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ON GENERALIZED M-PROJECTIVE ϕ-RECURRENT TRANS-SASAKIAN MANIFOLDS *

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Abstract. The aim of the present paper is to study generalized M-projective ϕ-recurrent trans-Sasakian manifold and its various geometric properties. First, we find the sufficient condition for generalized M-projective ϕ-recurrent trans-Sasakian manifold to become Einstein. Then non-existence of generalized M-projective ϕ-recurrent trans-Sasakian manifold has been shown under certain condition. Finally, the sufficient condition for super generalized Ricci-recurrent was also established.

Keywords: Trans-Sasakian manifold; M-projective curvature tensor; Generalized ϕ-recurrent; Einstein manifold; Super generalized Ricci-recurrent; Quasi-generalized Ricci-recurrent

1. Introduction

A new class of almost contact manifold was initiated by Oubina [14], called trans-Sasakian manifold, which is of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are respectively known as the cosymplectic, α-Sasakian and β-Kenmotsu manifold; where α, β are being smooth scalar functions. In particular, if $\alpha = 0, \beta = 1$ and $\alpha = 1, \beta = 0$ then a trans-Sasakian manifold will become a Kenmotsu and Sasakian manifold, respectively.

In 1971, Pokhariyal and Mishra [15] defined a new curvature called M-projective curvature tensor on Riemannian manifold. After that many researcher such as Ojha [12, 13], Singh [20], Choubey and Ojha [3] studied some properties of M-projective curvature in different manifolds.

The idea of local symmetry of a Riemannian manifold started by Cartan [1]. This idea has been used by many authors in several directions such as recurrent manifolds by Walker [24], semi-symmetric manifold by Szabo [22], pseudo-symmetric manifold by Chaki [2], pseudo-symmetric spaces by Deszcz [5], weakly symmetric

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manifold by Tamassy and Binh [23], weakly symmetric Riemannian spaces by Selberg [21]. Despite, the idea of pseudo-symmetric by Chaki and Deszcz and weak symmetry by Selberge and Tamassy and Binh are different. As a mild version of local symmetry, Takahashi [25] introduced the notion of ϕ-symmetry on a Sasakian manifold. For generalizing the idea of ϕ-symmetry, De et al. [8] introduced the concept of ϕ-recurrent Sasakian manifold. De [7] and Pal [11] studied generalized concircularly recurrent and generalized M-projectively recurrent Riemannian manifold. The purpose of this paper is to study generalized ϕ-recurrent trans-Sasakian manifold using M-projective curvature in place of Riemannian curvature i.e generalized M-projectively ϕ-recurrent trans-Sasakian manifold.

The paper is organized as follows:

In Section 2, we gave basic formulae of trans-Sasakian manifold and some relevant definitions. In Section 3, we studied generalized M-projectively ϕ-recurrent trans-Sasakian manifold and obtain a sufficient condition for such a manifold to be Einstein. Then, we found the condition such that the generalized M-projectively ϕ-recurrent trans-Sasakian manifold will not exist. Finally, we find different condition for such manifold to be super generalized Ricci recurrent and quasi-generalized Ricci-recurrent.

2. Preliminaries

In this section, we mention some basic formulae and definitions which will be used later.

Let M^m be an $m = (2n+1)$ dimensional almost contact metric manifold [4, 17] equipped with an almost contact metric structure $(\phi,\ \xi,\ \eta,\ g)$ consisting of a $(1,1)$ tensor field ϕ, a characteristic vector field ξ, a 1-form η and a Riemannian metric g. Then

$$\phi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \eta(\phi X) = 0, \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\xi, \xi) = 1, \phi \circ \xi = 0, \eta \circ \phi = 0, \tag{2.3}$$

for any X, Y in TM. From (2.1) and (2.2), it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), g(X, \xi) = \eta(X). \tag{2.4}$$

For an almost contact metric structure (ϕ, ξ, η, g) on M, we put

$$\Phi(X, Y) = g(X, \phi Y). \tag{2.5}$$

Let M^{2n+1} be almost contact manifold and consider the structure $(M \times \mathcal{R},\ \mathcal{J},\ \mathcal{G})$ belongs to the class W_4 of the Hermitian manifolds, we denote a vector field on

$M \times \mathcal{R}$ by $(X, f\frac{d}{dt})$, where X is tangent to M, t is the coordinates of $\mathcal{R}$ and f as C^∞ function on $M \times \mathcal{R}$. Define an almost complex structure [9]

$$\mathcal{J}\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

for any vector field X on $M \times \mathcal{R}$ and $\mathcal{G}$ is Hermitian metric on the product $M \times \mathcal{R}$. This may be expressed by the condition

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.6}$$

where ∇ is a Levi-civita connection and α, β are some smooth functions on M^{2n+1} and we say that trans-Sasakian structure is of type (α, β). From the above it is follows that

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \tag{2.7}$$

$$(\nabla_X \xi) = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{2.8}$$

On trans-Sasakian manifold M^{2n+1} with structure $(\phi,\ \xi,\ \eta,\ g)$, the following relations hold [4, 17]:

$$\begin{aligned} R(X,Y,\xi) \;=\; & (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + (Y\alpha)\phi X - (X\alpha)\phi Y \\ & +2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \end{aligned} \tag{2.9}$$

$$R(\xi, X, \xi) = (\alpha^2 - \beta^2 - \xi\beta)[\eta(X)\xi - X], \tag{2.10}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{2.11}$$

$$\eta(R(X,Y,\xi)) = \eta(R(\xi,Y,\xi)) = 0, \tag{2.12}$$

$$S(X,\xi) = [2n(\alpha^2 - \beta^2) - \xi\beta]\eta(X) - (2n-1)X\beta - (\phi X)\alpha, \tag{2.13}$$

$$S(\xi,\xi) = 2n(\alpha^2 - \beta^2 - \xi\beta), \tag{2.14}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n(\alpha^2 - \beta^2 - \xi\beta)\eta(X)\eta(Y), \tag{2.15}$$

$$Q\xi = (2n(\alpha^2 - \beta^2) - \xi\beta)\xi - (2n-1)grad\beta + \phi(grad\alpha), \tag{2.16}$$

$$S(X,Y) = g(QX,Y), \tag{2.17}$$

where R is the curvature tensor, S is the Ricci tensor, r is scalar curvature and Q being the symmetric endomorphism of the tangent space at each point corresponding to Ricci tensor S. Now, if we assume

$$\phi(grad\alpha) = (2n-1)grad\beta, \tag{2.18}$$

then [4, 17]

$$S(X,\xi) = 2n(\alpha^2-\beta^2)\eta(X), \tag{2.19}$$

$$S(\phi X,\phi Y) = S(X,Y) - 2n(\alpha^2-\beta^2)\eta(X)\eta(Y), \tag{2.20}$$

$$Q\xi = 2n(\alpha^2-\beta^2)\xi. \tag{2.21}$$

$$\begin{aligned}(\nabla_W S)(Y,\xi) &= 2n(\alpha^2-\beta^2)[-\alpha g(Y,\phi W) + \beta g(Y,W)] \\ &\quad +\alpha S(Y,\phi W) - \beta S(Y,W).\end{aligned} \tag{2.22}$$

Here, we are going to mention some definitions, which will be considered in the later results:

Definition 2.1. [9] A Riemannian manifold M^{2n+1} is said to be ϕ-symmetric, if the curvature tensor R satisfies the relation

$$\phi^2((\nabla_W R)(X,Y,Z)) = 0, for\ all\ X, Y\ and\ Z \in TM. \tag{2.23}$$

Definition 2.2. [9] A Riemannian manifold M^{2n+1} is said to be generalized Ricci-recurrent, if the Ricci tensor S satisfies the relation

$$(\nabla_W S)(X,Y) = A(W)S(X,Y) + B(W)g(X,Y), \tag{2.24}$$

for all X, Y and $W \in TM$ and A, B are the non-vanishing 1-forms.

Definition 2.3. [18] A Riemannian manifold M^{2n+1} is said to be super generalized Ricci-recurrent, if the Ricci tensor S satisfies the relation

$$(\nabla_W S)(X,Y) = A(W)S(X,Y) + B(W)g(X,Y) + C(W)\eta(X)\eta(Y), \tag{2.25}$$

for all $X,\ Y$ and $W \in TM$ and A, B and C are the non-vanishing 1-forms.

In specific, if $B(W) = C(W)$, then the relation (2.25) converted to the quasi-generalized Ricci-recurrent manifold [19].

3. Generalized M-projective ϕ-recurrent trans-Sasakian manifold

Definition 3.1. A trans-Sasakian manifold M^{2n+1} is said to be generalized M-projective ϕ-recurrent, if the M-projective curvature tensor M^* satisfies the relation

$$\begin{aligned} \phi^2((\nabla_W M^*)(X,Y,Z)) &= A(W)M^*(X,Y,Z) \\ &+B(W)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.1}$$

where A and B are two 1-forms, B is non-zero and defined by

$$g(W,\rho_1) = A(W), \; g(W,\rho_2) = B(W),$$

and

$$\begin{aligned} &M^*(X,Y,Z) \\ &= R(X,Y,Z) - \frac{1}{4n}\Big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\Big], \end{aligned} \tag{3.2}$$

for all $X, Y, Z, W \in TM$ and ρ_1, ρ_2 being vector fields associated to the 1-form A and B, respectively.

Theorem 3.1. *If a generalized M-projective ϕ-recurrent trans-Sasakian manifold M^{2n+1} satisfies $\phi(grad\alpha) = (2n-1)grad\beta$, then the associated 1-form A and B are related by the equation*

$$[2n(2n+1)(\alpha^2 - \beta^2) - r]A(W) + 8n^2B(W) - dr(W) = 0. \tag{3.3}$$

Proof. Let us consider that M^{2n+1} be a generalized M-projective ϕ-recurrent trans-Sasakian manifold. Then by virtue of the relation (2.1), the equation (3.1) becomes

$$\begin{aligned} &-(\nabla_W M^*)(X,Y,Z) + \eta((\nabla_W M^*)(X,Y,Z))\xi \\ &= A(W)M^*(X,Y,Z) + B(W)(g(Y,Z)X - g(X,Z)Y). \end{aligned} \tag{3.4}$$

From the above equation, it follows that

$$-g((\nabla_W R)(X,Y,Z),U) + g((\nabla_W R)(X,Y,Z),\xi)g(U,\xi)$$
$$+\frac{1}{4n}\Big[(\nabla_W S)(Y,Z)g(X,U) - (\nabla_W S)(X,Z)g(Y,U)$$
$$+g(Y,Z)(\nabla_W S)(X,U) - g(X,Z)(\nabla_W S)g(Y,U)\Big]$$
$$-\frac{1}{4n}\Big[(\nabla_W S)(Y,Z)g(X,\xi) - (\nabla_W S)(X,Z)g(Y,\xi)$$
$$+g(Y,Z)(\nabla_W S)(X,\xi) - g(X,Z)(\nabla_W S)g(Y,\xi)\Big]\eta(U)$$
$$= A(W)\Big[g(R(X,Y,Z),U) - \frac{1}{4n}\Big(S(Y,Z)g(X,U) - S(X,Z)g(Y,U)$$
$$+g(Y,Z)S(X,U) - g(X,Z)S(Y,U)\Big)\Big]$$
$$+B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \tag{3.5}$$

Let us suppose $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $X = U = e_i$ in the relation (3.5) and taking summation over i, $1 \leq i \leq 2n+1$, we obtain

$$-(\nabla_W S)(Y,Z) + \frac{1}{4n}\Big[(2n-1)(\nabla_W S)(Y,Z) + dr(W)g(Y,Z)\Big]$$
$$+\eta((\nabla_W R)(\xi,Y,Z)) - \frac{1}{4n}\Big[(\nabla_W S)(Y,Z) - (\nabla_W S)(\xi,Z)\eta(Y)$$
$$+g(Y,Z)(\nabla_W S)(\xi,\xi) - (\nabla_W)(S,\xi)\eta(Z)\Big]$$
$$= \frac{2n+1}{4n}A(W)S(Y,Z) + \Big[2nB(W) - \frac{r}{4n}A(W)\Big]\,g(Y,Z). \tag{3.6}$$

Putting $Z = \xi$ in the above equation, we can find

$$-(\nabla_W S)(Y,\xi) + \frac{1}{4n}\Big[(2n-1)(\nabla_W S)(Y,\xi) + dr(W)g(Y,\xi)\Big]$$
$$+\eta((\nabla_W R)(\xi,Y,\xi)) - \frac{1}{4n}\Big[(\nabla_W S)(Y,\xi) - (\nabla_W S)(\xi,\xi)\eta(Y)$$
$$+g(Y,\xi)(\nabla_W S)(\xi,\xi) - (\nabla_W)(Y,\xi)\Big]$$
$$= \frac{2n+1}{4n}A(W)S(Y,\xi) + \Big[2nB(W) - \frac{r}{4n}A(W)\Big]\eta(Z). \tag{3.7}$$

By virtue of the relations (2.10), (2.12) and (2.22), we obtain

$$\Big(-1+\frac{(2n-1)}{4n}\Big)(\nabla_W S)(Y,\xi)+\frac{dr(W)}{4n}\eta(Y)$$
$$=A(W)\left[S(Y,\xi)-\frac{1}{4n}\Big((2n-1)S(Y,\xi)+r\eta(Y)\Big)\right]$$
$$+2nB(W)\eta(Y). \tag{3.8}$$

Putting $Y=\xi$ and then using the equations (2.19) and (2.22), we have the relation (3.3). □

Theorem 3.2. *A generalized M-projective ϕ-recurrent trans-Sasakian manifold M^{2n+1} satisfying $\phi(grad\alpha)=(2n-1)grad\beta$ is an Einstein manifold.*

Proof. Let M^{2n+1} be a generalized M-projective ϕ-recurrent trans-Sasakian manifold. By making use of the equations (2.19) and (3.3) in the relation (3.8), one can easily found

$$(\nabla_W S)(Y,\xi)=0. \tag{3.9}$$

By virtue of the equation (2.22), the above equation becomes

$$2n(\alpha^2-\beta^2)[-\alpha g(Y,\phi W)+\beta g(Y,W)]+\alpha S(Y,\phi W)-\beta S(Y,W)=0. \tag{3.10}$$

Interchanging Y and W by ϕY and ϕW, respectively in the above relation and then using equations (2.1), (2.4), (2.17), (2.18) and (2.21), we get

$$S(Y,W)=2n(\alpha^2-\beta^2)g(Y,W)$$

and

$$S(\phi Y,W)=2n(\alpha^2-\beta^2)g(\phi Y,W). \tag{3.11}$$

Hence, it is Einstein. □

Theorem 3.3. *Let M^{2n+1} be an Einstein trans-Sasakian manifold with a constant scalar curvature satisfying $\phi(grad\alpha)=(2n-1)grad\beta$, then it can not be generalized M-projective ϕ-recurrent.*

Proof. Let M^{2n+1} be trans-Sasakian manifold. Since it is an Einstein manifold, hence with the help of relation (2.20), we can obtain

$$r=2n(2n+1)(\alpha^2-\beta^2). \tag{3.12}$$

Now, suppose if possible, M^{2n+1} is a generalized M-projective ϕ-recurrent. Then by virtue of above relation relation, the equation (3.3) implies that

$$dr(W)=8n^2B(W).$$

Also, since r is constant, therefore $dr(W) = 0$ and hence from the above relation, we can conclude

$$B(W) = 0,$$

which is a contradiction to the fact that for generalized M-projective ϕ-recurrent $B(W) \neq 0$. Thus we finished the proof. □

Theorem 3.4. *A generalized M-projective ϕ-recurrent trans-Sasakian manifold M^{2n+1} is super generalized Ricci-recurrent.*

Proof. Let M^{2n+1} be a generalized M-projective ϕ-recurrent trans-Sasakian manifold. Then taking contraction over Y and Z of the relation (3.5), we obtain

$$\begin{aligned} &-\frac{2n+1}{4n}(\nabla_W S)(X,U) - \frac{dr(W)}{4n}\eta(X)\eta(U) \\ &= \frac{2n+1}{4n}A(W)S(X,U) + \left[2nB(W) - \frac{dr(W)}{4n} - \frac{r}{4n}A(W)\right]g(X,U), \end{aligned} \tag{3.13}$$

which implies

$$\begin{aligned} (\nabla_W S)(X,U) &= -A(W)S(X,U) - \frac{dr(W)}{2n+1}\eta(X)\eta(U) \\ &\quad + \frac{1}{2n+1}\left[rA(W) + dr(W) - 8n^2B(W)\right]g(X,U), \end{aligned} \tag{3.14}$$

which shows that M^{2n+1} is a super generalized Ricci-recurrent. □

If we assume scalar curvature r is constant, then we can state the following corollary:

Corollary 3.1. *If a generalized M-projective ϕ-recurrent trans-Sasakian manifold M^{2n+1} is of constant scalar curvature, then it is generalized Ricci-recurrent.*

Next, if we consider

$$rA(W) - 8n^2B(W) = 0. \tag{3.15}$$

Then by the equation (3.14), we can write

$$\begin{aligned} (\nabla_W S)(X,U) &= -A(W)S(X,U) \\ &\quad - \frac{dr(W)}{2n+1}\left[g(X,U) + \eta(X)\eta(U)\right]. \end{aligned}$$

Thus we can state two other corollaries:

Corollary 3.2. *A generalized M-projective ϕ-recurrent trans-Sasakian manifold M^{2n+1} is quasi-generalized Ricci-recurrent, if the relation (3.15) hold.*

Corollary 3.3. *If a generalized M-projective ϕ-recurrent trans-Sasakian manifold M^{2n+1} is of constant scalar curvature and the relation (3.15) holds, then it is Ricci-recurrent.*

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