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ON ϕ-IDEAL WARD CONTINUITY

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Abstract. An ideal I is a family of subsets of positive integers $\mathbb{N}$ which is closed under taking finite unions and subsets of its elements. Let P denote the space whose elements are finite sets of distinct positive integers. Given any element σ of P, we denote by $p(\sigma)$ the sequence $\{p_n(\sigma)\}$ such that $p_n(\sigma) = 1$ for $n \in \sigma$ and $p_n(\sigma) = 0$ otherwise. Further $P_s = \left\{\sigma \in P : \sum_{n=1}^{\infty} p_n(\sigma) \leq s\right\}$, i.e.$P_s$ is the set of those σ whose support has cardinality at most s, and $\Phi = \{\phi = (\phi_n) : 0 < \phi_1 \leq \phi_n \leq \phi_{n+1} \text{ and } n\phi_{n+1} \leq (n+1)\phi_n\}$. A sequence (x_n) of points in $\mathbb{R}$ is called ϕ-ideal convergent (or I_ϕ -convergent) to a real number ℓ if for every $\varepsilon > 0$

$$\left\{ s \in \mathbb{N} : \frac{1}{\phi_s} \sum_{n\in\sigma,\sigma\in P_s} |x_n - \ell| \geq \varepsilon \right\} \in I.$$

We introduce ϕ-ideal ward continuity of a real function. A real function is ϕ-ideal ward continuous if it preserves ϕ-ideal quasi Cauchy sequences where a sequence (x_n) is called to be ϕ -ideal quasi Cauchy (or I_ϕ-quasi Cauchy) when $(\Delta x_n) = (x_{n+1} - x_n)$ is ϕ-ideal convergent to 0. i.e. a sequence (x_n) of points in $\mathbb{R}$ is called ϕ-ideal quasi Cauchy (or I_ϕ-quasi Cauchy) for every $\varepsilon > 0$ if

$$\left\{ s \in \mathbb{N} : \frac{1}{\phi_s} \sum_{n\in\sigma,\sigma\in P_s} |x_{n+1} - x_n| \geq \varepsilon \right\} \in I.$$

In this paper, we prove that any ϕ-ideal continuous function is uniformly continuous either on an interval or on a ϕ-ideal ward compact subset of $\mathbb{R}$. We also characterize the uniform continuity via ϕ-ideal quasi-Cauchy sequences.

Keywords: Ideal convergence, ideal continuity, ϕ-sequence, quasi-Cauchy sequence

1. Introduction

A real valued function is continuous on the set of real numbers if and only if it preserves Cauchy sequences. Using the idea of continuity of a real function and the

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idea of compactness in terms of sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: forward continuity [6], slowly oscillating continuity [9], statistical ward continuity [7], δ-ward continuity [12], ideal ward continuty [2, 13], N_θ-ward continuity [3, 4] and λ-statistical ward continuity [14]. The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. A sequence (x_n) of points in $\mathbb{R}$ is called quasi-Cauchy if (Δx_n) is a null sequence where $\Delta x_n = x_{n+1} - x_n$. In [1] Burton and Coleman named these sequences as "quasi-Cauchy" and in [8] Çakallı used the term "ward convergent to 0" sequences. In terms of quasi-Cauchy we restate the definitions of ward compactness and ward continuity as follows: a function f is ward continuous if it preserves quasi-Cauchy sequences, i.e.$(f(x_n))$ is quasi-Cauchy whenever (x_n) is, and a subset E of $\mathbb{R}$ is ward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x} = (x_n)$.

2. Preliminaries and Notations

First of all, some definitions and notation will be given in the following. Throughout this paper, $\mathbb{N}$, and $\mathbb{R}$ will denote the set of all positive integers, and the set of all real numbers, respectively. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of terms in $\mathbb{R}$.
It is known that a sequence (x_n) of points in $\mathbb{R}$, the set of real numbers, is slowly oscillating if

$$\lim_{\lambda \to 1^+} \overline{\lim}_n \max_{n+1 \leq k \leq [\lambda n]} |x_k - x_n| = 0,$$

where $[\lambda n]$ denotes the integer part of λn. This is equivalent to the following if $(x_m - x_n) \to 0$ whenever $1 \leq \frac{m}{n} \to 1$ as $m, n \to \infty$. Using $\varepsilon > 0$ s and δ s this is also equivalent to the case when for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \geq N(\varepsilon)$ and $n \leq m \leq (1 + \delta)n$ (see [9]).

A function defined on a subset E of $\mathbb{R}$ is called slowly oscillating continuous if it preserves slowly oscillating sequences, i.e.$(f(x_n))$ is slowly oscillating whenever (x_n) is.

The notion of ideal convergence, which is a generalization of ordinary convergence, and statistical convergence, was introduced by Kostyrko et al. [24] and also independently by Nuray and Ruckle in [25] who called it generalized statistical convergence (see [17, 18]) based on the structure of the admissible ideal I of subset of natural numbers $\mathbb{N}$.

A family of sets $I \subset P(\mathbb{N})$ (the power sets of $\mathbb{N}$) is said to be an *ideal* on $\mathbb{N}$ if and only if

(i) $\phi \in I$

(ii) for each $A, B \in I$, we have $A \cup B \in I$

(ii) for each $A \in I$ and each $B \subset A$, we have $B \in I$.

A non-empty family of sets $F \subset P(\mathbb{N})$ is said to be a *filter* on $\mathbb{N}$ if and only if

(i) $\phi \notin F$

(ii) for each $A, B \in F$, we have $A \cap B \in F$

(iii) each $A \in F$ and each $B \supset A$, we have $B \in F$.

An ideal I is called *non-trivial* ideal if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$.

A non-trivial ideal $I \subset P(\mathbb{N})$ is called

(i) *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$.

(ii) *maximal* if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

Recall that a sequence $\mathbf{x} = (x_n)$ of points in $\mathbb{R}$ is said to be I-convergent to the number ℓ if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\} \in I$. In this case we write $I\text{-}\lim x_n = \ell$. A sequence $\mathbf{x} = (x_n)$ of points in $\mathbb{R}$ is said to be I-quasi-Cauchy if $I - \lim_n (x_{n+1} - x_n) = 0$. We see that I-convergence of a sequence (x_n) implies I-quasi-Cauchyness of (x_n). We note that the definition of a quasi-Cauchy sequence is a special case of an ideal quasi-Cauchy sequences where I is taken as the finite subsets of the set of positive integers. Cakalli and Hazarika [2] introduced the concept of ideal quasi Cauchy sequences and proved some results related to ideal ward continuity and ideal ward compactness. For more details on ideal convergence we refer to [19, 20, 21, 22, 28].

Throughout this paper we assume I is a non-trivial admissible ideal in $\mathbb{N}$, also, $I(\mathbb{R})$ and ΔI_ϕ will denote the set of all I-convergent sequences and the set of all ϕ-ideal quasi-Cauchy sequences of points in $\mathbb{R}$, respectively.

Now we give the following interesting examples which show emphasis the interest in different research areas.

Example 2.1.[29] Let n be a positive integer. In a group of n people, each person selects at random and simultaneously another person of the group. All of the selected persons are then removed from the group, leaving a random number $n_1 < n$ of people which form a new group. The new group then repeats independently the selection and removal thus described, leaving $n_2 < n_1$ persons, and so forth until either one person remains, or no persons remain. Denote by pn the probability that, at the end of this iteration initiated with a group of n persons, one person remains. Then the sequence $\mathbf{p}= (p_1, p_2, .., p_n, ...)$ is a I_ϕ-quasi-Cauchy sequence, and $\lim_n p_n$ does not exist.

Example 2.2. [23] Let n be a positive integer. In a group of n people, each person selects independently and at random one of three subgroups to which to belong, resulting in three groups with random numbers n_1, n_2, n_3 of members; $n_1 + n_2 + n_3 = n$. Each of the subgroups is then partitioned independently in the same manner to form three sub subgroups, and so forth. Subgroups having no members or having only one member are removed from the process. Denote by t_n the expected value of the number of iterations up to complete removal, starting initially with a group of n people. Then the sequence $(t_1, \frac{t_2}{2}, \frac{t_3}{3}, ..., \frac{t_k}{k}, ...)$ is a bounded nonconvergent I_ϕ-quasi-Cauchy sequence.

Example 2.3.[23] Let $\mathbf{x} := (x_n)$ be a sequence such that for each nonnegative integer n, x_n is either 0 or 1. For each positive integer n set $a_n = \frac{x_1+x_2+...+x_n}{n}$. Then a_n is the arithmetic mean average of the sequence up to time or position n. Clearly for each n, $0 \leq a_n \leq 1$. (a_n) is a quasi-Cauchy sequence. i.e.the sequence of averages of 0 s and 1 s is always a quasi-Cauchy sequence.

Connor and Grosse-Erdman [16] gave sequential definitions of continuity for real functions calling G-continuity instead of A-continuity and their results covers the earlier works related to A-continuity where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace of s, space of all sequences, denoted by c_G, into $\mathbb{R}$. A sequence $\mathbf{x} = (x_n)$ is said to be G-convergent to ℓ if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the linear space c and st-lim denotes the statistical limit function $st\text{-}\lim \mathbf{x} = st\text{-}\lim_n x_n$ on the linear space $st(\mathbb{R})$. Also I-lim denotes the I-limit function $I\text{-}\lim \mathbf{x} = I\text{-}\lim_n x_n$ on the linear space $I(\mathbb{R})$. A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$. Recently, Cakalli gave new sequential definitions of compactness and slowly oscillating compactness in [9, 10].

3. ϕ-ideal ward continuity

Let P denote the space whose elements are finite sets of distinct positive integers. Given any element σ of P, we denote by $p(\sigma)$ the sequence $\{p_n(\sigma)\}$ such that $p_n(\sigma) = 1$ for $n \in \sigma$ and $p_n(\sigma) = 0$ otherwise. Further

$$P_s = \left\{ \sigma \in P : \sum_{n=1}^{\infty} p_n(\sigma) \leq s \right\},$$

i.e.P_s is the set of those σ whose support has cardinality at most s, and $\Phi = \{\phi = (\phi_n) : 0 < \phi_1 \leq \phi_n \leq \phi_{n+1} \text{ and } n\phi_{n+1} \leq (n+1)\phi_n\}$. A sequence (x_n) of points in $\mathbb{R}$ is called ϕ-ideal convergent (or I_ϕ-convergent) to a real number ℓ if for every $\varepsilon > 0$

$$\left\{ s \in \mathbb{N} : \frac{1}{\phi_s} \sum_{n\in\sigma,\sigma\in P_s} |x_n - \ell| \geq \varepsilon \right\} \in I.$$

We introduce ϕ-ideal ward continuity of a real function. A real function is ϕ-ideal ward continuous if it preserves ϕ-ideal quasi Cauchy sequences where a sequence (x_n) is called to be ϕ -ideal quasi Cauchy (or I_ϕ-quasi Cauchy) when $(\Delta x_n) = (x_{n+1} - x_n)$ is ϕ-ideal convergent to 0. i.e.a sequence (x_n) of points in $\mathbb{R}$ is called ϕ-ideal quasi Cauchy (or I_ϕ-quasi Cauchy) for every $\varepsilon > 0$ if

$$\left\{ s \in \mathbb{N} : \frac{1}{\phi_s} \sum_{n\in\sigma,\sigma\in P_s} |x_{n+1} - x_n| \geq \varepsilon \right\} \in I.$$

Now we give the definitions of I_ϕ-sequential compactness of a subset E and I_ϕ-sequential continuity of a function defined on E of $\mathbb{R}$ as follows.

Definition 3.1. A subset E of $\mathbb{R}$ is called I_ϕ-sequentially compact if whenever (x_n) is a sequence of points in E there is I_ϕ -convergent subsequence $\mathbf{y} = (y_k) = (x_{n_k})$ of (x_n) such that I_ϕ-$\lim \mathbf{y}$ is in E.

Definition 3.2. A function $f : E \to \mathbb{R}$ is I_ϕ-sequentially continuous at a point x_0 if, given a sequence (x_n) of points in E, I_ϕ-$\lim \mathbf{x} = x_0$ implies that I_ϕ-$\lim f(\mathbf{x}) = f(x_0)$.

Theorem 3.1. *A subset of $\mathbb{R}$ is sequentially compact if and only if it is I_ϕ-sequentially compact.*

Proof. The proof easily follows from Corollary 3 on page 597 in [10] and Theorem 1 in [2] so is omitted. □

Theorem 3.2. *Any I_ϕ-sequentially continuous function at a point x_0 is continuous at x_0 in the ordinary sense.*

Proof. Let f be any I_ϕ-sequentially continuous function at point x_0, Since any proper admissible ideal is a regular subsequential method, it follows from Theorem 13 on page 316 in [11] that f is continuous in the ordinary sense. □

Theorem 3.3. *Any continuous function at a point x_0 is I_ϕ-sequentially continuous at x_0.*

Proof. Proof of the theorem follows from Theorem 2.2 in [27]. □

Combining Theorem 3.1 and Theorem 3.2 we have the following:

Corollary 3.1. *A function is I_ϕ-sequentially continuous at a point x_0 if and only if it is continuous at x_0.*

Theorem 3.4. *If a function is δ-ward continuous on a subset E of $\mathbb{R}$, then it is I_ϕ-sequentially continuous on E.*

Proof. Let f be any δ-ward continuous function on E. It follows from Corollary 2 on page 399 in [12] that f is continuous. By Theorem 3.1 we obtain that f is I_ϕ-sequentially continuous on E. This completes the proof. □

Corollary 3.2. *If a function is quasi-slowly oscillating continuous on a subset E of $\mathbb{R}$, then it is I_ϕ-sequentially continuous on E.*

Proof. Let f be any quasi-slowly oscillating continuous on E. It follows from Theorem 3.2 in [15] that f is continuous. By Theorem 3.1 we deduce that f is I_ϕ-sequentially continuous on E. This completes the proof. □

We say that a sequence $\mathbf{x} = (x_n)$ is I_ϕ-ward convergent to a number ℓ if I_ϕ-$\lim_{n\to\infty} \Delta x_n = \ell$ where $\Delta x_n = x_{n+1} - x_n$. For the special case $\ell = 0$ we say that $\mathbf{x}$ is ϕ-ideal quasi-Cauchy, or I_ϕ-quasi-Cauchy, in place of I_ϕ-ward convergent to 0. Thus a sequence (x_n) of points of $\mathbb{R}$ is I_ϕ-quasi-Cauchy if (Δx_n) is I_ϕ-convergent to 0. We denote ΔI_ϕ the set of all ϕ-ideal quasi Cauchy sequences of points in $\mathbb{R}$ (see [22]).

Now we give the definition of I_ϕ-ward compactness of a subset of $\mathbb{R}$.

Definition 3.3. A subset E of $\mathbb{R}$ is called I_ϕ-ward compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in E there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ such that I_ϕ-$\lim_{k\to\infty} \Delta z_k = 0$.

We note that this definition of I_ϕ-ward compactness can not be obtained by any G-sequential compactness, i.e.by any summability matrix A, even by the summability matrix $A = (a_{nk})$ defined by $a_{nk} = -1$ if $k = n$ and $a_{kn} = 1$ if $k = n+1$ and

$$G(x) = I_\phi - \lim A\mathbf{x} = I_\phi - \lim_{k\to\infty} \sum_{n=1}^{\infty} a_{kn} x_n = I_\phi - \lim_{k\to\infty} \Delta x_k$$

(see [10] for the definition of G-sequential compactness). Despite that G-sequential compact subsets of $\mathbb{R}$ should include the singleton set $\{0\}$, I_ϕ-ward compact subsets of $\mathbb{R}$ do not have to include the singleton $\{0\}$.
We give the definition of I_ϕ-ward continuity of a real function.

Definition 3.4. A function f is called I_ϕ-ward continuous on E if I_ϕ-$\lim_{n\to\infty} \Delta f(x_n) = 0$ whenever I_ϕ-$\lim_{n\to\infty} \Delta x_n = 0$, for a sequence $\mathbf{x} = (x_n)$ of terms in E.

Theorem 3.5. *If a function f is uniformly continuous on a subset E of $\mathbb{R}$ and (x_n) is a quasi-Cauchy sequence of points in E, then $(f(x_n))$ is I_ϕ-quasi Cauchy.*

Proof. Let f be uniformly continuous on E. Suppose that there exists a quasi-Cauchy sequence (x_n) of points in E. To show that $(f(x_n))$ is I_ϕ-quasi Cauchy. i.e.for every $\varepsilon > 0$ we need to show that

$$\left\{ s \in \mathbb{N} : \frac{1}{\phi_s} \sum_{n\in\sigma,\sigma\in P_s} |f(x_{n+1}) - f(x_n)| < \varepsilon \right\} \in F.$$

As f is uniformly continuous on E, for this ε, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ for $x, y \in E$. Since (x_n) is quasi-Cauchy, there exists an ingeter $n_0 \in \mathbb{N}$ such that $|x_{n+1} - x_n| < \delta$ for $n \geq n_0$. Now we have

$$\frac{1}{\phi_s} \sum_{n\in\sigma,\sigma\in P_s} |f(x_{n+1}) - f(x_n)| < \varepsilon \text{ for } s \geq n_0.$$

Therefore

$$\left\{ s \in \mathbb{N} : \frac{1}{\phi_s} \sum_{n\in\sigma,\sigma\in P_s} |f(x_{n+1}) - f(x_n)| < \varepsilon \right\} \in F.$$

This completes the proof of the theorem. □

Theorem 3.6. *Let E be an I_ϕ-ward compact subset of $\mathbb{R}$ and let $f : E \to \mathbb{R}$ be an I_ϕ-ward continuous function on E. Then f is uniformly continuous on E.*

Proof. Suppose that f is not uniformly continuous on E so there exists an $\varepsilon > 0$ such that for any $\delta > 0$, for $x, y \in E$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. For each positive integer n we can find (x_n) and (y_n) are sequences of points in E such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Since E is I_ϕ-ward compact, there is a subsequence (x_{n_k}) of (x_n) such that $I_\phi\text{-}\lim_{k\to\infty} \Delta x_{n_k} = 0$. On the other hand there is a subsequence of $(y_{n_{k_j}})$ of (y_{n_k}) such that $I_\phi\text{-}\lim_{j\to\infty} \Delta y_{n_{k_j}} = 0$. It is clear that the corresponding sequence $(x_{n_{k_j}})$ is also I_ϕ -quasi Cauchy i.e.$I_\phi\text{-}\lim_{j\to\infty} \Delta x_{n_{k_j}} = 0$ because $(y_{n_{k_j}})$ is an I_ϕ-quasi-Cauchy sequence and we have

$$x_{n_{k_j}} - x_{n_{k_{j+1}}} = (x_{n_{k_j}} - y_{n_{k_j}}) + (y_{n_{k_j}} - y_{n_{k_{j+1}}}) + (y_{n_{k_{j+1}}} - x_{n_{k_{j+1}}}).$$

Now we define a sequence $\mathbf{z} = (z_j)$ be setting $z_1 = x_{n_{k_1}}$, $z_2 = y_{n_{k_1}}$, $z_3 = x_{n_{k_2}}$, $z_4 = y_{n_{k_2}}$, $z_5 = x_{n_{k_3}}$, $z_6 = y_{n_{k_3}}$, and so on. Thus (z_j) is an I_ϕ-quasi-Cauchy sequence while $(\Delta f(z_j))$ is a I_ϕ-quasi-Cauchy. This contradiction completes the proof of the theorem. □

Corollary 3.3. *If a function f is I_ϕ-ward continuous on a bounded subset E of $\mathbb{R}$, then it is uniformly continuous on E.*

Proof. The proof follows from the preceding theorem and Theorem 8 in [2]. □

We have much more below for a real function f defined on an interval that f is uniformly continuous if and only if $(f(x_n))$ is I_ϕ-quasi Cauchy whenever (x_n) is a quasi-Cauchy sequence if points in E. First we give the following lemma.

Lemma 3.1. *Let (ξ_n, η_n) is a sequence of ordered pairs of points in an interval such that $\lim_n |\xi_n - \eta_n| = 0$, then there exists an I_ϕ-quasi Cauchy sequence (x_n) with the property that for any positive integer i there exists a positive integer j such that $(\xi_i, \eta_i) = (x_{j-1}, x_j)$.*

Proof. The following proof is similar to that of [1], but we give it for completeness. For each positive integer k, we can fix $z_0^k, z_1^k, z_2^k, ..., z_{n_k}^k$ in E with $z_0^k = \eta_k, z_{n_k}^k = \xi_{k+1}$ and $|z_i^k - z_{i-1}^k| < \frac{1}{k}$ for $1 \leq i \leq n_k$. Now we write the sequence

$$(x_n) = (\xi_1, \eta_1, z_1^1, ..., z_{n_1-1}^1, \xi_2, \eta_2, z_1^2, ..., z_{n_2-1}^2, \xi_3, \eta_3, ..., \xi_k, \eta_k, z_1^k, ..., z_{n_k-1}^k, \xi_{k+1}, \eta_{k+1}, ...)$$

Then we obtain that for any positive integer i there exists a positive integer j such that $(\xi_i, \eta_i) = (x_{j-1}, x_i)$. The sequence constructed is a quasi-Cauchy sequence and it is an I_ϕ-quasi Cauchy sequence, since any quasi-Cauchy sequence is an I_ϕ-quasi Cauchy sequence. This completes the proof of the theorem. □

Theorem 3.7. *If a function defined on an interval E is I_ϕ-ward continuous, then it is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous on E. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$ there exist $x, y \in E$ with $|x-y| < \delta$ but $|f(x)-f(y)| \geq \varepsilon$. For every $n \in \mathbb{N}$ fix $\xi_n, \eta_n \in E$ with $|\xi_n - \eta_n| < \frac{1}{n}$ but $|f(\xi_n) - f(\eta_n)| \geq \varepsilon$. By the Lemma 3.1, there exists an I_ϕ-quasi Cauchy sequence (x_i) such that for any integer $i \geq 1$ there exists a j with $\xi_i = x_j$ and $\eta_i = x_{j+1}$. This implies that $|f(x_{j+1}) - f(x_j)| \geq \varepsilon$. Hence $(f(x_i))$ is not I_ϕ-quasi Cauchy. Thus f does not preserve I_ϕ-quasi Cauchy sequences. This established the proof of the theorem. □

Since the sequence constructed in Lemma 3.1 is also quasi-Cauchy, we see that the function f is uniformly continuous on an interval E if the sequence $(f(x_n))$ is I_ϕ-quasi Cauchy whenever (x_n) is a quasi-Cauchy sequence of points in E. Combining this with the Theorem 3.5, we have the following result.

Corollary 3.4. *If a function defined on an interval is I_ϕ-ward continuous, then it is ward continuous.*

Proof. The proof follows from the Theorem 3.7 and Theorem 5 in [12], so it is omitted. □

Corollary 3.5. *If a function defined on an interval is I_ϕ-ward continuous, then it is slowly oscillating continuous.*

Proof. The proof follows from the Theorem 3.7 and Theorem 5 in [12], so it is omitted. □

Cakalli [5] introduced the concept G-sequentially connected as, a non-empty subset E of $\mathbb{R}$ is called G-sequential connectedness if there are non-empty and disjoint G-sequentially closed subsets U and V such that $A \subseteq U \cup V$, and $A \cap U$ and $A \cap V$ are empty. As far as G-sequentially connectedness is considered, then we get the following results.

Theorem 3.8. *Any I_ϕ-sequentially continuous image of any I_ϕ -sequentially connected subset of $\mathbb{R}$ is I_ϕ-sequentially connected.*

Proof. The proof follows from the Theorem 1 in [5]. □

Theorem 3.9. *A subset of $\mathbb{R}$ is I_ϕ-sequentially connected if and only if it is connected in ordinary sense and so is an interval.*

Proof. The proof follows from the Corollary 1 in [5]. □

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