2k_2' &= 0, \\ -6k_3^2k_3' - 5k_1k_3k_1' - 5k_2k_3k_2' + k_3''' - k_1^2k_3' - k_2^2k_3' &= 0. \end{aligned} \tag{3.11}$$

$ii)$ *of $PAW(2)$ type if and only if*

$$\begin{aligned} ({k_2'}^2 + {k_3'}^2)\psi_1 &= k_1'(k_2'\psi_2 + k_3'\psi_3), \\ ({k_1'}^2 + {k_3'}^2)\psi_2 &= k_2'(k_1'\psi_1 + k_3'\psi_3), \\ ({k_1'}^2 + {k_2'}^2)\psi_3 &= k_3'(k_1'\psi_1 + k_2'\psi_2). \end{aligned} \tag{3.12}$$

iii) of $PAW(3)$ type if and only if

$$(3.13)\qquad \begin{aligned} (k_2^2+k_3^2)\psi_1 &= k_1(k_2\psi_2+k_3\psi_3),\\ (k_1^2+k_3^2)\psi_2 &= k_2(k_1\psi_1+k_3\psi_3),\\ (k_1^2+k_2^2)\psi_3 &= k_3(k_1\psi_1+k_2\psi_2). \end{aligned}$$

iv) of $PAW(4)$ type if and only if

$$(3.14)\qquad \begin{aligned} (\phi_2^2+\phi_3^2)\psi_1 &= \phi_1(\phi_2\psi_2+\phi_3\psi_3),\\ (\phi_1^2+\phi_3^2)\psi_2 &= \phi_2(\phi_1\psi_1+\phi_3\psi_3),\\ (\phi_1^2+\phi_2^2)\psi_3 &= \phi_3(\phi_1\psi_1+\phi_2\psi_2). \end{aligned}$$

v) of $PAW(5)$ type if and only if

$$(3.15)\qquad \begin{aligned} \psi_1 &= a_1k_1+b_1k_1',\\ \psi_2 &= a_1k_2+b_1k_2',\\ \psi_3 &= a_1k_3+b_1k_3'. \end{aligned}$$

vi) of $PAW(6)$ type if and only if

$$(3.16)\qquad \begin{aligned} \psi_1 &= a_2k_1+b_2\phi_1,\\ \psi_2 &= a_2k_2+b_2\phi_2,\\ \psi_3 &= a_2k_3+b_2\phi_3. \end{aligned}$$

vii) of $PAW(7)$ type if and only if

$$(3.17)\qquad \begin{aligned} \psi_1 &= a_3k_1'+b_3\phi_1,\\ \psi_2 &= a_3k_2'+b_3\phi_2,\\ \psi_3 &= a_3k_3'+b_3\phi_3. \end{aligned}$$

Proof. i) Let γ be of $PAW(1)$-type. Then from the equations (3.1) and (3.4), we have $\overline{N_4}=\psi_1M_1+\psi_2M_2+\psi_3M_3=0$. Since M_1,M_2,M_3 are linearly independent, we obtain $\psi_1=\psi_2=\psi_3=0$, which means

$$(3.18)\qquad \begin{aligned} -6k_1^2k_1'-5k_1k_2k_2'-5k_1k_3k_3'+k_1'''-k_1'k_2^2-k_1'k_3^2 &= 0,\\ -6k_2^2k_2'-5k_1k_2k_1'-5k_3k_2k_3'+k_2'''-k_1^2k_2'-k_3^2k_2' &= 0,\\ -6k_3^2k_3'-5k_1k_3k_1'-5k_2k_3k_2'+k_3'''-k_1^2k_3'-k_2^2k_3' &= 0. \end{aligned}$$

The sufficiency is trivial.

ii) Let γ be of $PAW(2)$-type. If we calculate $\left\|\overline{N_2}\right\|^2$ and $\left\langle \overline{N_2},\overline{N_4}\right\rangle$, by the use of equations (3.1) and (3.5), we get

(3.19)

$$(k_1'^2+k_2'^2+k_3'^2)(\psi_1M_1+\psi_2M_2+\psi_3M_3)=(k_1'\psi_1+k_2'\psi_2+k_3'\psi_3)(k_1'M_1+k_2'M_2+k_3'M_3),$$

which means

$$(3.20)\qquad \begin{aligned} (k_2'^2 + k_3'^2)\psi_1 &= k_1'(k_2'\psi_2 + k_3'\psi_3), \\ (k_1'^2 + k_3'^2)\psi_2 &= k_2'(k_1'\psi_1 + k_3'\psi_3), \\ (k_1'^2 + k_2'^2)\psi_2 &= k_3'(k_1'\psi_1 + k_2'\psi_2). \end{aligned}$$

Conversely, if the equations (3.20) are satisfied, by the equation (3.5), γ is of $PAW(2)$-type.

iii) Let γ be of $PAW(3)$-type. If we calculate $\left\|\overline{N_1}\right\|^2$, $\langle \overline{N_1}, \overline{N_4} \rangle$ and substitute them in the equation (3.6), we get

$$(3.21)\qquad \begin{aligned} &(k_1^2 + k_2^2 + k_3^2)(\psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3) = \\ &(k_1\psi_1 + k_2\psi_2 + k_3\psi_3)(k_1 M_1 + k_2 M_2 + k_3 M_3), \end{aligned}$$

which means

$$(3.22)\qquad \begin{aligned} (k_2^2 + k_3^2)\psi_1 &= k_1(k_2\psi_2 + k_3\psi_3), \\ (k_1^2 + k_3^2)\psi_2 &= k_2(k_1\psi_1 + k_3\psi_3), \\ (k_1^2 + k_2^2)\psi_3 &= k_3(k_1\psi_1 + k_2\psi_2). \end{aligned}$$

Conversely, if the equations (3.22) are satisfied, by the equation (3.6), γ is of $PAW(3)$-type.

iv) Let γ be of $PAW(4)$-type. If we calculate $\left\|\overline{N_3}\right\|^2$, $\langle \overline{N_3}, \overline{N_4} \rangle$ and substitute them in (3.7), we get

$$(3.23)\qquad \begin{aligned} &(\phi_1^2 + \phi_2^2 + \phi_3^2)(\psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3) = \\ &(\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3)(\phi_1 M_1 + \phi_2 M_2 + \phi_3 M_3), \end{aligned}$$

which means

$$(3.24)\qquad \begin{aligned} (\phi_2^2 + \phi_3^2)\psi_1 &= \phi_1(\phi_2\psi_2 + \phi_3\psi_3), \\ (\phi_1^2 + \phi_3^2)\psi_2 &= \phi_2(\phi_1\psi_1 + \phi_3\psi_3), \\ (\phi_1^2 + \phi_2^2)\psi_3 &= \phi_3(\phi_1\psi_1 + \phi_2\psi_2). \end{aligned}$$

Conversely, if the equations (3.24) are satisfied, by the equation (3.7), γ is of $PAW(4)$-type.

v) Let γ be of $PAW(5)$-type. In view of the equations (3.1) and (3.8), we can write

$$(3.25)\qquad \psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3 = a_1(k_1 M_1 + k_2 M_2 + k_3 M_3) + b_1(k_1' M_1 + k_2' M_2 + k_3' M_3),$$

which gives us

$$(3.26)\qquad \begin{aligned} \psi_1 &= a_1 k_1 + b_1 k_1', \\ \psi_2 &= a_1 k_2 + b_1 k_2', \\ \psi_3 &= a_1 k_3 + b_1 k_3'. \end{aligned}$$

Conversely, if the equations (3.26) are satisfied, by the equation (3.8), γ is of $PAW(5)$-type.

vi) Let γ be of $PAW(6)$-type. In view of the equations (3.1) and (3.9), we can write

$$\psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3 = a_2(k_1 M_1 + k_2 M_2 + k_3 M_3) + b_2(\phi_1 M_1 + \phi_2 M_2 + \phi_3 M_3), \tag{3.27}$$

that means

$$\begin{aligned} \psi_1 &= a_2 k_1 + b_2 \phi_1, \\ \psi_2 &= a_2 k_2 + b_2 \phi_2, \\ \psi_3 &= a_2 k_3 + b_2 \phi_3. \end{aligned} \tag{3.28}$$

Conversely, if the equations (3.28) are satisfied, by the equation (3.9), γ is of $PAW(6)$-type.

vii) Let γ be of $PAW(7)$-type. In view of equations (3.1) and (3.10), we can write

$$\psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3 = a_3(k_1' M_1 + k_2' M_2 + k_3' M_3) + b_3(\phi_1 M_1 + \phi_2 M_2 + \phi_3 M_3), \tag{3.29}$$

which means

$$\begin{aligned} \psi_1 &= a_3 k_1' + b_3 \phi_1, \\ \psi_2 &= a_3 k_2' + b_3 \phi_2, \\ \psi_3 &= a_3 k_3' + b_3 \phi_3. \end{aligned} \tag{3.30}$$

Conversely, if the equations (3.30) are satisfied, by the equation (3.10), γ is of $PAW(7)$-type. □

From now on, we consider space curves whose curvatures k_1 is non-zero constant, k_2 and k_3 are not constants. We give curvature conditions of such a curve to be of $PAW(k)$-type. In this case, we obtain;

$$\begin{aligned} \overline{N_1} &= k_1 M_1 + k_2 M_2 + k_3 M_3, \\ \overline{N_2} &= k_2' M_2 + k_3' M_3, \\ \overline{N_3} &= \phi_{11} M_1 + \phi_{21} M_2 + \phi_{31} M_3, \\ \overline{N_4} &= \psi_{11} M_1 + \psi_{21} M_2 + \psi_{31} M_3, \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} \phi_{11} &= -k_1^3 - k_1 k_2^2 - k_1 k_3^2 \ , \\ \phi_{21} &= k_2'' - k_2^3 - k_1^2 k_2 - k_3^2 k_2, \\ \phi_{31} &= k_3'' - k_3^3 - k_1^2 k_3 - k_2^2 k_3, \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} \psi_{11} &= -5k_1k_2k_2' - 5k_1k_3k_3', \\ \psi_{21} &= -6k_2^2k_2' - 5k_2k_3k_3' + k_2''' - k_1^2k_2' - k_3^2k_2', \\ \psi_{31} &= -6k_3^2k_3' - 5k_2k_3k_2' + k_3''' - k_1^2k_3' - k_2^2k_3'. \end{aligned} \tag{3.33}$$

Proposition 3.1. *Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with non-zero constant k_1. Then γ is*

i) of $PAW(1)$-type if and only if

$$-k_2^2 = k_3^2 + c, \tag{3.34}$$

and

$$\begin{aligned} -6k_2^2k_2' - 5k_3k_2k_3' + k_2''' - k_1^2k_2' - k_3^2k_2' &= 0, \\ -6k_3^2k_3' - 5k_2k_3k_2' + k_3''' - k_1^2k_3' - k_2^2k_3' &= 0. \end{aligned} \tag{3.35}$$

ii) of $PAW(2)$-type if and only if

$$-k_2^2 = k_3^2 + c, \tag{3.36}$$

and

$$k_3'\psi_{21} = k_2'\psi_{31}. \tag{3.37}$$

iii) of $PAW(3)$-type if and only if

$$\begin{aligned} (k_2^2 + k_3^2)\psi_{11} &= k_1(k_2\psi_{21} + k_3\psi_{31}), \\ (k_1^2 + k_3^2)\psi_{21} &= k_2(k_1\psi_{11} + k_3\psi_{31}), \\ (k_1^2 + k_2^2)\psi_{31} &= k_3(k_1\psi_{11} + k_2\psi_{21}). \end{aligned} \tag{3.38}$$

iv) of $PAW(4)$-type if and only if

$$\begin{aligned} \left(\phi_{21}^2 + \phi_{31}^2\right)\psi_{11} &= \phi_{11}(\phi_{21}\psi_{21} + \phi_{31}\psi_{31}), \\ (\phi_{11}^2 + \phi_{31}^2)\psi_{21} &= \phi_{21}(\phi_{11}\psi_{11} + \phi_{31}\psi_{31}), \\ (\phi_{11}^2 + \phi_{21}^2)\psi_{31} &= \phi_{31}(\phi_{11}\psi_{11} + \phi_{21}\psi_{21}). \end{aligned} \tag{3.39}$$

v) of $PAW(5)$-type if and only if

$$\begin{aligned} \psi_{11} &= a_1k_1, \\ \psi_{21} &= a_1k_2 + b_1k_2', \\ \psi_{31} &= a_1k_3 + b_1k_3', \end{aligned} \tag{3.40}$$

and

$$a_1 = -\frac{5}{2}(\kappa^2)'. \tag{3.41}$$

vi) of $PAW(6)$-type if and only if

$$\begin{aligned} \psi_{11} &= a_2 k_1 + b_2 \phi_{11}, \\ \psi_{21} &= a_2 k_2 + b_2 \phi_{21}, \\ \psi_{31} &= a_2 k_3 + b_2 \phi_{31}, \end{aligned} \tag{3.42}$$

and

$$a_2 - b_2 \kappa^2 = -\frac{5}{2}(\kappa^2)'. \tag{3.43}$$

vii) of $PAW(7)$-type if and only if

$$\begin{aligned} \psi_{11} &= b_3 \phi_{11}, \\ \psi_{21} &= a_3 k_2' + b_3 \phi_{21}, \\ \psi_{31} &= a_3 k_3' + b_3 \phi_{31}, \end{aligned} \tag{3.44}$$

and

$$\frac{5}{2}(\kappa^2)' = b_3 \kappa^2. \tag{3.45}$$

Proof. i) Let γ be of $PAW(1)$-type. Using the equations (3.4), (3.31) and (3.33), we obtain

$$\psi_{11} = \psi_{21} = \psi_{31} = 0, \tag{3.46}$$

that means

$$\begin{aligned} -5k_1 k_2 k_2' - 5k_1 k_3 k_3' &= 0, \\ -6k_2^2 k_2' - 5k_3 k_2 k_3' + k_2''' - k_1^2 k_2' - k_3^2 k_2' &= 0, \\ -6k_3^2 k_3' - 5k_2 k_3 k_2' + k_3''' - k_1^2 k_3' - k_2^2 k_3' &= 0. \end{aligned} \tag{3.47}$$

If we solve the equation (3.47), we get

$$-k_2^2 = k_3^2 + c, \tag{3.48}$$

where c is an arbitrary constant. Converse proposition is trivial.

ii) Let γ be of $PAW(2)$-type. Using the equations (3.5), (3.31) and (3.33), we obtain

$$\begin{aligned} (k_2'^2 + k_3'^2)\psi_{11} &= 0, \\ k_3'^2 \psi_{21} &= k_2' k_3' \psi_{31}, \\ k_2'^2 \psi_{31} &= k_2' k_3' \psi_{21}. \end{aligned} \tag{3.49}$$

Since k_2 and k_3 are not constants the solution of the first equation of the system (3.49) is

$$\psi_{11} = 0,$$

which corresponds to

$$-k_2^2 = k_3^2 + c.$$

If we simplify the second and the third equations of the system (3.49), we obtain

$$k_3'\psi_{21} = k_2'\psi_{31}.$$

Converse proposition is trivial.

iii) Let γ be of $PAW(3)$-type. Substituting the equations (3.31) and (3.33) in (3.6), we get the solution. Converse proposition is trivial.

iv) Let γ be of $PAW(4)$-type. Substituting the equations (3.31), (3.32) and (3.33) in (3.7), we get the solution. Converse proposition is trivial.

v) Let γ be of $PAW(5)$-type. Using the equation (3.8), (3.31) and (3.33), we get

$$\begin{aligned} \psi_{11} &= a_1k_1, \\ \psi_{21} &= a_1k_2 + b_1k_2', \\ \psi_{31} &= a_1k_3 + b_1k_3'. \end{aligned} \tag{3.50}$$

From the first equation of the system (3.50), we obtain

$$-5k_1k_2k_2' - 5k_1k_3k_3' = a_1k_1,$$

which corresponds to

$$a_1 = -5k_2k_2' - 5k_3k_3'.$$

Using $k_1^2 + k_2^2 + k_3^2 = \kappa^2$ and solving the last equation, we obtain

$$a_1 = -\frac{5}{2}(\kappa^2)'.$$

Converse proposition is trivial.

vi) Let γ be of $PAW(6)$-type. Substituting the equations (3.31), (3.32) and (3.33) in (3.9), we obtain the equations (3.42). Substituting the equations (3.32) and (3.33) in (3.42), we get

$$k_1(a_2) + k_1(-b_2k_1^2 - b_2k_2^2 - b_2k_3^2) = k_1(-5k_2k_2' - 5k_3k_3'),$$

which means

$$a_2 - b_2\kappa^2 = -\frac{5}{2}(\kappa^2)'.$$

Converse proposition is trivial.

vii) Let γ be of $PAW(7)$-type. Substituting the equations (3.31), (3.32) and (3.33) in (3.10), we obtain the equations(3.44). Substituting the equations (3.32) and (3.33) in (3.44), we get

$$(-5k_1k_2k_2' - 5k_1k_3k_3') = b_3(-k_1^3 - k_1k_2^2 - k_1k_3^2).$$

If we divide both side with $-k_1$, we obtain

$$5(k_2k_2' + k_3k_3' + k_1k_1') = b_3(k_1^2 + k_2^2 + k_3^2),$$

which means

$$\frac{5}{2}(\kappa^2)' = b_3\kappa^2.$$

Converse proposition is trivial. $\square$

Corollary 3.1. *Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with non-zero constant k_1. If γ is of $PAW(1)$-type, then γ is of $PAW(2)$-type.*

Now, let's assume that k_2 is non-zero constant, k_1 and k_3 are not constants. In this case, we obtain;

$$\begin{aligned} \overline{N_1} &= k_1M_1 + k_2M_2 + k_3M_3, \\ \overline{N_2} &= k_1'M_1 + k_3'M_3, \\ \overline{N_3} &= \phi_{12}M_1 + \phi_{22}M_2 + \phi_{32}M_3, \\ \overline{N_4} &= \psi_{12}M_1 + \psi_{22}M_2 + \psi_{32}M_3, \end{aligned} \tag{3.51}$$

where

$$\begin{aligned} \phi_{12} &= k_1'' - k_1^3 - k_1k_2^2 - k_1k_3^2, \\ \phi_{22} &= -k_2^3 - k_1^2k_2 - k_3^2k_2, \\ \phi_{32} &= k_3'' - k_3^3 - k_1^2k_3 - k_2^2k_3, \end{aligned} \tag{3.52}$$

and

$$\begin{aligned} \psi_{12} &= -6k_1^2k_1' - 5k_1k_3k_3' + k_1''' - k_1'k_2^2 - k_1'k_3^2, \\ \psi_{22} &= -5k_1k_2k_1' - 5k_3k_2k_3', \\ \psi_{32} &= -6k_3^2k_3' - 5k_1k_3k_1' + k_3''' - k_1^2k_3' - k_2^2k_3'. \end{aligned} \tag{3.53}$$

Proposition 3.2. *Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with non-zero constant k_2. Then γ is*

i) of $PAW(1)$-type if and only if

$$-k_1^2 = k_3^2 + c,$$

and

$$\begin{aligned} -6k_1^2k_1' - 5k_1k_3k_3' + k_1''' - k_1'k_2^2 - k_1'k_3^2 &= 0, \\ -6k_3^2k_3' - 5k_1k_3k_1' + k_3''' - k_1^2k_3' - k_2^2k_3' &= 0. \end{aligned}$$

ii) of $PAW(2)$-type if and only if

$$\begin{aligned} -k_1^2 &= k_3^2 + c, \\ k_1'\psi_{32} &= k_3'\psi_{12}. \end{aligned}$$

iii) of $PAW(3)$-type if and only if

$$\begin{aligned}(k_2^2+k_3^2)\psi_{12} &= k_1(k_2\psi_{22}+k_3\psi_{32}),\\(k_1^2+k_3^2)\psi_{22} &= k_2(k_1\psi_{12}+k_3\psi_{32}),\\(k_1^2+k_2^2)\psi_{32} &= k_3(k_1\psi_{12}+k_2\psi_{22}).\end{aligned}$$

iv) of $PAW(4)$-type if and only if

$$\begin{aligned}(\phi_{22}^2+\phi_{32}^2)\psi_{12} &= \phi_{12}(\phi_{22}\psi_{22}+\phi_{32}\psi_{32}),\\(\phi_{12}^2+\phi_{32}^2)\psi_{22} &= \phi_{22}(\phi_{12}\psi_{12}+\phi_{32}\psi_{32}),\\(\phi_{12}^2+\phi_{22}^2)\psi_{32} &= \phi_{32}(\phi_{12}\psi_{12}+\phi_{22}\psi_{22}).\end{aligned}$$

v) of $PAW(5)$-type if and only if

$$\begin{aligned}\psi_{12} &= a_1k_1+b_1k_1',\\\psi_{22} &= a_1k_2,\\\psi_{32} &= a_1k_3+b_1k_3',\end{aligned}$$

and

$$a_1=-\frac{5}{2}(\kappa^2)'.$$

vi) of $PAW(6)$-type if and only if

$$\begin{aligned}\psi_{12} &= a_2k_1+b_2\phi_{12},\\\psi_{22} &= a_2k_2+b_2\phi_{22},\\\psi_{32} &= a_2k_3+b_2\phi_{32},\end{aligned}\tag{3.54}$$

and

$$a_2-b_2\kappa^2=-\frac{5}{2}(\kappa^2)'.$$

vii) of $PAW(7)$-type if and only if

$$\psi_{12} = a_3k_1'+b_3\phi_{12},\tag{3.55}$$

$$\psi_{22} = b_3\phi_{22},$$

$$\psi_{32} = a_3k_3'+b_3\phi_{32},\tag{3.56}$$

and

$$\frac{5}{2}(\kappa^2)'=b_3\kappa^2.$$

Proof. *i)* Let γ be of $PAW(1)$-type. Using the equations (3.4), (3.51) and (3.53), we obtain

$$\psi_{11}=\psi_{22}=\psi_{32}=0,$$

that means

$$
\begin{array}{rcl}
-6k_1^2k_1' - 5k_1k_3k_3' + k_1''' - k_1'k_2^2 - k_1'k_3^2 &=& 0, \\
-5k_1k_2k_1' - 5k_2k_3k_3' &=& 0, \\
-6k_3^2k_3' - 5k_1k_3k_1' + k_3''' - k_1^2k_3' - k_2^2k_3' &=& 0.
\end{array} \tag{3.57}
$$

If we solve the equation (3.57), we get

$$-k_1^2 = k_3^2 + c,$$

where c is an arbitrary constant. Converse proposition is trivial.

ii) Let γ be of $PAW(2)$-type. Using the equations (3.5), (3.51) and (3.53), we obtain

$$
\begin{array}{rcl}
k_3'^2\psi_{12} &=& k_1'k_3'\psi_{32}, \\
(k_1'^2 + k_3'^2)\psi_{22} &=& 0, \\
k_1'^2\psi_{32} &=& k_1'k_3'\psi_{12}.
\end{array} \tag{3.58}
$$

Since k_1 and k_3 are not constant, the solution of the second equation of the system (3.58) is

$$\psi_{22} = 0,$$

which corresponds to

$$-k_1^2 = k_3^2 + c.$$

If we simplify the first and the third equations of the system (3.58), we obtain

$$k_1'\psi_{32} = k_3'\psi_{12}.$$

Converse proposition is trivial.

iii) Let γ be of $PAW(3)$-type. Substituting the equations (3.51) and (3.53) in (3.6), we get the solution. Converse proposition is trivial.

iv) Let γ be of $PAW(4)$-type. Substituting the equations (3.51), (3.52) and (3.53) in (3.7), we get the solution. Converse proposition is trivial.

v) Let γ be of $PAW(5)$-type. Using (3.8), equations (3.51) and (3.53), we get

$$
\begin{array}{rcl}
\psi_{12} &=& a_1k_1 + b_1k_1', \\
\psi_{22} &=& a_1k_2, \\
\psi_{32} &=& a_1k_3 + b_1k_3'.
\end{array} \tag{3.59}
$$

From the second equation of the system (3.59), we obtain

$$-5k_1k_2k_1' - 5k_3k_2k_3' = a_1k_2,$$

which corresponds to

$$a_1 = -5k_1k_1' - 5k_3k_3'.$$

Using $k_1^2+k_2^2+k_3^2=\kappa^2$ and solving the last equation, we obtain

$$a_1=-\frac{5}{2}(\kappa^2)'.$$

Converse proposition is trivial.

vi) Let γ be of $PAW(6)$-type. Substituting the equations (3.51), (3.52) and (3.53) in (3.9), we obtain the equations (3.54). Substituting the equations (3.52) and (3.53) in (3.54), we get

$$k_2\left(a_2\right)+k_2\left(-b_2k_2^2-b_2k_1^2-b_2k_3^2\right)=k_2\left(-5k_1k_1'-5k_3k_3'\right),$$

which means

$$a_2-b_2\kappa^2=-\frac{5}{2}(\kappa^2)'.$$

Converse proposition is trivial.

vii) Let γ be of $PAW(7)$-type. Substituting the equations (3.51), (3.52) and (3.53) in (3.10), we obtain the equations

$$\psi_{22}=b_3\phi_{22}, \tag{3.60}$$

(3.55) and (3.56). Substituting the equations (3.52) and (3.53) in the last equation, we get

$$-5k_1k_2k_1'-5k_3k_2k_3'=-b_3k_2(k_2^2+k_1^2+k_3^2).$$

If we divide both side with k_2, we obtain

$$5(k_1k_1'+k_3k_3'+k_2k_2')=b_3\kappa^2$$

which means

$$\frac{5}{2}(\kappa^2)'=b_3\kappa^2.$$

Converse proposition is trivial. □

Corollary 3.2. *Let $\gamma:I\subseteq\mathbb{R}\to\mathbb{E}^4$ be a unit speed curve with non-zero constant k_2. If γ is of $PAW(1)$-type, then γ is of $PAW(2)$-type.*

Now, let's assume that k_3 is non-zero constant, k_1 and k_2 are not constants. In this case, we obtain;

$$\begin{aligned}\overline{N_1}&=k_1M_1+k_2M_2+k_3M_3,\\ \overline{N_2}&=k_1'M_1+k_2'M_2,\\ \overline{N_3}&=\phi_{13}M_1+\phi_{23}M_2+\phi_{33}M_3,\\ \overline{N_4}&=\psi_{13}M_1+\psi_{23}M_2+\psi_{33}M_3,\end{aligned} \tag{3.61}$$

where

$$\begin{aligned}\phi_{13}&=k_1''-k_1^3-k_1k_2^2-k_1k_3^2,\\ \phi_{23}&=k_2''-k_2^3-k_1^2k_2-k_3^2k_2,\\ \phi_{33}&=-k_3^3-k_1^2k_3-k_2^2k_3,\end{aligned} \tag{3.62}$$

and

$$\begin{aligned} \psi_{13} &= -6k_1^2k_1' - 5k_1k_2k_2' + k_1''' - k_1'k_2^2 - k_1'k_3^2, \\ \psi_{23} &= -6k_2^2k_2' - 5k_1k_2k_1' + k_2''' - k_1^2k_2' - k_3^2k_2', \\ \psi_{33} &= -5k_1k_3k_1' - 5k_2k_3k_2'. \end{aligned} \tag{3.63}$$

Proposition 3.3. *Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with non-zero constant k_3. Then γ is*

i) of $PAW(1)$-type if and only if

$$-k_1^2 = k_2^2 + c,$$

and

$$\begin{aligned} -6k_1^2k_1' - 5k_1k_2k_2' + k_1''' - k_1'k_2^2 - k_1'k_3^2 &= 0, \\ -6k_2^2k_2' - 5k_1k_2k_1' + k_2''' - k_1^2k_2' - k_3^2k_2' &= 0. \end{aligned}$$

ii) of $PAW(2)$-type if and only if

$$\begin{aligned} k_2'\psi_{13} &= k_1'\psi_{23}, \\ -k_1^2 &= k_2^2 + c. \end{aligned}$$

iii) of $PAW(3)$-type if and only if

$$\begin{aligned} (k_2^2 + k_3^2)\psi_{13} &= k_1(k_2\psi_{23} + k_3\psi_{33}), \\ (k_1^2 + k_3^2)\psi_{23} &= k_2(k_1\psi_{13} + k_3\psi_{33}), \\ (k_1^2 + k_2^2)\psi_{33} &= k_3(k_1\psi_{13} + k_2\psi_{23}). \end{aligned}$$

vi) of $PAW(4)$-type if and only if

$$\begin{aligned} (\phi_{23}^2 + \phi_{33}^2)\psi_{13} &= \phi_{13}(\phi_{23}\psi_{23} + \phi_{33}\psi_{33}), \\ (\phi_{13}^2 + \phi_{33}^2)\psi_{23} &= \phi_{23}(\phi_{13}\psi_{13} + \phi_{33}\psi_{33}), \\ (\phi_{13}^2 + \phi_{23}^2)\psi_{33} &= \phi_{33}(\phi_{13}\psi_{13} + \phi_{23}\psi_{23}). \end{aligned}$$

v) of $PAW(5)$-type if and only if

$$\begin{aligned} \psi_{13} &= a_1k_1 + b_1k_1', \\ \psi_{23} &= a_1k_2 + b_1k_2', \\ \psi_{33} &= a_1k_3, \end{aligned}$$

and

$$a_1 = -\frac{5}{2}(\kappa^2)'.$$

vi) of $PAW(6)$-type if and only if

$$\begin{aligned} \psi_{13} &= a_2 k_1 + b_2 \phi_{13}, \\ \psi_{23} &= a_2 k_2 + b_2 \phi_{23}, \\ \psi_{33} &= a_2 k_3 + b_2 \phi_{33}, \end{aligned} \tag{3.64}$$

and

$$a_2 - b_2 \kappa^2 = -\frac{5}{2}(\kappa^2)'.$$

vii) of $PAW(7)$-type if and only if

$$\begin{aligned} \psi_{13} &= a_3 k_1' + b_3 \phi_{13}, \\ \psi_{23} &= a_3 k_2' + b_3 \phi_{23}, \\ \psi_{33} &= b_3 \phi_{33}, \end{aligned} \tag{3.65}$$

and

$$\frac{5}{2}(\kappa^2)' = b_3 \kappa^2.$$

Proof. *i)* Let γ be of $PAW(1)$-type. Using the equations (3.4), (3.61) and (3.63), we obtain

$$\psi_{13} = \psi_{23} = \psi_{33} = 0,$$

that means

$$\begin{aligned} -6k_1^2 k_1' - 5k_1 k_2 k_2' + k_1''' - k_1' k_2^2 - k_1' k_3^2 &= 0, \\ -6k_2^2 k_2' - 5k_1 k_2 k_1' + k_2''' - k_1^2 k_2' - k_3^2 k_2' &= 0, \\ -5k_1 k_3 k_1' - 5k_2 k_3 k_2' &= 0. \end{aligned} \tag{3.66}$$

If we solve the equation (3.66), we get

$$-k_1^2 = k_2^2 + c,$$

where c is an arbitrary constant. Converse proposition is trivial.

ii) Let γ be of $PAW(2)$-type. Using the equations (3.5), (3.61) and (3.63), we obtain

$$\begin{aligned} k_2'^2 \psi_{13} &= k_1' k_2' \psi_{23}, \\ k_1'^2 \psi_{23} &= k_1' k_2' \psi_{13}, \\ (k_1'^2 + k_2'^2)\psi_{33} &= 0. \end{aligned} \tag{3.67}$$

Since k_1 and k_2 are not constant, the solution of the third equation of the system (3.67) is

$$\psi_{33} = 0,$$

which corresponds to

$$-k_1^2 = k_2^2 + c.$$

If we simplify the first and the second equations of the system (3.67), we obtain

$$k_2'\psi_{13} = k_1'\psi_{23}.$$

Converse proposition is trivial.

iii) Let γ be of $PAW(3)$-type. Substituting the equations (3.61) and (3.63) in (3.6), we get the solution. Converse proposition is trivial.

iv) Let γ be of $PAW(4)$-type. Substituting the equations (3.61), (3.62) and (3.63) in (3.7), we get the solution. Converse proposition is trivial.

v) Let γ be of $PAW(5)$-type. Using the equations (3.8), (3.61) and (3.63), we get

$$\begin{aligned} \psi_{13} &= a_1k_1 + b_1k_1', \\ \psi_{23} &= a_1k_2 + b_1k_2', \\ \psi_{33} &= a_1k_3. \end{aligned} \tag{3.68}$$

From the third equation of the system (3.68), we obtain

$$-5k_1k_3k_1' - 5k_2k_3k_2' = a_1k_3,$$

which corresponds to

$$a_1 = -5k_1k_1' - 5k_2k_2'.$$

Using $k_1^2 + k_2^2 + k_3^2 = \kappa^2$ and solving the last equation, we obtain

$$a_1 = -\frac{5}{2}(\kappa^2)'.$$

Converse proposition is trivial.

vi) Let γ be of $PAW(6)$-type. Substituting the equations (3.61), (3.62) and (3.63) in (3.9), we obtain the equations (3.64). Substituting the equations (3.62) and (3.63) in (3.64), we get

$$k_3\left(a_2\right) + k_3\left(-b_2k_2^2 - b_2k_1^2 - b_2k_3^2\right) = k_3\left(-5k_1k_1' - 5k_2k_2'\right),$$

which means

$$a_2 - b_2\kappa^2 = -\frac{5}{2}(\kappa^2)'.$$

Converse proposition is trivial.

vii) Let γ be of $PAW(7)$-type. Since k_3 is non-zero constant, substituting the equations (3.61), (3.62) and (3.63) in (3.10), we obtain the equations

$$\psi_{33} = b_3\phi_{33},$$

and (3.65). Substituting the equations (3.62) and (3.63) in the last equation, we get

$$-5k_1k_2k_1' - 5k_2k_3k_2' = -b_3k_3(k_2^2 + k_1^2 + k_3^2).$$

If we divide both side with k_3, we obtain

$$5(k_1k_1' + k_3k_3' + k_2k_2') = b_3\kappa^2,$$

which means

$$\frac{5}{2}(\kappa^2)' = b_3\kappa^2.$$

Converse proposition is trivial. □

Corollary 3.3. *Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with non-zero constant k_3. If γ is of $PAW(1)$-type, then γ is of $PAW(2)$-type.*

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İlim Kişi
Department of Mathematics
Kocaeli University
41380 Kocaeli, Turkey
ilim.ayvaz@kocaeli.edu.tr

Sezgin Büyükkütük
Department of Mathematics
Kocaeli University
41380 Kocaeli, Turkey
sezgin.buyukkutuk@kocaeli.edu.tr

Deepmala (Corresponding author)
SQC & OR Unit
Indian Statistical Institute
203 B. T. Road, Kolkata-700 108, India
dmrai23@gmail.com, deepmaladm23@gmail.com

Günay Öztürk
Department of Mathematics
Kocaeli University
41380 Kocaeli, Turkey
ogunay@kocaeli.edu.tr