FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 5 (2016), 1041–1049
DOI:10.22190/FUMI1605041T

ON THE BASE SPACE OF AN ALMOST PARACONTACT SUBMERSION

Tshikunguila Tshikuna-Matamba

Abstract. The purpose of this note is to describe the base space of an almost paracontact submersion. Here the base space is an almost para-Hermitian manifold. So, the paper intertwines paracontact and para-Hermitian structures via the theory of submersions.
Keywords: Riemannian submersions; Almost para-Hermitian manifolds; Almost paracontact metric manifolds; Almost paracontact metric submersions

1. Introduction

Almost paracontact metric submersions are Riemannian submersions whose total space is endowed with almost paracontact metric structure. They were introduced by G*ü*nd*ü*zalp and Sahin [5] who considered the case of semi-Riemannian manifolds. Their study focused on the transfer of the structure from the total to the base space, the latter being also a paracontact metric manifold, extending the study of Watson [15]. In [13], we studied the geometry of the fibres regarding their structures and implications on the total and the base space.

Regarding the similarity between contact and paracontact structure, as indicated by Sato [8, 9], it seems interesting to examine the same similarities via submersions. In the present paper, we describe the structure of the base space when it is endowed with an almost para-Hermitian metric.

The paper is organized as follows. Section §2 is devoted to the preliminaries on manifolds where we consider almost para-Hermitian structures. Following Gray and Hervella [4], we have adapted the defining relations of almost para-Hermitian structures which can be also found in [10]. Almost paracontact structures are reviewed. In Section §3, we treat the case of almost paracontact metric submersions where, after recalling fundamental properties, we have examined the structure of the base space according to that of the total space. Note that, as in [15], this class of submersions will be called *almost paracontact submersions of type II.*

Received April 25, 2016; accepted June 14, 2016
2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53B20, 53B25, 53B26, 53B30

2. Preliminaries on manifolds

2.1. Almost para-Hermitian manifolds

Let M^{2m} be a smooth manifold of even dimension $2m$. Consider an almost para complex structure J such that $J^2 = \mathbb{I}$, where $\mathbb{I}$ is the identity transformation. If there exists on M a metric tensor g such that $g(JD, JE) = -g(D, E)$, then the couple (g, J) is called an almost para complex metric structure (or an almost para-Hermitian metric). So, (M^{2m}, g, J) is an almost para-Hermitian manifold.

As in the case of almost Hermitian manifolds, see for example [14, 16], the fundamental $2-$form Ω, of the structure (g, J) is given by
$\Omega(D, E) = g(D, JE)$. If further, J is parallel along the Levi-Civita connection ∇, (meaning that $\nabla J = 0$), then the manifold is said to be para-Kählerian.

Let us note some remarkable classes of almost para-Hermitian structures susceptible to be used in this study.

Following Gray and Hervella [4], see also [10], an almost para-Hermitian manifold is called:

(1) para-Kählerian if $\nabla J = 0$;

(2) almost para- Kählerian if $d\Omega(D, E, G) = 0$;

(3) quasi para- Kählerian if $(\nabla_D \Omega)(E, G) + (\nabla_{JD}\Omega)(JE, G) = 0$;

(4) nearly para- Kählerian if $(\nabla_D \Omega)(D, E) = 0$.

2.2. Almost paracontact metric manifolds

Let M be a differentiable manifold of dimension $2m + 1$. An almost paracontact structure on M is a triple (φ, ξ, η), where:

(1) ξ is a characteristic vector field,

(2) η is a $1-$form such that $\eta(\xi) = 1$, and

(3) φ is a tensor field of type $(1, 1)$ satisfying

$$\varphi^2 = \mathbb{I} - \eta \otimes \xi, \tag{2.1}$$

where $\mathbb{I}$ is the identity transformation. If M is equipped with a Riemannian metric g such that

$$g(\varphi D, \varphi E) = -g(D, E) + \eta(D)\eta(E), \tag{2.2}$$

then (g, φ, ξ, η) is called an almost paracontact metric structure. So, the quintuple $(M^{2m+1}, g, \varphi, \xi, \eta)$ is an almost paracontact metric manifold. As in the case of almost contact metric manifolds, any almost paracontact metric manifold admits a fundamental 2−form, ϕ, defined by

$$\phi(D, E) = g(D, \varphi E).$$

Moreover $\eta \circ \varphi = 0$, $\eta(D) = g(D, \xi)$ and $\varphi\xi = 0$.

For some remarkable classes, we have the following defining relations.

An almost paracontact manifold is said to be:

(1) *normal* if $N_\varphi - 2d\eta \otimes \xi = 0$, where N_φ is the Nijenhuis tensor of φ.

(2) *para-contact* if $\phi = d\eta$,

(3) *para-K-contact* if it is para-contact and ξ is Killing,

(4) *para-cosymplectic* if $\nabla\eta = 0$ and $\nabla\phi = 0$,

(5) *almost para-cosymplectic* if $d\phi = 0$ and $d\eta = 0$,

(6) *para-Sasakian* if $\phi = d\eta$ and M is normal,

(7) *quasi para-Sasakian* if $d\phi = 0$ and M is normal,

(8) *quasi para-K-cosymplectic* if

$(\nabla_D\varphi)E + (\nabla_{\varphi D}\varphi)\varphi E - \eta(E)(\nabla_{\varphi D}\xi) = 0$;

(9) *almost para-Kenmotsu* if $d\phi(D, E, G) = \frac{2}{3}\mathcal{G}\left\{\eta(D)\phi(E, G)\right\}$, where $\mathcal{G}$ denotes the cyclic sum over D, E, G;

(10) *para-Kenmotsu* if $d\phi(D, E, G) = \frac{2}{3}\mathcal{G}\left\{\eta(D)\phi(E, G)\right\}$, $d\eta = 0$ and is normal;

(11) *quasi para-Kenmotsu* if

$(\nabla_D\phi)(E, G) + (\nabla_{\varphi D}\phi)(\varphi E, G) = \eta(E)\phi(G, D) + 2\eta(G)\phi(D, E)$ and $d\eta = 0$;

(12) *nearly para-Kenmotsu* if $(\nabla_D\varphi)D = -\eta(D)\varphi D$ and $d\eta = 0$;

(13) *nearly para-cosymplectic* if $(\nabla_D\varphi)D = 0$;

(14) *closely para-cosymplectic* if $(\nabla_D\varphi)D = 0$ and $d\eta = 0$;

Following [5, 10], it is known that

$N^{(1)}(D, E) = N_\varphi(D, E) - 2d\eta(D, E)\xi$,

$N^{(2)}(D, E) = (\mathcal{L}_{\varphi D}\eta)E - (\mathcal{L}_{\varphi E}\eta)D$, where $\mathcal{L}$ denotes the Lie derivative.

Moreover, if $N^{(1)} = 0$ then $N^{(2)} = 0$. The vanishing of the tensor $N^{(1)}$ means that the manifold is normal.

Note that almost paracontact metric manifolds have been studied by Dacko [2], Dacko and Olszak [3], Kaneyuki and Williams [6], Sato [8, 9], Zamkovoy [17] among others.

Some Examples

Following A.M. Blaga [1], let $M = \left\{(x, y, z) \in \mathbb{R}^3 : z \neq 0\right\}$ and setting $\eta = -\frac{1}{z}dz$, $\xi = -z\frac{\partial}{\partial z}$;

Note by $\mathcal{M}_{2m+1}(\mathbb{R})$ the set of $(2m + 1)$ real matrices.

Taking $\varphi \in \mathcal{M}_{2m+1}(\mathbb{R})$ such that

$$\varphi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

it is easy to verify that (φ, ξ, η) is an almost paracontact structure.

Now, considering $\varphi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have that (φ, ξ, η) is an almost contact structure.

In [3], Dacko and Olszak constructed an example of para-cosymplectic structure in the following way. Let (N, J, G) be a para-Kählerian manifold. Consider the structure (φ, ξ, η, g) defined on the product manifold $M = N \times \mathbb{R}$ by $\varphi = (J, 0)$, $\eta = dt$, $\xi = \frac{\partial}{\partial t}$ and $g = G \times dt^2$ where t is the Cartesian coordinate on $\mathbb{R}$. Then (φ, ξ, η, g) is para-cosymplectic.

Following the previous example, others can be constructed by taking (N, J, G) in the various classes of almost para-Hermitian manifolds such as: almost para-Kählerian, quasi para- Kählerian and so on. Thus, we obtain the defining relations of almost para-cosymplectic and quasi para-cosymplectic respectively.

3. Almost paracontact submersions of type II

Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ and $(M'^{2m'}, g', J')$ be an almost paracontact metric manifold and an almost para-Hermitian one. By an *almost paracontact metric submersion of type II* in the sense of [15] see also [11, 12], one understands a Riemannian submersion

$$\pi : M^{2m+1} \to M'^{2m'}$$

satisfying $\pi_* \varphi = J' \pi_*$.

Recall that the tangent bundle $T(M)$ of the total space M has an orthogonal decomposition

$$T(M) = V(M) \oplus H(M),$$

where $V(M)$ is the vertical distribution while $H(M)$ designates the horizontal one. In [7], O'Neill has defined two configuration tensors T and A, of the total space of a Riemannian submersion by setting

$$T_D E = \mathcal{H} \nabla_{\mathcal{V}D} \mathcal{V} E + \mathcal{V} \nabla_{\mathcal{V}D} \mathcal{H} E;$$

$$A_D E = \mathcal{V} \nabla_{\mathcal{H}D} \mathcal{H} E + \mathcal{H} \nabla_{\mathcal{H}D} \mathcal{V} E.$$

Here, $\mathcal{H}$ and $\mathcal{V}$ designate the horizontal and vertical projections respectively.

A vector field X of the horizontal distribution is called a basic vector field if it is $\pi-$related to a vector field X_* of the base space M'. Such a vector field means that $\pi_* X = X_*$.

On the base space, tensors and other objects will be denoted by a prime ' while those tangent to the fibres will be specified by a carret ˆ. For instance, $\hat{N}_J$ denotes the Nijenhuis tensor of J on the fibres and $N_{J'}$ on the base space. Herein, vector fields tangent to the fibres will be denoted by U, V and W.

The fibres of almost paracontact submersions, say of type I, have been treated in [13]. Let us say something about the structure of the fibres of the under consideration type of submersions.

Proposition 3.1. *Let $\pi : M^{2m+1} \longrightarrow M^{2m'}$ be an almost paracontact metric submersion of type II. Then, the fibres are almost paracontact metric manifolds.*

Proof. Clearly the fibres are $2(m-m')+1$-dimensional submanifolds. Let (g, φ, ξ, η) be the almost paracontact metric structure of the total space; note by $(\hat{g}, \hat{\varphi}, \hat{\xi}, \hat{\eta})$ its restriction on the fibres. The problem is to show that $(\hat{g}, \hat{\varphi}, \hat{\xi}, \hat{\eta})$ is an almost paracontact metric structure. Indeed, let U and V be two vertical vector fields tangent to the fibres. We have

$\hat{\varphi}^2 U = U - \hat{\eta}(U)\hat{\xi}$;

$\hat{\eta}(\hat{\xi}) = \hat{g}(\hat{\xi}, \hat{\xi}) = g(\xi, \xi) = 1$;

$\hat{g}(\hat{\varphi}U, \hat{\varphi}V) = -\hat{g}(U, V) + \hat{\eta}(U)\hat{\eta}(V)$. □

Next, we overview some of the fundamental properties of this type of submersions, which also appear in [15].

3.1. Fundamental properties

Proposition 3.2. *Let $\pi : M^{2m+1} \longrightarrow M^{2m'}$ be an almost paracontact metric submersion of type II. Then,*

(1) $\pi^*\Omega' = \phi$;

(2) $U \in V(M)$ implies that $\varphi U \in V(M)$;

(3) $X \in H(M)$ implies that $\varphi X \in H(M)$;

(4) $\xi \in ker\pi_*$;

(5) $X \in H(M)$ implies that $\eta(X) = 0$;

(6) $\pi_* N^{(1)} = N_{J'}$.

Proof. Let X and Y be basic vector fields. We have

$$\begin{aligned} \pi^*\Omega'(X, Y) &= \Omega'(\pi_* X, \pi_* Y) \\ &= g'(\pi_* X, J'\pi_* Y) \\ &= g'(\pi_* X, \pi_* \varphi Y) \end{aligned}$$

$$
\begin{aligned}
&= \pi^* g'(X, \varphi Y) \\
&= g(X, \varphi Y) \\
&= \phi(X, Y).
\end{aligned}
$$

Thus $\pi^*\Omega'(X, Y) = \phi(X, Y)$ which establishes (1).

It is known that if U is vertical, then it is in the kernel of π_*.
But $\pi_*\varphi U = J'\pi_* U$. Since $\pi_* U = 0$, then $\pi_*\varphi U = 0$ which shows that φU is vertical because it is in the kernel of π_*. This is the proof of assertion (2).

Concerning assertion (3), let X be horizontal and U vertical. At assertion (2), we have shown that φU is vertical so that $g(X, \varphi U) = 0$. On the other hand, $g(X, \varphi U) = -g(\varphi X, U)$; therefore φX is horizontal because it is orthogonal to U which is vertical.

Let us examine the case of assertion (4). It is clear that $J'\pi_*\xi = \pi_*\varphi\xi = 0$ because $\varphi\xi = 0$. This is to explain why $J'\pi_*\xi = 0$ which implies that $\pi_*\xi = 0$ and so $\xi \in ker\pi_*$.

To establish assertion (5), remember that $\eta(X) = g(X, \xi) = 0$ since ξ is vertical and X is horizontal.

Regarding assertion (6), since π_* is an isometry on horizontal vector fields, one has

$$\pi_* N^{(1)}(X, Y) = \pi_* N_\varphi(X, Y) - \pi_*(2d\eta(X, Y))\pi_*\xi.$$

As $\pi_*\xi = 0$, then

$$\pi_* N^{(1)}(X, Y) = \pi_* N_\varphi(X, Y).$$

With $\pi_*\varphi = J'\pi_*$ in mind, we have

$$
\begin{aligned}
\pi_* N_\varphi(X, Y) &= N_{J'}(\pi_* X, \pi_* Y) \\
&= N_{J'}(X_*, Y_*)
\end{aligned}
$$

Thus $\pi_* N^{(1)}(X, Y) = N_{J'}(X_*, Y_*)$ from which $\pi_* N^{(1)} = N_{J'}$ follows. $\square$

Proposition 3.3. *Let $\pi : M^{2m+1} \longrightarrow M^{2m'}$ be an almost paracontact metric submersion of type II. Then,*

(1) φX is basic associated to $J'X_*$ if X is basic;

(2) $\mathcal{H}(\nabla_X\varphi)Y$ is basic associated to $(\nabla'_{X_*}J')Y_*$ when X and Y are basic.

Proof. (1) We have shown that $\varphi X \in H(M)$ because $X \in H(M)$.
Since $\pi_*\varphi X = J'\pi_* X$, and $\pi_* X = X_*$, then $\pi_* J'X = J'X_*$ which shows that φX is basic associated to $J'X_*$.

According to [7], $\mathcal{H}(\nabla_X)Y$ is basic associated to $(\nabla'_{X_*}Y_*)$, we then have $\mathcal{H}(\nabla_X\varphi)Y$ is basic associated to $(\nabla'_{X_*}J')Y_*$. $\square$

3.2. Structure of the base space

Proposition 3.4. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost paracontact metric submersion of type II. If the total space is para-cosymplectic, quasi para-Sasakian or para-Kenmotsu, then the base space is para-Kählerian.*

Proof. In this proposition, the proof consist in showing that $d\Omega' = 0 = N_{J'}$. Let X, Y and Z be three basic vector fields. For a para-cosymplectic manifold, we can refer to its defining relation $\nabla_X \varphi = 0$ which gives $(\nabla'_{X_*})J' = 0$ and this is the defining relation of a para-Kähler structure on the base space.

Concerning the quasi para-Sasakian structure, the defining relation $d\phi = 0$ gives $d\Omega' = 0$. Since $N^{(1)} = 0$ then $N_{J'} = 0$. We then reach the defining relation of a para-Kähler structure on the base space.

Consider the case of a para-Kenmotsu manifold, which is defined by
$d\phi(D, E, G) = \frac{2}{3}\mathcal{G}\left\{\eta(D)\phi(E, G)\right\}$, $d\eta = 0$ and $N^{(1)} = 0$,

These relations become $d\phi(X, Y, Z) = \frac{2}{3}\mathcal{G}\left\{\eta(X)\phi(Y, Z)\right\}$ and $N_{J'} = 0$ on the base space. Since η vanishes on horizontal vector fields, we have
$d\phi(X, Y, Z) = 0$, which gives $d\Omega' = 0$. On the other hand,

$$N_{J'}(X_*, Y_*) = N^{(1)}(X, Y) = 0.$$

Therefore, the base space is defined by $d\Omega' = 0 = N_{J'}$, which are the defining relations of the para-Kähler structure. $\square$

Proposition 3.5. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost paracontact metric submersion of type II. If the total space is almost para-cosymplectic or an almost para-Kenmotsu manifold, then the base space is almost para-Kählerian.*

Proof. As in the preceding proposition, the problem is to show that $d\Omega' = 0$ which is the defining relation of an almost para-Kähler structure.

Let the total space M be endowed with an almost para-cosymplectic structure. As in the preceding proposition, on the base space, its defining relation gives $d\Omega' = 0$ which defines an almost para-Kähler structure.

Concerning the case of almost para-Kenmotsu structure, we also have
$d\Omega' = 0$ because $d\phi = 0$ which gives $d\Omega' = 0$ as already established. $\square$

Proposition 3.6. *Assume that $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ is an almost paracontact metric submersion of type II. If the total space is nearly para-cosymplectic, nearly para-Kenmotsu or closely para-cosymplectic, then the base space is nearly para-Kählerian.*

Proof. To establish that the base space is nearly para-Kählerian, we have to show that $(\nabla'_{X_*} J')X_* = 0$.

Note that a nearly para-Kähler structure is defined by $(\nabla_D \Omega)(D, E) = 0$ which can be expressed as $g(E, (\nabla_D J)D) = 0$. With this, we see that $(\nabla_D J)D$ is orthogonal to E. But on the base space, since X is horizontal,
$g(X, (\nabla'_{X_*} J')X_*) = 0$ implies that $(\nabla'_{X_*} J')X_* = 0$.

Let us consider the case of nearly para-cosymplectic, defined by
$(\nabla_D \varphi)D = 0$. It is clear that on the base space one has
$(\nabla'_{X_*} J')X_* = 0$ defining a nearly para-Kähler structure.

In the same way, a closely para-cosymplectic structure is defined by
$(\nabla_D \varphi)D = 0 = d\eta$ so that on the base space we have $(\nabla'_{X_*} J')X_* = 0$.

Consider the case of nearly para-Kenmotsu structure which verifies
$(\nabla_D \varphi)D = -\eta(D)\varphi D$ and $d\eta = 0$. On the base space, this condition becomes $(\nabla'_{X_*} J')X_* = 0$ because $\eta(X) = 0$, we then get the nearly para-Kähler structure. □

Proposition 3.7. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost paracontact metric submersion of type II. If the total space is quasi para-K-cosymplectic or a quasi para-Kenmotsu manifold, then the base space is quasi para-Kählerian.*

Proof. A quasi para-Kähler structure means that

$$(\nabla'_{X_*} J')Y_* + (\nabla'_{J'X_*} J')J'Y_* = 0.$$

If the total space is quasi para-K-cosymplectic, as in the preceding cases, the base space verifies $(\nabla'_{X_*} J')Y_* + (\nabla'_{J'X_*} J')J'Y_* = 0$ because of the vanishing of η on horizontal vector fields. Thus one obtains the defining relation of a quasi para-Kähler structure.

Considering the case of a quasi para-Kenmotsu manifold, we have
$(\nabla'_{X_*} \Omega')(Y_*, Z_*) + (\nabla'_{J'X_*} \Omega')(J'Y_*, Z_*) = 0$ which is the defining relation of a quasi para-Kähler structure. □

REFERENCES

1. Blaga, A.M.: *$\eta-$Ricci solitons on para-Kenmotsu manifolds*, arXiv:1402.0223 v1[math.DG] 2 Feb.2014.
2. Dacko, P.: *On almost para-cosymplectic manifolds*, Tsukuba J. Math. **28** (2004), 193-213.
3. Dacko, P. and Olszak, Z.: *On weakly para-cosymplectic manifolds of dimension 3*, J. Geom. Physics, **57** (2007), 561-570.
4. Gray, A. and Hervella, L.M.: *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl., **123** (1980), 35-58.
5. Gunduzalp, Y. and Sahin, B.: *Paracontact semi-Riemannian submersions*, Turkish J. Math., **37**(2013), 114-128.
6. Kaneyuki, S. and Williams, F.L.: *Almost paracontact and paraHodge structures on manifolds*, Nagoya Math. J., **99**(1985), 173-187.

7. O'Neill, B.: *The fundamental equations of a submersion*, Michigan Math.J., **13** (1966), 459-469.
8. Sato, I.: *On a structure similar to the almost contact structure I*, Tensor, N.S. **30** (1976), 219-224.
9. Sato, I.: *On a structure similar to the almost contact structure II*, Tensor, N.S. **31** (1977), 199-205.
10. Shukla, S.S. and Shankar Uma, V.: *Paracomplex Paracontact Pseudo-Riemannian Submersions*, Hindawi Publ. Corporation Geometry, Vol. 2014, Article ID 616487.
11. Tshikuna-Matamba, T.: *Almost contact metric submersions in Kenmotsu geometry*, J. Math. Univ. Tokushima, **48** (2014), 1-16.
12. Tshikuna-Matamba, T.: *A note on Riemannian submersions with umbilical fibres*, Journal of Progressive Research in Mathematics, **6**(2)(2016), 778-784.
13. Tshikuna-Matamba, T.: *On the fibres of an almost paracontact metric submersion*, J. Math. Univ. Tokushima, **50** (2016), to appear.
14. Tshikuna-Matamba, T.: *A note on curvature tensors in almost Hermitian and contact geometry*, Facta Universitatis (Niš), Ser. Math. Inform., 31(3) (2016), 751–760.
15. Watson, B.: *The differential geometry of two types of almost contact metric submersions, in The Math. Heritage of C.F. Gauss, (Ed. G.M. Rassias),* World Sci. Publ. Co. Singapore, 1991, pp. 827-861.
16. Watson, B.: *Superminimal fibres in an almost Hermitian submersion*, Boll. Un. Mat. Ital., **(8)I-B** (2000), 159-172.
17. Zamkovoy, S.: *Cannonical connections on paracontact manifolds*, Ann. Global Anal. Geometry, **36** (2009), 37-60.

Tshikunguila Tshikuna-Matamba
Departement de Mathematiques
Institut Superieur Pedagogique de Kananga
B.P. 282-Kananga
Republique Democratique du Congo.
tshikmat@gmail.com