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ON MAXIMAL FUNCTION AND V-CONJUGATION

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Abstract. In this paper we prove that on the 3-series field H^1 cannot be defined by means of the V-conjugation. In other words, the norms $\|f\|_{H^1}$ and $\|\tilde{f}\|_{L^1}$ are not equivalent in the case of the 3-series field. This gives a new proof to the result of Memić [7], which answers a question raised by P. Simon [13]. Also, we prove that the mean value of function $f \in L^1(G)$ on the coset $I_{N-1}(x)$ is dominated by either $\sigma_{M_{N-1}}$ or σ_{M_N} on some translated element.

Keywords: Maximal function; V-conjugation; Vilenkin groups

1. Introduction and Preliminaries

The definition of Hardy spaces is possible in several ways. The only question is which of these possibilities is useful in respect of the Vilenkin-Fourier analysis.

For bounded locally compact Vilenkin groups, the Hardy space can be characterized in two equivalent ways. The first one is by atomic structure. The atomic decomposition is a useful characterization of Hardy spaces by the help of which some duality theorems and martingale inequalities can be proved [19]. It is known [9] that in the investigations with respect to the Vilenkin system the boundedness condition plays an important part. In the bounded case the H^1 space is atomic.

The second way of characterizations of Hardy spaces is by maximal function. In the theory of trigonometric series it is well known that the classical H^1 space contains exactly those L^1-functions, whose (trigonometric) conjugate function is integrable. We investigate the analogous question for the Vilenkin system.

We use standard notations and many formulae contained in [4], where the maximal function and maximal operator were substantially studied.

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Let $(m_0, m_1, \ldots, m_n, \ldots)$ be a bounded sequence of integers not less than 2. Put $m := \max_n m_n$.

Let $G := \prod_{n=0}^{\infty} \mathbb{Z}_{m_n}$, where $\mathbb{Z}_{m_n}$ denotes the discrete group of order m_n, with addition *mod* m_n. Each element from G can be represented as a sequence $(x_n)_n$, where $x_n \in \{0, 1, \ldots, m_n - 1\}$. Addition in G is obtained coordinatewise.

The group of integers of the p-series field is similarly defined by $\prod_{n=0}^{\infty} \mathbb{Z}_p$, where $p \geq 2$ is an integer.

The topology on G is generated by the subgroups
$I_n := \{x = (x_i)_i \in G, x_i = 0 \text{ for } i < n\}$, and their translations $I_n(y) := \{x = (x_i)_i \in G, x_i = y_i \text{ for } i < n\}$.

The basis $(e_n)_n$ is formed by elements $e_n = (\delta_{in})_i$.

Define the sequence $(M_n)_n$ as follows: $M_0 = 1$ and $M_{n+1} = m_n M_n$.

If $|I_n|$ denotes the normalized product measure of I_n then it can be easily seen that $|I_n| = M_n^{-1}$.

For every nonnegative integer n, there exists a unique sequence $(n_i)_i$ so that $n = \sum_{i=0}^{\infty} n_i M_i$.

The generalized Rademacher functions are defined by

$$r_n(x) := e^{\frac{2\pi i x_n}{m_n}}, n \in \mathbb{N} \cup \{0\}, x \in G,$$

and the system of Vilenkin functions by

$$\psi_n(x) := \prod_{i=0}^{\infty} r^{n_i}(x), n \in \mathbb{N} \cup \{0\}, x \in G.$$

The Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the Vilenkin system are respectively defined as follows

$$\hat{f}(n) = \int f(x)\bar{\psi}_n(x)dx,$$

$$S_n f = \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$D_n = \sum_{k=0}^{n-1} \psi_k,$$

$$\sigma_n f = \frac{1}{n}\sum_{k=1}^{n} S_k f,$$

$$K_n = \frac{1}{n}\sum_{k=1}^{n} D_k,$$

for every $f \in L^1(G)$.

It can be easily seen that

$$S_n f(y) = \int D_n(y-x) f(x) dx,$$

$$\sigma_n f(y) = \int K_n(y-x) f(x) dx,$$

and

$$D_{M_n}(x) = M_n 1_{I_n}(x).$$

We introduce the maximal function and the maximal operator:

$$f^*(x) = \sup_n |I_n|^{-1} |\int_{I_n(x)} f(t) dt|,$$

$$\sigma^* f(x) = \sup_n |\sigma_n f(x)|.$$

We say that operator T is of type (Y, X) if there exist an absolute constant $C > 0$ for which $\|Tf\|_Y \leqslant C\|f\|_X$ for all $f \in X$.
T is of weak type (L^1, L^1) if there exist an absolute constant $C > 0$ for which

$$\mu(Tf > \lambda) \leqslant C\|f\|_1/\lambda,$$

for all $\lambda > 0$ and $f \in L^1(G)$.

It is known that the operator which maps function f to the maximal function f^* is of weak type (L^1, L^1), and of type (L^p, L^p) for all $1 < p \leq \infty$ [10].

Generally speaking, the Hardy space $H^p(G)$, $p > 0$ consists of integrable functions f for which $f^* \in L^p(G)$. $H^p(G)$ is a Banach space with the norm

$$\|f\|_{H^p} := \|f^*\|_p.$$

Many equivalent norms are defined in [13], [19] and [1]. The boundedness of σ^* from H^1 to L^1 for bounded groups was established by [3](see also [13]). However, P. Simon proved in [13] that the boundedness of the group is necessary and sufficient for the boundedness of the maximal operator σ^*. We remark that in the so-called positive case, i.e. for non-negative functions the reverse of Fujiis inequality can be proved in a simple way (see e.g. Schipp - Wade - Simon - Pál [11]). Theorem 1 of Fujii was extended by F. Weisz in [17], [18] and [19] on (H^p, L^p), for $p > \frac{1}{2}$. In

[20] F. Weisz proved that σ^* is bounded from $H^{\frac{1}{2}}$ to the so-called weak $L^{\frac{1}{2}}$. Later U. Goginava in [6] (see also [2]) has shown by means of a counterexample that σ^* cannot have type $(H^{\frac{1}{2}}, L^{\frac{1}{2}})$. We remark that from this it follows by interpolation that σ^* is not bounded from the Hardy space H^p to the space weak-L^p for any $0 < p < \frac{1}{2}$. Furthermore, we are concerned only with the one-dimensional case, however, there are many works by Gát, Goginava, Simon, Weisz, etc. with respect to the analogous questions in the two- or multi-dimensional case.

In [13] it was also proved that that the V-conjugate function is bounded from H^1 to L^1. However, we prove that for a conveniently chosen sequence of functions the inverse inequality does not hold. This gives new proof to the result of Memić [7], which answers a question raised by P. Simon [13].

The V-conjugate of an integrable function f defined in [13] has the form:

$$\tilde{f} := \sum_{k=0}^{\infty} f * L_k D_{M_k},$$

where

$$L_k := -\sum_{j=1}^{\triangle_k} r_k^j + \sum_{j=\triangle_k+1}^{m_k-1} r_k^j,$$

$\triangle_k = [\frac{m_k-1}{2}]$, if $m_k > 2$, and $\triangle_k = 1$, if $m_k = 2$.

In [13] Theorem 4 it has been proved that the V-conjugation is of type (H^1, L^1).

The question on whether $\|\tilde{f}\|_{L^1}$ provides a new norm on H^1 was also mentioned in [13].

Here we also give the expression of L_k by Simon [14]:

a)

$$L_k(x) = 1 - \frac{1}{i}\frac{(-1)^{x_k}}{sin\frac{\pi x_k}{m_k}} + \frac{exp\left(-\frac{\pi x_k i}{m_k}\right)}{i}\left(sin\frac{x_k \pi}{m_k}\right)^{-1}$$

$$(x_k \neq 0, m_k \equiv 1(2));$$

b) for $m_k \equiv 0\ (2) (m_k > 2)\ (x_k \neq 0)$ we have

$$L_k(x) = 1 + \frac{1}{i}\frac{exp\left(-\frac{\pi x_k i}{m_k}\right)[1-(-1)^{x_k}]}{sin\frac{\pi x_k}{m_k}}; \tag{1.1}$$

$c)$ in the case $m_k = 2$ we have $L_k(x) = -(-1)^{x_k}$ $(x \in G)$.

For $s \in \{0, 1, ..., m_k - 1\}$ we write

$$L_k(se_k) = 1 + \frac{i(-1)^s}{sin\frac{s\pi}{m_k}} - \frac{icos\frac{s\pi}{m_k} + sin\frac{s\pi}{m_k}}{sin\frac{s\pi}{m_k}}.$$

2. Main results

Theorem 2.1. *On the group of integers of the 3-series field, the norms* $\|f\|_{H^1}$ *and* $\|\tilde{f}\|_{L^1}$ *are not equivalent.*

Proof. We construct the sequence of functions $(f_n)_n$ as follows:

$$f_n = 1_{I_1(e_0)} - 1_{I_1(-e_0)} + 2 \cdot \sum_{k=0}^{n-1} 3^k (1_{I_{k+2}(e_{k+1})} - 1_{I_{k+2}(-e_{k+1})}).$$

We use the expression of L_k proved in [14]

$$L_k(se_k) = 1 - \frac{1}{i}\frac{(-1)^s}{\sin\dfrac{\pi s}{m_k}} + \frac{\exp\left(-\dfrac{\pi s i}{m_k}\right)}{i}\left(\sin\frac{s\pi}{m_k}\right)^{-1},$$

for

$$(s \neq 0, m_k \equiv 1(2)).$$

so we have for $s \in \{0, 1, 2\}$

$$L_k(se_k) = 1 + \frac{i(-1)^s}{\sin\dfrac{s\pi}{3}} - \frac{i\cos\dfrac{s\pi}{3} + \sin\dfrac{s\pi}{3}}{\sin\dfrac{s\pi}{3}} = \frac{i(-1)^s}{\sin\dfrac{s\pi}{3}} - i\frac{\cos\dfrac{s\pi}{3}}{\sin\dfrac{s\pi}{3}}.$$

If s is even, we have

$$L_k(se_k) = i\frac{1 - \cos\dfrac{s\pi}{3}}{\sin\dfrac{s\pi}{3}} = 2i\frac{\sin^2\dfrac{s\pi}{6}}{2\sin\dfrac{s\pi}{6}\cos\dfrac{s\pi}{6}} = i\tan\frac{s\pi}{6}.$$

If s is odd, we obtain

$$L_k(se_k) = -i\frac{1+cos\dfrac{s\pi}{3}}{\sin\dfrac{s\pi}{3}} = -2i\frac{\cos^2\dfrac{s\pi}{6}}{2\sin\dfrac{s\pi}{6}\cos\dfrac{s\pi}{6}} = -i\cot\frac{s\pi}{6}.$$

Let $\beta_s = \tan\frac{\pi s}{6}$ when s is even, and $\beta_s = -\cot\frac{\pi s}{6}$, when s is odd. This yields

$$\begin{aligned}
(f * L_k D_{M_k})(y) &= \int f(x)(L_k D_{M_k})(y-x)dx \\
&= M_k \int\limits_{I_k(y)} f(x)L_k(y-x)dx \\
&= M_k \sum_{s=0}^{m_{k-1}} \int\limits_{(y-x)\in(s\cdot e_k+I_{k+1})} f(x)L_k(y-x)dx \\
&= M_k \sum_{s=0}^{m_{k-1}} i\beta_s \int\limits_{t\in(s\cdot e_k+I_{k+1})} f(y-t)dt \\
&= i\sum_{s=0}^{m_{k-1}} \beta_s a_s^k(y),
\end{aligned}$$

where $a_s^k(y) = M_k \int\limits_{s\cdot e_k+I_{k+1}} f(y-t)dt$.

Notice that if s is odd then

$$-\beta_s = \cot\frac{\pi s}{6} = \tan\left(\frac{\pi}{2} - \frac{\pi s}{6}\right) = \tan\left(\frac{3-s}{6}\pi\right) = \beta_{3-s}.$$

This is clearly also valid when s is even. Therefore,

$$\sum_{s=0}^{2} \beta_s a_s^k = \sum_{s=1}^{2} \beta_s a_s^k = \beta_1(a_1^k - a_{-1}^k).$$

Let $t \in I_{k+2}(e_{k+1}) \cup I_{k+2}(-e_{k+1})$ and $0 \le k \le n-1$, then

$$L_{k+1}D_{M_{k+1}} * f_n(t) = i\beta_1(a_1^{k+1}(t) - a_{-1}^{k+1}(t)) = -2\cdot 3^k i\beta_1.$$

If $t \in I_{k+2}$ and $0 \le k \le n-1$, then

$$L_{k+1}D_{M_{k+1}} * f_n(t) = 4\cdot 3^k i\beta_1.$$

Moreover,

$$L_s D_{M_s} * f_n(t) = 0,$$

if $t \in I_{k+2}(e_{k+1}) \cup I_{k+2}(-e_{k+1})$ and $s \geq k+2$. Also,

$$L_0 * f_n(t) = -i\beta_1,$$

for every $t \in I_1(e_0) \cup I_1(-e_0)$, and

$$L_0 * f_n(t) = 2i\beta_1,$$

if $t \in I_1$.

Using these facts, we obtain that

$$\tilde{f}_n(t) = -i\beta_1,$$

for every $t \in I_1(e_0) \cup I_1(-e_0)$.

On $I_1 \setminus I_{n+1}$,

$$\tilde{f}_n(t) = \sum_{s=0}^{n} L_s D_{M_s} * f_n(t) = \sum_{s=0}^{k+1} L_s D_{M_s} * f_n(t),$$

if $t \in I_{k+1} \setminus I_{k+2}$. It follows

$$\tilde{f}_n(t) = 2i\beta_1 + 4i\beta_1 \sum_{s=1}^{k} 3^{s-1} - i2 \cdot 3^k \beta_1 = 2i\beta_1(1 + 3^k - 1 - 3^k) = 0.$$

Finally, on I_{n+1},

$$\tilde{f}_n(t) = \sum_{s=0}^{n} L_s D_{M_s} * f_n(t) = 2i\beta_1 + 4i\beta_1 \sum_{s=1}^{n} 3^{s-1} = 2i\beta_1 3^n.$$

It follows that

$$\|\tilde{f}_n\|_{L^1} = \frac{4}{3}\beta_1.$$

It is also easily seen that

$$\|f\|_{H^1} = \|f_n^*\|_1 = (\frac{2}{3} + 4\sum_{k=0}^{n-1} \frac{3^k}{3^{k+2}})\beta_1 = (\frac{2}{3} + \frac{4}{9}n)\beta_1.$$

□

The second result does not give a characterization of Hardy spaces, but it is connected with Lemma 2.2 [7] in which it was proved that there exists a constant

$C > 0$, only depending on the sequence $(m_n)_n$, such that

$$|f * L_n D_{M_n}| \leq C \sup_{s \in \{0,\ldots,m_n-1\}} |S_{M_{n+1}} f(x + s e_n) - S_{M_n} f(x)|.$$

Namely, we prove that the mean value of f on the coset $I_{N-1}(x)$ is dominated by either $\sigma_{M_{N-1}}$ or σ_{M_N} on some translated element.

Theorem 2.2. *Let $x \in G$, $N \in \mathbb{N}$. Then,*

$$|S_{M_{N-1}} f(x)| \leq 4 \max(|\sigma_{M_{N-1}} f(x + j e_{N-1})|, |\sigma_{M_N} f(x + j e_{N-1})|),$$

for at least some $j \in \{0, 1, \ldots, m_{N-1} - 1\}$, where j depends on x.

Proof. Notice that the expression in the right side is constant on each I_N-coset.

Assume that for some $x \in G$, $N \in \mathbb{N}$,

$$|S_{M_{N-1}} f(x)| > 4 \max(|\sigma_{M_{N-1}} f(x + j e_{N-1})|, |\sigma_{M_N} f(x + j e_{N-1})|),$$

for every $j \in \{0, 1, \ldots, m_{N-1} - 1\}$. In [4, Lemma 2.6], it was proved that for $z \in G \setminus I_N$, $K_{M_N}(z) = \frac{M_t}{1 - r(i e_t)}$, if $z - i e_t \in I_N$, $i = 0, 1, \ldots, m_t - 1$, and $K_{M_N}(z) = 0$ otherwise. It was also noticed in the proof of [4, Theorem 2.1] that $K_{M_N}(z) = \frac{M_N - 1}{2}$, if $z \in I_N$.

Following the proof of [4, Theorem 2.1], we have

$$\sigma_{M_N} f(x) = \sum_{t=0}^{N-1} \sum_{i=1}^{m_t - 1} \frac{M_t}{1 - r_t(i e_t)} \int_{I_N(x + i e_t)} f(y) dy + \frac{M_N - 1}{2} \int_{I_N(x)} f(y) dy. \tag{2.1}$$

From our assumption, it follows that

$$\left| \sum_{j=0}^{m_{N-1}-1} \sigma_{M_N} f(x + j e_{N-1}) \right| < \frac{m_{N-1}}{4} |S_{M_{N-1}} f(x)|.$$

Applying formula (2.1), we get

$$\sum_{j=0}^{m_{N-1}-1} \sigma_{M_N} f(x+je_{N-1}) = \sum_{j=0}^{m_{N-1}-1} \frac{M_N-1}{2}\int_{I_N(x+je_{N-1})} f(y)dy$$
$$+\sum_{t=0}^{N-1}\sum_{i=1}^{m_t-1} \frac{M_t}{1-r_t(ie_t)} \sum_{j=0}^{m_{N-1}-1} \int_{I_N(x+ie_t+je_{N-1})} f(y)dy$$
$$= \frac{M_N-1}{2}\int_{I_{N-1}(x)} f(y)dy + \sum_{t=0}^{N-1}\sum_{i=1}^{m_t-1} \frac{M_t}{1-r_t(ie_t)}\int_{I_{N-1}(x+ie_t)} f(y)dy$$
$$= (\frac{m_{N-1}}{2}-\frac{1}{2M_{N-1}})S_{M_{N-1}} + \sum_{t=0}^{N-2}\sum_{i=1}^{m_t-1} \frac{M_t}{1-r_t(ie_t)}\int_{I_{N-1}(x+ie_t)} f(y)dy$$
$$+\sum_{i=1}^{m_{N-1}-1} \frac{M_{N-1}}{1-r_{N-1}(ie_{N-1})}\int_{I_{N-1}(x)} f(y)dy.$$

Using the formula

$$\sum_{i=1}^{m_{N-1}-1} \frac{1}{1-r_{N-1}(ie_{N-1})} = \frac{m_{N-1}-1}{2},$$

obtained in the proof of [4, Lemma 2.3], we get from the assumption and (2.1) that

$$\begin{aligned}
\sum_{j=0}^{m_{N-1}-1} & \sigma_{M_N} f(x+je_{N-1}) = (\frac{m_{N-1}}{2}-\frac{1}{2M_{N-1}})S_{M_{N-1}} \\
+ & \sum_{t=0}^{N-2}\sum_{i=1}^{m_t-1} \frac{M_t}{1-r_t(ie_t)}\int_{I_{N-1}(x+ie_t)} f(y)dy \\
+ & \frac{m_{N-1}-1}{2}M_{N-1}\int_{I_{N-1}(x)} f(y)dy \\
= & (\frac{m_{N-1}}{2}-\frac{1}{2M_{N-1}})S_{M_{N-1}} \\
+ & \sum_{t=0}^{N-2}\sum_{i=1}^{m_t-1} \frac{M_t}{1-r_t(ie_t)}\int_{I_{N-1}(x+ie_t)} f(y)dy \\
+ & \frac{M_{N-1}-1}{2}\int_{I_{N-1}(x)} f(y)dy + \frac{m_{N-1}-2}{2}M_{N-1}\int_{I_{N-1}(x)} f(y)dy \\
+ & \frac{1}{2}\int_{I_{N-1}(x)} f(y)dy \\
= & (m_{N-1}-1)S_{M_{N-1}}f(x) + \sigma_{M_{N-1}}f(x),
\end{aligned}$$

hence,

$$|(m_{N-1}-1)S_{M_{N-1}}f(x)+\sigma_{M_{N-1}}f(x)| < \frac{m_{N-1}}{4}|S_{M_{N-1}}f(x)|,$$

which leads to the contradicting fact

$$|\sigma_{M_{N-1}}f(x)| > \frac{m_{N-1}}{4}|S_{M_{N-1}}f(x)|.$$

□

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