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ENDPOINT ESTIMATES FOR MULTILINEAR COMMUTATOR OF INTEGRAL OPERATOR

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Abstract. In this paper, we prove the endpoint estimates for some multilinear commutator of certain integral operators.
Keywords: Multilinear commutator, integral operator.

1. Introduction

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator. It is well known that the commutator $[b, T]$ is defined as follows:

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberb and Weiss (see [2]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$, $(1 < p < \infty)$. In [5], E.Harboure, C.Segovia and J.L.Torrea proved the boundedness properties of the commutators for the extreme values of p (also see [1]). In this paper, we will introduce the multilinear commutator of certain integral operator and prove the boundedness properties of the operator for the extreme cases. The integral operator include the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

First, let us introduce some notations (see [3][8-10]). In this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a function b, let $b_Q = |Q|^{-1} \int_Q b(x)dx$ and $b(Q) = \int_Q b(x)dx$, the sharp function of b is defined by

$$b^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy.$$

It is well-known that (see [3])

$$b^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |b(y) - c| dy.$$

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We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^{\infty}(R^n)$ and define $||b||_{BMO} = ||b^{\#}||_{L^{\infty}}$. It has been known that (see [9])

$$||b - b_{2^kQ}||_{BMO} \leqslant Ck||b||_{BMO}.$$

We also define the central $BMO(R^n)$ space by $CMO(R^n)$, which is the space of those functions $f \in L_{loc}(R^n)$ such that

$$||f||_{CMO} = \sup_{r>1} |Q(0,r)|^{-1} \int_Q |f(y) - f_Q| dy < \infty.$$

It is well-known that

$$||f||_{CMO} \approx \sup_{r>1} \inf_{c \in C} |Q(0,r)|^{-1} \int_Q |f(x) - c| dx.$$

The A_1 weight is defined by (see [3])

$$A_1 = \{0 < \omega \in L^1_{loc} : \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \omega(y) dy \leqslant c\omega(x), a.e.\}.$$

Definition 1. A function a is called a $H^1(R^n)-$atom, if there exists a cube Q, such that

1) *supp* $a \subset Q = Q(x_0, r)$,
2) $||a||_{L^{\infty}} \leq |Q|^{-1}$,
3) $\int_{R^n} a(x)dx = 0$.

It is well known that the Hardy space $H^1(R^n)$ has the atomic decomposition characterization (see [3][8][10]).

Definition 2. Let $0 < \delta < n$ and $1 < p < n/\delta$. We shall call $B^{\delta}_p(R^n)$ the space of those functions f on R^n such that

$$||f||_{B^{\delta}_p} = \sup_{r>1} r^{-n(1/p - \delta/n)} ||f\chi_{Q(0,r)}||_{L^p} < \infty.$$

Definition 3. Suppose b_j $(j = 1, \cdots, m)$ are the fixed locally integrable functions on R^n. Let $F_t(x, y)$ be the function defined on $R^n \times R^n \times [0, +\infty)$. Set

$$S_t(f)(x) = \int_{R^n} F_t(x, y) f(y) dy$$

and

$$S^{\vec{b}}_t(f)(x) = \int_{R^n} \prod_{j=1}^{m} (b_j(x) - b_j(y)) F_t(x, y) f(y) dy$$

for every bounded and compactly supported function f. Let H be the Banach space $H = \{h : ||h|| < \infty\}$ such that, for each fixed $x \in R^n$, $S_t(f)(x)$ and $S^{\vec{b}}_t(f)(x)$ may be viewed as the mappings from $[0, +\infty)$ to H. The multilinear commutator related to S_t is defined by

$$T_\delta^{\vec{b}}(f)(x) = ||S_t^{\vec{b}}(f)(x)||,$$

where F_t satisfies: for fixed $\epsilon > 0$ and $0 < \delta < n$,

$$||F_t(x, y)|| \leqslant C|x - y|^{-n+\delta}$$

and

$$||F_t(x, y) - F_t(x, z)|| + ||F_t(y, x) - F_t(z, x)|| \leqslant C|y - z|^\epsilon |x - z|^{-n-\epsilon+\delta}$$

if $2|y - z| \leq |x - z|$ and $2|y - u| \leq |x - u|$. We also define $T_\delta(f)(x) = ||S_t(f)(x)||$.

Note that when $b_1 = \cdots = b_m$, $T_{\vec{b}}$ is just the m order commutator (see [1][13]). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1][4][5-7]).

Given a positive integer m and $1 \leqslant j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

2. Theorems and Proofs

We begin with a preliminary lemma.

Lemma.(see [3]) Let $w \in A_1$. Then $w\chi_Q \in A_1$ for any cube Q.

Theorem 1. Let $0 < \delta < n$ and $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in BMO(R^n)$ for $1 \leq j \leq m$. Suppose that T_δ is bounded from $L^u(w)$ to $L^v(w)$ for all u, v with $1 < u < v/\delta$, $1/v = 1/u - \delta/n$ and $w \in A_1$. Then $T_\delta^{\vec{b}}$ is bounded from $L^{n/\delta}(R^n)$ to $BMO(R^n)$.

Proof. It is only to prove that there exist a constant C_Q such that

$$\frac{1}{|Q|} \int_Q |T_\delta^{\vec{b}}(f)(x) - C_Q| dx \leq C||f||_{L^{n/\delta}}.$$

Fix a cube $Q = Q(x_0, d)$, we decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$, $f_2 = f\chi_{(R^n \setminus 2Q)}$.

When $m = 1$, set $(b_1)_Q = |Q|^{-1} \int_Q b_1(y) dy$, we have

$$S_t^{b_1}(f)(x) = (b_1(x) - (b_1)_Q) S_t(f)(x) - S_t((b_1 - (b_1)_Q) f_1)(x) - S_t((b_1 - (b_1)_Q) f_2)(x),$$

so

$$\begin{aligned}
& |T_\delta^{b_1}(f)(x) - T_\delta(((b_1)_Q - b_1)f_2)(x_0)| \\
= & \left| ||S_t^{b_1}(f)(x)|| - ||S_t(((b_1)_Q - b_1)f_2)(x_0)|| \right| \\
\leq & ||S_t^{b_1}(f)(x) - S_t(((b_1)_Q - b_1)f_2)(x_0)|| \\
\leq & ||(b_1(x) - (b_1)_Q)S_t(f)(x)|| + ||S_t((b_1 - (b_1)_Q)f_1)(x)|| \\
& + ||S_t((b_1 - (b_1)_Q)f_2)(x) - S_t((b_1 - (b_1)_Q)f_2)(x_0)|| \\
= & A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, set $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $1/q + 1/q' = 1$, by the Hölder's inequality and Lemma, we get

$$\begin{aligned}
\frac{1}{Q}\int_Q |A(x)|dx &\leq \left(\frac{1}{|Q|}\int_Q |b_1(x) - (b_1)_Q|^{q'} dx\right)^{1/q'} \left(\frac{1}{|Q|}\int_{R^n} |T_\delta(f)(x)|^q \chi_Q(x)dx\right)^{1/q} \\
&\leq C||b_1||_{BMO} \frac{1}{|Q|^q} \left(\int_{R^n} |f(x)|^p \chi_Q(x)dx\right)^{1/p} \\
&\leq C||b_1||_{BMO} \frac{1}{|Q|^q} ||f||_{L^{n/\delta}} |Q|^{(1-(\delta p/n))/p} \\
&\leq C||b_1||_{BMO} ||f||_{L^{n/\delta}}.
\end{aligned}$$

For $B(x)$, taking $1 < r < n/\delta$ and $1/s = 1/r - \delta/n$, by the Hölder's inequality, we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |B(x)|dx &\leq \left(\frac{1}{|Q|}\int_{R^n} (T_\delta((b_1(x) - (b_1)_Q)f_1)(x))^s dx\right)^{1/s} \\
&\leq C|Q|^{-1/s} ||(b_1 - (b_1)_Q)f\chi_{2Q}||_{L^r} \\
&\leq C\left(\frac{1}{|2Q|}\int_{2Q} |b_1(x) - (b_1)_Q|^s dx\right)^{1/s} ||f||_{L^{n/\delta}} \\
&\leq C||b_1||_{BMO} ||f||_{L^{n/\delta}}.
\end{aligned}$$

For $C(x)$, by the Minkowski's inequality, we obtain

$$\begin{aligned}
C(x) &\leq \int_{(2Q)^c} |b_1(y) - (b_1)_Q||f(y)|||F_t(x,y) - F_t(x_0,y)||dy \\
&\leq C\int_{(2Q)^c} |b_1(y) - (b_1)_Q||f(y)| \frac{|x - x_0|^\epsilon}{|y - x_0|^{n+\epsilon-\delta}} dy \\
&\leq C\sum_{k=1}^{\infty} \int_{2^k d < |x_0 - y| < 2^{k+1}d} |b_1(y) - (b_1)_Q||f(y)| \frac{|x - x_0|^\epsilon}{|y - x_0|^{n+\epsilon-\delta}} dy \\
&\leq C\sum_{k=1}^{\infty} \int_{2^k d < |x_0 - y| < 2^{k+1}d} \frac{d^\epsilon}{(2^k d)^{n+\epsilon-\delta}} |b_1(y) - (b_1)_Q||f(y)|dy
\end{aligned}$$

$$\le C\sum_{k=1}^{\infty}2^{-k\varepsilon}\left(\int_{2^{k+1}Q}|f(y)|^{n/\delta}dy\right)^{\delta/n}$$
$$\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(y)-(b_1)_Q|^{\frac{n}{n-\delta}}dy\right)^{1-\delta/n}$$
$$\le C\sum_{k=1}^{\infty}k2^{-k\varepsilon}||b_1||_{BMO}||f||_{L^{n/\delta}}$$
$$\le C||b_1||_{BMO}||f||_{L^{n/\delta}},$$

thus

$$\frac{1}{|Q|}\int_Q|C(x)|dx\leqslant C||b_1||_{BMO}||f||_{L^{n/\delta}}.$$

This completes the proof of the case $m=1$.

When $m>1$, set $\vec{b}_Q=((b_1)_Q,\cdots,(b_m)_Q)\in R^n$, where $(b_j)_Q=|Q|^{-1}\int_Q b_j(y)dy,\ 1\le j\le m$, we have

$$\begin{aligned}S_t^{\vec{b}}(f)(x) &= (b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)S_t(f)(x)\\ &\quad+(-1)^mS_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f)(x)\\ &\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma\int_{R^n}(\vec{b}(y)-\vec{b}_Q)_{\sigma^c}F_t(x,y)f(y)dy\\ &= (b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)S_t(f)(x)\\ &\quad+(-1)^mS_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)\\ &\quad+(-1)^mS_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x)\\ &\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma S_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x),\end{aligned}$$

thus

$$\begin{aligned}&|T_\delta^{\vec{b}}(f)(x)-T_\delta(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|\\ &\le ||(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)S_t(f)(x)||\\ &+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(\vec{b}(x)-\vec{b}_Q)_\sigma S_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)||\\ &+||S_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)||\\ &+||S_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x)-S_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x_0)||\\ &=I_1(x)+I_2(x)+I_3(x)+I_4(x).\end{aligned}$$

For $I_1(x)$, taking $1<p<n/\delta$, and $1/q=1/p-\delta/n$, by the Hölder's inequality, we have

$$\frac{1}{|Q|}\int_Q I_1(x)dx \leqslant \left(\frac{1}{|Q|}\int_Q |\prod_{j=1}^{m}(b_j(x)-(b_j)_Q)|^{q'}dx\right)^{1/q'} \left(\frac{1}{|Q|}\int_Q |T_\delta(f)(x)|^q dx\right)^{1/q}$$
$$\le C||\vec{b}||_{BMO}|Q|^{-1/q}\left(\int_Q |f(x)|^p dx\right)^{1/p}$$
$$\le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}|Q|^{-1/q}\|Q|^{(1-(\delta p/n))/p}$$
$$\le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.$$

For $I_2(x)$, taking $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, then

$$\frac{1}{|Q|}\int_Q I_2(x)dx$$
$$\le \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma|^{q'}dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_Q |T_\delta((\vec{b}-\vec{b}_Q)_{\sigma^c})f)(x)|^q dx\right)^{1/q}$$
$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}|Q|^{-1/q}\left(\int_{R^n}|(b(\vec{x})-\vec{b}_Q)_{\sigma^c})f(x)|^p\chi_Q(x)dx\right)^{1/p}$$
$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^q dx\right)^{1/q}||f||_{L^{n/\delta}}$$
$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO}||f||_{L^{n/\delta}}$$
$$\le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.$$

For $I_3(x)$, taking $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, we get

$$\frac{1}{|Q|}\int_Q I_3(x)dx$$
$$\le \left(\frac{1}{|Q|}\int_Q |T_\delta((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^q dx\right)^{1/q}$$
$$\le C|Q|^{-1/q}||((b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f_1(x)||_{L^p}$$
$$\le C\left(\frac{1}{|2Q|}\int_{2Q}|(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)|^q dx\right)^{1/q}||f||_{L^{n/\delta}}$$
$$\le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.$$

For $I_4(x)$, we have

$$
\begin{aligned}
I_4(x) &\le \int_{R^n} |\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)||f(y)\chi_{(2Q)^c}(y)|||F_t(x,y)-F_t(x_0,y)||dy \\
&\le C\int_{(2Q)^c} |\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)||f(y)|\frac{|x-x_0|^{\epsilon}}{|y-x_0|^{n+\epsilon-\delta}}dy \\
&\le C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|x-x_0|^{\varepsilon}|x_0-y|^{-(n+\varepsilon-\delta)}|\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)||f(y)|dy \\
&\le C\sum_{k=1}^{\infty}2^{-k\varepsilon}\left(\int_{2^{k+1}Q}|f(y)|^{n/\delta}dy\right)^{\delta/n} \\
&\quad\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)|^{\frac{n}{n-\delta}}dy\right)^{1-\delta/n} \\
&\le C\sum_{k=1}^{\infty}k^m2^{-k\varepsilon}\prod_{j=1}^{m}||b_j||_{BMO}||f||_{L^{n/\delta}} \\
&\le C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}
$$

Thus

$$
\frac{1}{|Q|}\int_Q |I_4(x)|dx \leqslant C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
$$

This completes the proof of Theorem 1.

Theorem 2. Let $0<\delta<n$, $1<p<n/\delta$ and $\vec{b}=(b_1,\cdots,b_m)$ with $b_j\in BMO(R^n)$ for $1\le j\le m$. Suppose that T_δ is bounded from $L^u(w)$ to $L^v(w)$ for all u,v with $1<u<v/\delta$, $1/v=1/u-\delta/n$ and $w\in A_1$. Then $T_\delta^{\vec{b}}$ is bounded from $B_p^\delta(R^n)$ to $CMO(R^n)$.

Proof. It suffices to prove that there exist constant C_Q, such that

$$
\frac{1}{|Q|}\int_Q |T_\delta^{\vec{b}}(f)(x)-C_Q|dx \leqslant C||f||_{B_p^\delta}
$$

holds for any cube $Q=Q(0,d)$ with $d>1$. Fix a cube $Q=Q(0,d)$ with $d>1$. Set $f_1=f\chi_{2Q}$, $f_2=f\chi_{R^n\setminus 2Q}$ and $\vec{b}_Q=((b_1)_Q,\cdots,(b_m)_Q)$, where $(b_j)_Q=|Q|^{-1}\int_Q|b_j(y)|dy$, $1\le j\le m$, we have

$$
\begin{aligned}
&|T_\delta^{\vec{b}}(f)(x) - T_\delta(((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)| \\
&\le ||(b_1(x) - (b_1)_Q)\cdots(b_m(x) - (b_m)_Q)S_t(f)(x)|| \\
&+ \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} ||(\vec{b}(x) - \vec{b}_Q)_\sigma S_t((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)|| \\
&+ ||S_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_1)(x)|| \\
&+ ||S_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_2)(x) - S_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_2)(x_0)|| \\
&= H_1(x) + H_2(x) + H_3(x) + H_4(x).
\end{aligned}
$$

Taking $1 < p < n/\delta$, $1/s = 1/r - \delta/n$, by the Hölder's inequality and Lemma, we have

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q H_1(x)dx \\
&\leqslant \left(\frac{1}{|Q|}\int_Q |\prod_{j=1}^m (b_j(x) - (b_j)_Q)|^{q'} dx\right)^{1/q'} \left(\frac{1}{|Q|}\int_Q |T_\delta(f)(x)|^q dx\right)^{1/q} \\
&\le C||\vec{b}||_{BMO}|Q|^{-1/q}\left(\int_Q |f(x)|^p dx\right)^{1/p} \\
&\le C||\vec{b}||_{BMO} d^{-n(1/p-\delta/n)}||f\chi_Q||_{L^p} \\
&\le C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}
$$

For $H_2(x)$, taking $1 < p < n/\delta$, $1/s = 1/r - \delta/n$, and $1/s' + 1/s = 1$, then

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q H_2(x)dx \\
&\le \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{s'} dx\right)^{1/s'} \left(\frac{1}{|Q|}\int_Q |T_\delta((\vec{b} - \vec{b}_Q)_{\sigma^c})f)(x)|^s dx\right)^{1/s} \\
&\le C\sum_{j=1}^{m-1} ||\vec{b}_\sigma||_{BMO}|Q|^{-1/s}\left(\int_{R^n} |(b(\vec{x}) - \vec{b}_Q)_{\sigma^c})f(x)|^r \chi_Q dx\right)^{1/r} \\
&\qquad \le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||\vec{b}_\sigma||_{BMO}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c}|^{pr/(p-r)} dx\right)^{(p-r)/pr} \\
&\qquad\qquad \times |Q|^{(\delta/n-1/p)}||f\chi_Q||_{L^p} \\
&\qquad \le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO} d^{-n(1/p-\delta/n)}||f\chi_Q||_{L^p} \\
&\qquad \le C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}
$$

For $H_3(x)$, taking $1 < p < n/\delta$, $1/s = 1/r - \delta/n$ and $1/s' + 1/s = 1$, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q H_3(x)dx \\
&\le \left(\frac{1}{|Q|}\int_Q |T_\delta((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^s dx\right)^{1/s} \\
&\le C|Q|^{-1/s}||((b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f\chi_{2Q}||_{L^r} \\
&\le C\left(\frac{1}{|Q|}\int_{2Q}|(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)|^{pr/(p-r)}dx\right)^{(p-r)/pr} \\
&\qquad \times d^{-n(1/p-\delta/n)}||f\chi_{2Q}||_{L^p} \\
&\le C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}$$

For $H_4(x)$, we have

$$\begin{aligned}
H_4(x) &\le \int_{R^n}|\prod_{j=1}^m(b_j(y)-b(b_j)_Q)||f(y)\chi_{(2Q)^c}(y)|||F_t(x,y)-F_t(x_0,y)||dy \\
&\le C\int_{(2Q)^c}|\prod_{j=1}^m(b_j(y)-(b_j)_Q)||f(y)|\frac{|x-x_0|^\varepsilon}{|x_0-y|^{(n+\varepsilon-\delta)}}dy \\
&\le C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|x-x_0|^\varepsilon|x_0-y|^{-(n+\varepsilon-\delta)}|\prod_{j=1}^m(b_j(y)-(b_j)_Q)||f(y)|dy \\
&\le C\sum_{k=1}^\infty\left(\int_{R^n}|f(y)\chi_{2^{k+1}Q}(y)|^p dy\right)^{1/p} \\
&\qquad \times\left(\int_{2^{k+1}Q}|\prod_{j=1}^m(b_j(y)-(b_1)_Q)|^{\frac{p}{p-1}}dy\right)^{1-1/p} \\
&\le C\sum_{k=1}^\infty k^m 2^{-k\varepsilon}\prod_{j=1}^m||b_j||_{BMO}(2^{k+1}d)^{-n(1/p-\delta/n)}||f\chi_{2^{k+1}Q}||_{L^{n/\delta}} \\
&\le C||\vec{b}||_{BMO}||f||_{B_p^\delta},
\end{aligned}$$

thus

$$\frac{1}{|Q|}\int_Q |H_4(x)|dx \leqslant C||\vec{b}||_{BMO}||f||_{B_p^\delta}.$$

This completes the proof of Theorem 2.

Theorem 3. Let $0 < \delta < n$ and $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in BMO(R^n)$ for $1 \le j \le m$. If for any $H^1(R^n)$-atom a supported on certain cube Q and $u \in Q$, there is

$$\sum_{j=1}^m\sum_{\sigma\in C_j^m}\int_{(2Q)^c}\left(|(b(x)-b_Q)_{\sigma^c}|\left|\left|\int_Q(\vec{b}(y)-\vec{b}_Q)_\sigma a(y)dy\ F_t(x,u)\right|\right|\right)^{n/(n-\delta)}dx \leqslant C,$$

then $T_\delta^{\vec{b}}$ is bounded from $H^1(R^n)$ to $L^{n/(n-\delta)}(R^n)$.

Proof. Let a be an atom supported in some cube Q. We write

$$\int_{R^n}|T_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx=\int_{2Q}|T_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx+\int_{(2Q)^c}|T_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx=I+II.$$

For I, taking $1<p<n/\delta$ and $1/q=1/p-\delta/n$, we have

$$I\leqslant ||T_\delta^{\vec{b}}(a)||_{L^q}^{n/(n-\delta)}|2Q|^{1-n/((n-\delta)q)}\leqslant C||a||_{L^p}^{n/(n-\delta)}|Q|^{1-n/((n-\delta)q)}\le C.$$

For II, we first calculate $S_t^{\vec{b}}(a)(x)$, we have

$$\begin{aligned}T_\delta^{\vec{b}}(a)(x)&=||S_t^{\vec{b}}(a)(x)||\le \left\|\prod_{j=1}^m(b_j(x)-(b_j)_Q)\int_{|x-y|\le t}F_t(x,y)a(y)dy\right\|\\&+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left\|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_{|x-y|\le t}(F_t(x,y)-F_t(x,u))(\vec{b}(y)-\vec{b}_Q)_\sigma a(y)dy\right\|\\&+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left\|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_{|x-y|\le t}F_t(x,u)(\vec{b}(y)-\vec{b}_Q)_\sigma a(y)dy\right\|\\&=A(x)+B(x)+C(x).\end{aligned}$$

For $A(x)$, we have

$$\begin{aligned}A(x)&\le\int_{|x-y|\le t}||F_t(x,y)-F_t(x,u)|||a(y)|dy\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\&\le C\int_{|x-y|\le t}\frac{|y-u|^\varepsilon}{|x-u|^{n+\varepsilon-\delta}}|a(y)|dy\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\&\le C||a||_{L^\infty}\sum_{k=0}^\infty\int_{2^{-k-1}t\le|x-y|\le 2^{-k}t}\frac{|y-u|^\varepsilon}{|x-u|^{n+\varepsilon-\delta}}dy\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\&\le C||a||_{L^\infty}\sum_{k=0}^\infty\frac{t^\varepsilon}{|x-u|^{n+\varepsilon-\delta}}(2^{-k}t)^n\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\&\le C\frac{t^{n+\varepsilon}}{|x-u|^{n+\varepsilon-\delta}}||a||_{L^\infty}\prod_{j=1}^m|b_j(x)-(b_j)_Q|.\end{aligned}$$

Thus

$$
\begin{aligned}
&\left(\int_{(2Q)^c}(A_(x))^{n/(n-\delta)}dx\right)^{(n-\delta)/n}\\
&\leq C||a||_{L^\infty}\left[\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left(\frac{t^{n+\varepsilon}}{(x-u)^{n+\varepsilon-\delta}}\prod_{j=1}^{m}|b_j(x)-(b_j)_Q|\right)^{n/(n-\delta)}dx\right]^{(n-\delta)/n}\\
&\leq C||a||_{L^\infty}t^{n+\varepsilon}\sum_{k=1}^{\infty}\frac{1}{(2^kt)^{n+\varepsilon-\delta}}\left[\int_{2^{k+1}Q}\left(\prod_{j=1}^{m}|b_j(x)-(b_j)_Q|\right)^{n/(n-\delta)}dx\right]^{\frac{n-\delta}{n}}\\
&\leq C||a||_{L^\infty}t^{n+\varepsilon}\sum_{k=1}^{\infty}\frac{1}{(2^kt)^{n+\varepsilon-\delta}}(2^{k+1}t)^{n-\delta}\\
&\qquad\times\left(\frac{1}{|2^{k+1}Q|}\int_{(2^{k+1}Q)}\left(\prod_{j=1}^{m}|b_j(x)-(b_j)_Q|\right)^{n/(n-\delta)}dx\right)^{\frac{n-\delta}{n}}\\
&\leq C||\vec{b}||_{BMO}.
\end{aligned}
$$

For $B(x)$, we have

$$
\begin{aligned}
B(x)&\leq\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_{|x-y|\leq t}||F_t(x,y)-F_t(x,u)||(\vec{b}(y)-\vec{b}_Q)_\sigma a(y)dy\right|\\
&\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}||a||_{L^\infty}(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_{|x-y|\leq t}\frac{|y-u|^\varepsilon}{|x-u|^{n+\varepsilon-\delta}}(\vec{b}(y)-\vec{b}_Q)_\sigma dy\\
&\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}||a||_{L^\infty}(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\\
&\qquad\times\sum_{k=0}^{\infty}\int_{2^{-k-1}t\leq|x-y|\leq 2^{-k}t}\frac{|y-u|^\varepsilon}{|x-u|^{n+\varepsilon-\delta}}(\vec{b}(y)-\vec{b}_Q)_\sigma dy\\
&\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}||a||_{L^\infty}(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\\
&\qquad\times\sum_{k=0}^{\infty}\frac{(2^{-k}t)^\varepsilon}{|x-u|^{n+\varepsilon-\delta}}\int_{2^{-k-1}t\leq|x-y|\leq 2^{-k}t}(\vec{b}(y)-\vec{b}_Q)_\sigma dy
\end{aligned}
$$

$$\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}||a||_{L^\infty}(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}$$

$$\times\frac{1}{|x-u|^{n+\varepsilon-\delta}}\sum_{k=0}^{\infty}(2^{-k}t)^{\varepsilon}(2^{-k}t)^{n}\left(\frac{1}{|2^{-k}Q|}\int_{2^{-k}Q}(\vec{b}(y)-\vec{b}_Q)_{\sigma}^2dy\right)^{1/2}$$

$$\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}||a||_{L^\infty}(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}||\vec{b}_\sigma||_{BMO}\frac{t^{n+\varepsilon}}{|x-u|^{n+\varepsilon-\delta}},$$

thus

$$\left(\int_{(2Q)^c}(B(x))^{n/(n-\delta)}dx\right)^{(n-\delta)/n}$$

$$\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||a||_{L^\infty}t^{n+\varepsilon}$$

$$\times\sum_{k=1}^{\infty}\left(\int_{2^{k+1}Q\setminus 2^kQ}\left(\frac{1}{|x-u|^{n+\varepsilon-\delta}}(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\right)^{n/(n-\delta)}dx\right)^{(n-\delta)/n}$$

$$\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||a||_{L^\infty}t^{n+\varepsilon}\sum_{k=1}^{\infty}\frac{1}{(2^kt)^{n+\varepsilon-\delta}}(2^{k+1}t)^{n-\delta}$$

$$\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\right|^{n/(n-\delta)}dx\right)^{(n-\delta)/n}$$

$$\leq C||\vec{b}||_{BMO}.$$

So, if

$$\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\int_{(2Q)^c}\left(|(b(x)-b_Q)_{\sigma^c}|\left|\left|\int_Q(\vec{b}(y)-\vec{b}_Q)_\sigma a(y)dy\ F_t(x,u)\right|\right|\right)^{n/(n-\delta)}dx\leq C,$$

then

$$\int_{R^n}|T_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx\leq C.$$

This completes the proof of the Theorem 3.

3. Applications

Now we give some applications of theorems in this paper.

Application 1. Littlewood-Paley operator.

Fixed $0<\delta<n$ and $\varepsilon>0$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \leqslant C(1+|x|)^{-(n+1-\delta)}$,

(3) $|\psi(x+y) - \psi(x)| \leqslant C|y|^\varepsilon (1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$.

The Littlewood-Paley multilinear operators are defined by

$$g_{\psi,\delta}^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y))\psi_t(x-y)f(y)dy$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define

$$g_{\psi,\delta}(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which is the Littlewood-Paley operator(see [10]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\},$$

then, for each fixed $x \in R^n$, $F_t^{\vec{b}}(f)(x)$ and $F_t^{\vec{b}}(f)(x,y)$ may be viewed as the mappings from $[0, +\infty)$ to H, and it is clear that

$$g_{\psi,\delta}^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||, \quad g_{\psi,\delta}(f)(x) = ||F_t(f)(x)||.$$

It is easily to see that $g_{\psi,\delta}$ satisfies the conditions of Theorems 1, 2 and 3 (see [5-7]), thus Theorems 1, 2 and 3 hold for $g_{\psi,\delta}^{\vec{b}}$.

Application 2. Marcinkiewicz operator.

Fixed $0 < \delta < n$ and $0 < \gamma \leqslant 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\mu_{\Omega,\delta}^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{|x-y|\leqslant t} \prod_{j=1}^m (b_j(x) - b_j(y)) \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy.$$

Set

$$F_t(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy.$$

We also define

$$\mu_{\Omega,\delta}(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz operator (see [11]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}.$$

Then, it is clear that

$$\mu^{\vec{b}}_{\Omega,\delta}(f)(x) = ||F^{\vec{b}}_t(f)(x)||, \quad \mu_{\Omega,\delta}(f)(x) = ||F_t(f)(x)||,$$

It is easily to see that $\mu_{\Omega,\delta}$ satisfies the conditions of Theorems 1, 2 and 3 (see [7][11]), thus Theorems 1, 2 and 3 hold for $\mu^{\vec{b}}_{\Omega,\delta}$.

Application 3. Bochner-Riesz operator .

Let $\eta > (n-1)/2$, $B^\eta_t(f)\hat{}(\xi) = (1 - t^2|\xi|^2)^\eta_+ \hat{f}(\xi)$ and $B^\eta_t(z) = t^{-n}B^\eta(z/t)$ for $t > 0$. Set

$$F^{\vec{b}}_{\eta,t}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y)) B^\eta_t(x-y) f(y)dy.$$

The maximal Bochner-Riesz multilinear commutator are defined by

$$B^{\vec{b}}_{\eta,*}(f)(x) = \sup_{t>0} |B^{\vec{b}}_{\eta,t}(f)(x)|.$$

We also define that

$$B_{\eta,*}(f)(x) = \sup_{t>0} |B^\eta_t(f)(x)|,$$

which is the maximal Bochner-Riesz operator(see [7][12]). Let H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B^{\vec{b}}_{\eta,*}(f)(x) = ||B^{\vec{b}}_{\eta,t}(f)(x)||, \quad B^\eta_*(f)(x) = ||B^\eta_t(f)(x)||.$$

It is easily to see that $B^{\vec{b}}_{\eta,*}$ satisfies the conditions of Theorems 1, 2 and 3 with $\delta = 0$ (see [12]), thus Theorems 1, 2 and 3 hold for $B^{\vec{b}}_{\eta,*}$.

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