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FIXED POINT THEOREMS FOR TWO PAIRS OF MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. In this paper, a general fixed point theorem for two pairs of weakly compatible mappings satisfying a ϕ - implicit relation different from the type from [16] is proved. As applications, we obtain the sufficient conditions for the existence of fixed points for a sequence of mappings in partial metric spaces.

Keywords: Fixed point theorem; Weakly compatible mapping; Partial metric space

1. Introduction

In 1994, Matthews [12] introduced the concept of partial metric spaces as a part of the study of denotational semantics of dataflow networks and proved the Banach contraction principle in such spaces.

Many authors studied the fixed points for mappings satisfying contractive conditions in complete partial metric spaces. Quite recently, in [2], [4], [5], [6], [10], some fixed point theorems under various contractive conditions in complete partial metric spaces are proved.

In [10] some fixed point theorems for particular pairs of mappings in partial metric spaces are proved, which generalize some results by [4], [5] and other papers. In [2], other results for pairs of mappings are obtained.

In 1994, Pant [13] introduced the notion of R - weakly commutativity, which is equivalent to commutativity at coincidence points. Jungck [9] defined f and g to be weakly compatible if $fx = gx$ implies $fgx = gfx$. Thus, f and g are weakly compatible if and only of f and g are R - weakly commuting.

In [10], some fixed point theorems for two weakly compatible self mappings in partial metric spaces are proved.

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [14], [15] and

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in other papers. This method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, convex metric spaces, ultra - metric spaces, compact metric spaces, in two and three metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings, and also, it is used in the study of fixed points for mappings satisfying a contractive condition of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, ordered metric spaces and G - metric spaces. With this method the proofs of some fixed point theorems are more simple. As well, the method allows the study of local and global properties of fixed point structures.

The study of fixed points of self mappings in partial metric spaces for mappings satisfying an implicit relation is initiated in [16].

In [3], Altun and Turkoglu introduced a new type of implicit relation satisfying a ϕ - map.

Recently, new results for coupled functions are published in [7] and [8].

The purpose of this paper is to prove a general fixed point for two pairs of weakly compatible mappings satisfying a new type of ϕ - implicit relation. As application we prove a fixed point theorem for a sequence of mappings in complete partial metric spaces.

2. Preliminaries

Definition 2.1. ([12]) Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}_+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:

$(P_1) : p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$,

$(P_2) : p(x,x) \leq p(x,y)$,

$(P_3) : p(x,y) = p(y,x)$,

$(P_4) : p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X, p) is called a partial metric space.

If $p(x,y) = 0$, then by (P_1) and (P_2), $x = y$, but the converse does not always hold.

Each partial metric p on X generates a T_0 - topology τ_p which has as base the family of open p - balls $\{B_p(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0\}$, where $B_p(x,y) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a limit $x \in X$ with respect to τ_p if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

If p is a partial metric on X, then the function $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ defines a metric on X.

Definition 2.2. ([12]) Let (X, p) be a partial metric space.

a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

b) (X, p) is said to be complete if every Cauchy sequence in (X, p) converges with respect to τ_p to a point $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = p(x, x)$.

Lemma 2.1. *([12], [5]) Let (X, p) be a complete partial metric space. Then:*

(1) *A sequence in X is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in (X, p^s).*

(2) *A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Further,*

$$\lim_{n\to\infty} p^s(x_n, x) = 0 \text{ if and only if } p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{m,n\to\infty} p(x_n, x_m), \tag{2.1}$$

where $\{x_n\}$ is a Cauchy sequence which converges to a point x.

Lemma 2.2. *Let (X, p) be a partial metric space and $\{x_n\}$ a sequence in X. If $\lim_{n\to\infty} x_n = x$ and $p(x, x) = 0$, then $\lim_{n\to\infty} p(x_n, y) = p(x, y)$, $\forall y \in X$.*

Proof. By (P_4),

$$p(x, y) \le p(x, x_n) + p(x_n, y).$$

Hence

$$p(x, y) - p(x, x_n) \le p(x_n, y) \le p(x_n, x) + p(x, y).$$

Letting n tends to infinity we obtain

$$\lim_{n\to\infty} p(x_n, y) = p(x, y).$$

□

Definition 2.3. Let X be a nonempty set and $f, g : X \to X$ such that $w = fx = gx$ for some $x \in X$. Then x is said to be a coincidence point of f and g and w is a point of coincidence of f and g.

3. Implicit relations

Definition 3.1. A function $\varphi : [0, \infty) \to [0, \infty)$ is a ϕ - function, $\varphi \in \phi$, if φ is nondecreasing function such that $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for all $t > 0$ and $\varphi(0) = 0$.

Remark 3.1. Since $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$, then $\lim_{n\to\infty} \varphi^n(t) = 0$. Then, as in [11], $\varphi(t) < t$ for $t > 0$ and $\varphi(0) = 0$.

Definition 3.2. Let $\mathcal{F}_\phi$ be the set of all continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ such that:

(F_1) : F is nonincreasing in variables $t_2, ..., t_6$,

(F_2) : There exists a function $\varphi \in \phi$ such that:

(F_{2a}) : $F(u, v, v, u, u+v, v) \leq 0$ and

(F_{2b}) : $F(u, v, u, v, v, u+v) \leq 0$,

implies $u \leq \varphi(v)$.

In the following examples, the proof of property (F_1) is easy.

Example 3.1. $F(t_1, ..., t_6) = t_1 - k\max\{t_2, ..., t_6\}$, where $k \in \left[0, \frac{1}{2}\right)$.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, v, u, u+v, u) = u - k(u+v) \leq 0$, which implies $u \leq \frac{k}{1-k}v$ and (F_{2a}) is satisfied for $\varphi(t) = \frac{k}{1-k}t$.

Similarly, $F(u, v, u, v, v, u+v) \leq 0$ implies $u \leq \varphi(v)$.

Example 3.2. $F(t_1, ..., t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \frac{t_3+2t_4}{3}, \frac{t_5+t_6}{3}\right\}$, where $k \in [0, 1)$.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, v, u, u+v, u) = u - k\max\left\{u, v, \frac{v+2u}{3}, \frac{u+2v}{3}\right\} \leq 0$. If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq kv$ and (F_{2a}) is satisfied for $\varphi(t) = kt$.

Similarly, $F(u, v, u, v, v, u+v) \leq 0$ implies $u \leq \varphi(v)$.

Example 3.3. $F(t_1, ..., t_6) = t_1 - k\max\{at_2, b(t_3+2t_4), b(t_4+t_5+t_6)\}$, where $a \in (0, 1)$ and $b \in \left(0, \frac{1}{4}\right)$.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, v, u, u+v, u) = u - \max\{av, b(v+2u), 2b(u+v)\} \leq 0$. If $u > v$, then $u(1 - \max\{a, 4b\}) \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq \max\{a, 4b\}v$ and (F_{2a}) is satisfied for $\varphi(t) = \max\{a, 4b\}t$.

Similarly, $F(u, v, u, v, v, u+v) \leq 0$ implies $u \leq \varphi(v)$.

Example 3.4. $F(t_1, ..., t_6) = t_1 - k\max\{t_2, t_3+t_4, t_5+t_6\}$, where $k \in \left[0, \frac{1}{3}\right)$.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, v, u, u+v, u) = u - k(u+2v) \leq 0$, which implies $u \leq \frac{2k}{1-k}v$ and (F_{2a}) is satisfied for $\varphi(t) = \frac{2k}{1-k}t$.

Similarly, $F(u, v, u, v, v, u+v) \leq 0$ implies $u \leq \varphi(v)$.

Example 3.5. $F(t_1, ..., t_6) = t_1^2 - a\max\{t_2^2, t_3^2, t_4^2\} - bt_5t_6$, where $a, b \geq 0$ and $a+2b < 1$.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, v, u, u+v, u) = u^2 - k\max\left\{u^2, v^2\right\} - bv(u+v) \leq 0$. If $u > v$, then $u^2[1-(a+2b)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq \sqrt{a+2b}v$ and (F_{2a}) is satisfied for $\varphi(t) = \sqrt{a+2b}t$.

Similarly, $F(u, v, u, v, v, u+v) \leq 0$ implies $u \leq \varphi(v)$.

Example 3.6. $F(t_1,...,t_6) = t_1^3 - at_1^2t_2 - bt_1t_2^2 - ct_2t_3t_4 - dt_1t_5t_6$, where $a,b,c,d \geq 0$ and $a+b+c+2d < 1$.

(F_2) : Let $u,v \geq 0$ be and

$$F(u,v,v,u,u+v,u) = u^3 - au^2v - buv^2 - cuv^2 - du^2(u+v) \leq 0.$$

If $u > v$, then $u^3[1-(a+b+c+2d)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq \sqrt[3]{a+b+c+2d}v$ and (F_{2a}) is satisfied for $\varphi(t) = \sqrt[3]{a+b+c+2d}t$.

Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq \varphi(v)$.

Example 3.7. $F(t_1,...,t_6) = t_1 - \varphi(at_2+bt_3+ct_4+dt_5+et_6)$, where $a,b,c,d,e \geq 0$ and $a+b+c+2d+e < 1$.

(F_2) : Let $u,v \geq 0$ be and

$$F(u,v,v,u,u+v,u) = u - \varphi(av+bv+cu+d(u+v)+eu) \leq 0.$$

If $u > v$, then $u[1-\varphi((a+b+c+2d+e)v)] \leq 0$, which implies $u \leq \varphi((a+b+c+2d+e)u) \leq \varphi(u) < u$, a contradiction. Hence $u \leq v$, which implies $u \leq \varphi(v)$.

Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq \varphi(v)$.

Example 3.8. $F(t_1,...,t_6) = t_1 - \varphi(at_2+bt_3+c\max\{t_4+t_5,t_6\})$, where $a,b,c \geq 0$ and $a+b+3c < 1$.

(F_2) : Let $u,v \geq 0$ be and

$$F(u,v,v,u,u+v,u) = u - \varphi(av+bv+c\max\{2u+v,u\}) \leq 0.$$

If $u > v$, then $u \leq \varphi((a+b+3c)u) \leq \varphi(u) < u$, a contradiction. Hence $u \leq v$, which implies $u \leq \varphi(v)$.

Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq \varphi(v)$.

Example 3.9. $F(t_1,...,t_6) = t_1 - \varphi(at_2+bt_3+c\max\{3t_4,t_5+t_6\})$, where $a,b,c \geq 0$ and $a+b+3c < 1$.

(F_2) : Let $u,v \geq 0$ be and

$$F(u,v,v,u,u+v,u) = u - \varphi(av+bv+c\max\{3u,2u+v\}) \leq 0.$$

If $u > v$, then $u \leq \varphi((a+b+3c)u) \leq \varphi(u) < u$, a contradiction. Hence $u \leq v$, which implies $u \leq \varphi(v)$.

Similarly, $F(u,v,u,v,v,u+v) \leq 0$ implies $u \leq \varphi(v)$.

4. Main results

Lemma 4.1. [1] Let f and g be weakly compatible self mappings of a nonempty set X. If f and g have a unique point of coincidence $w = fx = gx$ for some $x \in X$, then w is the unique common fixed point of f and g.

Theorem 4.1. *Let (X,p) be a partial metric space and A,B,S,T be self mappings of X such that*

$$F(p(Ax,By),p(Sx,Ty),p(Sx,Ax),\\ p(Ty,By),p(Sx,By),p(Ty,Ax))\leq 0, \tag{4.1}$$

for all $x,y\in X$, where $F\in\mathcal{F}_\phi$. If there exist $u,v\in X$ such that $Au=Su$ and $Bv=Tv$, then there exists $t\in X$ such that t is the unique point of coincidence of A and S, as well the unique point of coincidence of B and T.

Proof. First we prove that $Su=Tv$. Suppose that $Su\neq Tv$. Then by (4.1) we get

$$F(p(Au,Bv),p(Su,Tv),p(Su,Au),\\ p(Tv,Bv),p(Su,Bv),p(Tv,Au))\leq 0,$$

$$F(p(Su,Tv),p(Su,Tv),p(Su,Tv),\\ p(Su,Tv),p(Su,Tv),p(Su,Tv))\leq 0.$$

By (P_2) and (F_1) we obtain

$$F(p(Su,Tv),p(Su,Tv),p(Su,Tv),\\ p(Su,Tv),2p(Su,Tv),p(Su,Tv))\leq 0.$$

By (F_{2a}) we get

$$p(Su,Tv)\leq\phi(p(Su,Tv))<p(Su,Tv),$$

a contradiction. Hence, $p(Su,Tv)=0$ and $Su=Tv=Au=Bv=t$ for some t of X.

Assuming that there exists $w\neq u$ such that $Aw=Sw$ and $Aw\neq Au$, then by (4.1) we obtain

$$F(p(Aw,Bv),p(Sw,Tv),p(Sw,Aw),\\ p(Tv,Bv),p(Sw,Bv),p(Tv,Aw))\leq 0,$$

$$F(p(Sw,Tv),p(Sw,Tv),p(Sw,Sw),\\ p(Tv,Tv),p(Sw,Tv),p(Sw,Tv))\leq 0.$$

By (F_1), (P_2) and (F_{2a}) we obtain

$$p(Sw,Tv)\leq\phi(p(Sw,Tv))<p(Sw,Tv),$$

if $p(Sw,Tv)\neq 0$, a contradiction. Hence, $p(Sw,Tv)=0$, which implies $Sw=Aw=Tv=Bv=Su=t$. Similarly one proves that t is the unique point of coincidence of B and T. □

Theorem 4.2. *Let (X,p) be a partial complete metric space and A,B,S,T be self mappings of X such that $AX \subset TX$ and $BX \subset SX$. If the inequality (4.1) holds for all $x,y \in X$, where $F \in \mathcal{F}_\phi$ and one of AX, BX, SX, TX is a closed subset of (X,p), then:*

(i) *A and S have a coincidence point,*

(ii) *B and T have a coincidence point.*

Moreover, if the pairs (A,S) and (B,T) are weakly compatible, then A,B,S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $AX \subset TX$, there exists $x_1 \in X$ such that $Tx_1 = Ax_0$. Since $BX \subset SX$, there exists $x_2 \in X$ such that $Sx_2 = Bx_1$. Continuing this process, we construct the sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = Tx_{2n+1} = Ax_{2n},\ y_{2n+1} = Sx_{2n+2} = Bx_{2n+1},\ n \in \mathbb{N}. \tag{4.2}$$

We prove that $\{y_n\}$ is a Cauchy sequence in (X,p). By (4.1) for $x = x_{2n}$ and $y = x_{2n+1}$ we have successively

$$\begin{gathered} F(p(Ax_{2n}, Bx_{2n+1}), p(Sx_{2n}, Tx_{2n+1}), p(Sx_{2n}, Ax_{2n}), \\ p(Tx_{2n+1}, Bx_{2n+1}), p(Sx_{2n}, Bx_{2n+1}), p(Tx_{2n+1}, Ax_{2n})) \leq 0, \end{gathered}$$

$$\begin{gathered} F(p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n}), p(y_{2n-1}, y_{2n}), \\ p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n+1}), p(y_{2n}, y_{2n})) \leq 0. \end{gathered} \tag{4.3}$$

Since by (P_4),

$$\begin{aligned} p(y_{2n-1}, y_{2n+1}) &\leq p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1}) - p(y_{2n}, y_{2n}) \\ &\leq p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1}) \end{aligned}$$

and by (P_2),

$$p(y_{2n}, y_{2n}) \leq p(y_{2n-1}, y_{2n}),$$

by (4.2) and (F_1) we obtain

$$\begin{gathered} F(p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n}), p(y_{2n-1}, y_{2n}), \\ p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n})) \leq 0. \end{gathered}$$

By (F_{2a}) we obtain

$$p(y_{2n}, y_{2n+1}) \leq \phi(p(y_{2n-1}, y_{2n})).$$

By (4.1) for $x = x_{2n+2}$ and $y = x_{2n+1}$, we obtain

$$\begin{gathered} F(p(Ax_{2n+2}, Bx_{2n+1}), p(Sx_{2n+2}, Tx_{2n+1}), p(Sx_{2n+2}, Ax_{2n+2}), \\ p(Tx_{2n+1}, Bx_{2n+1}), p(Sx_{2n+2}, Bx_{2n+1}), p(Tx_{2n+1}, Ax_{2n+2})) \leq 0, \end{gathered}$$

$$F(p(y_{2n+2},y_{2n+1}),p(y_{2n+1},y_{2n}),p(y_{2n+1},y_{2n+2}),\\ p(y_{2n},y_{2n+1}),p(y_{2n+1},y_{2n+1}),p(y_{2n},y_{2n+2}))\leq 0. \tag{4.4}$$

Since by (P_4),

$$p(y_{2n},y_{2n+2})\leq p(y_{2n},y_{2n+1})+p(y_{2n+1},y_{2n+2})$$

and by (P_2),

$$p(y_{2n+1},y_{2n+1})\leq p(y_{2n},y_{2n+1}),$$

by (4.4) and (F_1) we obtain

$$F(p(y_{2n+2},y_{2n+1}),p(y_{2n},y_{2n+1}),p(y_{2n+1},y_{2n+2}),\\ p(y_{2n},y_{2n+1}),p(y_{2n},y_{2n+1}),p(y_{2n},y_{2n+1})+p(y_{2n+1},y_{2n+2}))\leq 0.$$

By (F_{2b}) we obtain

$$p(y_{2n+2},y_{2n+1})\leq\phi(p(y_{2n+1},y_{2n}))$$

which implies

$$p(y_n,y_{n+1})\leq\phi(p(y_{n-1},y_n))\leq ...\leq\phi^n(p(y_1,y_0)).$$

For $n,m\in\mathbb{N}$ with $m>n$, by repeated use of (P_4), we have that

$$\begin{aligned} p(y_n,y_m) &\leq p(y_n,y_{n+1})+p(y_{n+1},y_{n+2})+...+p(y_{m-1},y_m)\\ &\leq \sum_{k=n}^{m-1}\phi^k(p(y_0,y_1)). \end{aligned}$$

Since $\sum\limits_{k=0}^{\infty}\phi^k(p(y_0,y_1))<\infty$, then $\lim_{n\to\infty}\sum\limits_{k=n}^{m-1}\phi^k(p(y_0,y_1))=0$ and $\lim_{n,m\to\infty}p(y_n,y_m)=0$ and so

$$p^s(y_n,y_m)\leq 2p(y_n,y_m)\to 0 \text{ as } n,m\to\infty.$$

This implies that $\{y_n\}$ is a Cauchy sequence in the metric space (X,p^s). Since (X,p) is complete, by Lemma 2.1, (X,p^s) is complete. Therefore, there exists $y\in X$ such that $\lim_{n\to\infty}p^s(y_n,y)=0$ and by (2.1)

$$p(y,y)=\lim_{n\to\infty}p(y_n,y)=\lim_{n,m\to\infty}p(y_n,y_m)=0.$$

This implies that $\lim_{n\to\infty}p(y_{2n},y)=\lim_{n\to\infty}p(y_{2n-1},y)=0$ and $\lim_{n\to\infty}p(Sx_{2n},y)=0$.

Now we can suppose, without loss of generality, that SX is a closed subset of the partial metric space (X,p). Then there exists $u\in X$ such that $y=Su$.

By (4.1) with $x = u$ and $y = x_{2n+1}$, we have

$$F(p(Au, Bx_{2n+1}), p(Su, Tx_{2n+1}), p(Su, Au), \\ p(Tx_{2n+1}, Bx_{2n+1}), p(Su, Bx_{2n+1}), p(Tx_{2n+1}, Au)) \leq 0,$$

$$F(p(Au, y_{2n+1}), p(Su, y_{2n}), p(Su, Au), \\ p(y_{2n}, y_{2n+1}), p(Su, y_{2n+1}), p(y_{2n}, Au)) \leq 0. \tag{4.5}$$

Letting n tends to infinity we obtain by Lemma 2.2 that

$$F(p(y, Au), 0, p(y, Au), 0, 0, p(y, Au)) \leq 0,$$

which implies by (F_{2b}) that $p(y, Au) \leq \phi(0) = 0$, i.e. $y = Au = Su$ and u is a coincidence point of A and S.

Since $AX \subset TX$, $y \in TX$, hence there exists $v \in X$ such that $y = Tv$. By (4.1) for $x = u$ and $y = v$ we obtain

$$F(p(Au, Bv), p(Su, Tv), p(Su, Au), p(Tv, Bv), p(Su, Bv), p(Tv, Au)) \leq 0,$$

$$F(p(y, Bv), 0, 0, p(y, Bv), p(y, Bv), 0) \leq 0.$$

By (F_{2a}) it follows that $p(y, Bv) = 0$, i.e. $y = Bv = Tv$ and v is a coincidence point of T and B. By Theorem 4.1, y is the unique point of coincidence of (A, S) and (B, T).

Moreover, if (A, S) and (B, T) are weakly compatible, y is the unique common fixed point A, B, S and T. □

If $A = B$ and $S = T$, we obtain the following result:

Corollary 4.1. *Let (X, p) be a partial complete metric space and A, S be self mappings of X with $AX \subset SX$ such that*

$$F(p(Ax, Ay), p(Sx, Sy), p(Sx, Ax), \\ p(Sy, Ay), p(Sx, Ay), p(Sy, Ax)) \leq 0, \tag{4.6}$$

for all $x, y \in X$, where $F \in \mathcal{F}_\phi$. If one of AX or SX is a closed set in X, then A and S have a coincidence point.

Moreover, if (A, S) is weakly compatible, then A and S have a coincidence point.

For a function $f : (X, p) \to (X, p)$ we denote

$$Fix(f) = \{x \in X : x = fx\}.$$

Theorem 4.3. *Let A, B, S and T be self mappings of a partial metric space (X, p). If the inequality (4.1) holds for all $x, y \in X$, then*

$$[Fix(S) \cap Fix(T)] \cap Fix(A) = [Fix(S) \cap Fix(T)] \cap Fix(B).$$

Proof. Let $x \in [Fix\,(S) \cap Fix\,(T)] \cap Fix\,(A)$. Then by (4.1) we have

$$F(p(Ax, Bx), p\,(Sx, Tx)\,, p\,(Sx, Ax)\,, p\,(Tx, Bx)\,, p\,(Sx, Bx)\,, p\,(Tx, Ax)) \leq 0,$$

$$F(p(x, Bx), p\,(x, x)\,, p\,(x, x)\,, p\,(x, Bx)\,, p\,(x, Bx)\,, p\,(x, x)) \leq 0.$$

By (P_2),

$$p\,(x, x) \leq p\,(x, Bx)$$

and by (F_1) we obtain

$$\begin{gathered} F(p\,(x, Bx)\,, p\,(x, Bx)\,, p\,(x, Bx)\,, p\,(x, Bx)\,, \\ p\,(x, Bx) + p\,(x, Bx)\,, p\,(x, Bx)) \leq 0, \end{gathered}$$

which implies by (F_{2a})

$$p\,(x, Bx) \leq \phi\,(p\,(x, Bx)) < p\,(x, Bx)\,,$$

a contradiction if $p\,(x, Bx) \neq 0$. Hence $p\,(x, Bx) = 0$, i.e. $x = Bx$. Then

$$[Fix\,(S) \cap Fix\,(T)] \cap Fix\,(A) \subset [Fix\,(S) \cap Fix\,(T)] \cap Fix\,(B)\,.$$

Similarly

$$[Fix\,(S) \cap Fix\,(T)] \cap Fix\,(B) \subset [Fix\,(S) \cap Fix\,(T)] \cap Fix\,(A)\,.$$

□

Theorems 4.2 and 4.3 imply:

Corollary 4.2. *Let S, T and $\{T_i\}_{i\in\mathbb{N}^*}$ be self mappings of a partial metric space such that:*

a) $T_2\,(X) \subset S\,(X)$ *and* $T_1\,(X) \subset T\,(X)$,

b) *one of* $T_1\,(X)$, $T_2\,(X)$, $S\,(X)$, $T\,(X)$ *is a closed subset of* X *and*

c) *the following inequality:*

$$\begin{gathered} F(p\,(T_i x, T_{i+1} y)\,, p\,(Sx, Ty)\,, p\,(Sx, T_i x)\,, \\ p\,(Ty, T_{i+1} y)\,, p\,(Sx, T_{i+1} y)\,, p\,(Ty, T_i x)) \leq 0 \end{gathered}$$

holds for all $x, y \in X$, $\forall i \in \mathbb{N}^*$ and $F \in \mathcal{F}_\phi$. Then S, T and $\{T_i\}_{i\in\mathbb{N}^*}$ have a unique common fixed point.

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REFERENCES

1. M. Abbas and B. E. Rhoades: *Common fixed points results for noncommuting mappings without continuity in generalized metric spaces.* Appl. Math. Comput. **215** (2009), 262–269.
2. T. Abdeljawad, E. Karapinar and K. Tas: *Existence and uniqueness of a common fixed point on a partial metric space.* Appl. Math. Lett. **24** (2011), 1900–1904.
3. I. Altun and D. Turkoglu: *Some fixed point theorems for weakly compatible mappings satisfying an implicit relation.* Taiwanese J. Math. **13** (4) (2009), 1291–1304.
4. I. Altun, F. Sola and H. Simsek: *Generalized contractions on partial metric spaces.* Topology Appl. **157** (18) (2010), 2778–2785.
5. R. P. Chi, E. Karapinar and T. D. Thanh: *A generalized contraction principle in partial metric spaces.* Math. Comput. Modelling **55** (2012), 1673–1681.
6. L. Cirić, B. Samet, H. Aydi and C. Vetro: *Common fixed point of generalized contraction on partial metric spaces and applications.* Appl. Math. Comput. **218** (2011), 2398–2406.
7. B. Deshpande and A. Handa: *Common coupled xed point theorems for two hybrid pairs of mappings satisfying an implicit relation.* Sarajevo J. Math. **11** (23) (1) (2015), 85–100.
8. B. Deshpande, A. Handa and C. Kothari: *Employing weak $\phi - \psi$ contraction in common coupled fixed point results for hybrid pairs of mappings satisfying (EA) Property.* IJMAA **3** (4-F) (2015), 29–44.
9. G. Jungck: *Common fixed points for noncommuting nonself mappings on non-numeric spaces.* Far East J. Math. Sci. **4** (2) (1996), 195–215.
10. Z. Kadelburg, H. K. Nashine and S. Radenović: *Fixed point results under various contractive conditions in partial metric spaces.* RACSAM **107** (2013), 241–256.
11. J. Matkowski: *Fixed point theorems for mappings with a contractive iterate at a point.* Proc. Amer. Math. Soc. **62**, 2 (1997), 344–348.
12. S. G. Matthews: *Partial metric topology.* Proc. 8^{th} Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. **728** (1994), 183–192.
13. R. P. Pant: *Common fixed point theorems for noncommuting mappings.* J. Math. Anal. Appl. **188** (1994), 436–440.
14. V. Popa: *Fixed point theorems for implicit contractive mappings.* Stud. Cerc. Ştiinţ., Ser. Mat., Univ. Bacău **7** (1997), 127–134.
15. V. Popa: *On some fixed point theorems for compatible mappings satisfying an implicit relation.* Demonstr. Math. **32** (1) (1999), 157–163.
16. C. Vetro and F. Vetro: *Common fixed points of mappings satisfying implicit relations in partial metric spaces.* J. Nonlinear Sci. Appl. **6** (2013), 152–161.

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