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ON DECOMPOSABLE AND WARPED PRODUCT GENERALIZED QUASI EINSTEIN MANIFOLDS

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Abstract. The object of the present paper is to study decomposable and warped product generalized quasi Einstein manifolds.
Keywords: Einstein manifold; Warped product; Ricci tensor; Generalized quasi-Einstein manifolds

1. Introduction

A Riemannian manifold (M^n, g), $n = \dim M \geq 2$, is said to be an Einstein manifold if the following condition

$$R_{ij} = \frac{r}{n} g_{ij} \tag{1.1}$$

holds on M, where R_{ij} and r denote the Ricci tensor and the scalar curvature of (M^n, g), respectively. According to Besse([2], p. 132), (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also, Einstein manifolds form a natural subclass of various classes of Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([2], p. 432-433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation:

$$R_{ij} = \lambda g_{ij} + \mu A_i A_j, \tag{1.2}$$

where $\lambda,\ \mu \in \mathbb{R}$ and A_i is a non-zero covariant vector. Moreover, different structures on Einstein manifolds have been studied by several authors.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi-Einstein manifold if its Ricci tensor R_{ij} of type (0,2) is not identically zero and satisfies the condition (1.2).

It is to be noted that Chaki and Maity [5] also introduced the notion of quasi-Einstein manifolds in a different way. They have taken λ and μ as scalars and

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the non-zero covariant vector A_i as a unit covariant vector. Such an n-dimensional manifold is denoted by the symbol $(QE)_n$. Quasi-Einstein manifolds have been studied by several authors such as De and Ghosh ([9], [10], [11], [12]), Ghosh, De and Binh [16], De and De [8], Debnath and Konar [14], Bejan and Binh [1] and many others.

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations, as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean manifolds. For instance, the Robertson-Walker space-time are quasi-Einstein manifolds. Also, quasi-Einstein manifold can be taken as a model of the perfect fluid space-time in general relativity [12]. So quasi-Einstein manifolds have some importance in the general theory of relativity.

Quasi-Einstein manifolds have been generalized by several authors in several ways such as generalized quasi-Einstein manifolds ([6], [10], [19]), super quasi Einstein manifolds ([7], [13], [21]), N(k)-quasi-Einstein manifolds ([17], [20], [25]) and many others. Also in [24] quasi-Einstein warped products have been studied by Sular and Özgür.

In a recent paper De and Ghosh [10] introduced the notion of generalized quasi Einstein manifolds. A non-flat Riemannian manifold is called a generalized quasi Einstein manifold if its Ricci tensor R_{ij} of type (0,2) is non-zero and satisfies the condition

$$R_{ij} = \lambda g_{ij} + \mu A_i A_j + \nu B_i B_j, \tag{1.3}$$

where λ, μ and ν are certain non-zero scalars and A_i, B_i are two orthogonal unit covariant vectors such that $g^{ij}A_iA_j = 1$, $g^{ij}B_iB_j = 1$ and $g^{ij}A_iB_j = 0$. The vectors A_i and B_i are called the generators of the manifold and λ, μ and ν are called the associated scalars. Such a manifold is denoted by $G(QE)_n$. If $\nu = 0$, then the manifold reduces to a quasi Einstein manifold. $G(QE)_n$ arose during the study of 2-quasi umbilical hypersurface of a Euclidean space [10]. In 2011, De and Mallick [15] prove the existence of $G(QE)_n$ by several examples. Motivated by the above studies, the authors study the decomposability and warped product of $G(QE)_n$.

The paper is organized as follows:
First, we state some examples of $G(QE)_n$. Then in Section 3, we study a decomposable generalized quasi Einstein manifold. Section 4 deals with a $G(QE)_n$ warped product manifold. Finally, we consider a $G(QE)_n$ warped product manifold, base of which is unit dimensional.

2. Examples of $G(QE)_n$

Example 2.1. [15] A 2-quasi-umbilical hypersurface of a space of constant curvature is a $G(QE)_n$, which is not a quasi-Einstein manifold.

Example 2.2. [15] A quasi-umbilical hypersurface of a Sasakian space form is a $G(QE)_n$, which is not a quasi-Einstein manifold.

Example 2.3. De and Mallick [15] considered a Riemannian metric g on R^4 by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2. \tag{2.1}$$

Then they showed that (M^4, g) is a generalized quasi-Einstein manifold, which is not a quasi-Einstein manifold.

Example 2.4. [22] Özgür and Sular assumed an isometrically immersed surface $\bar{M}$ in E^3 with non-zero distinct principal curvatures λ and μ. Then they considered the hypersurface $M = \bar{M} \times E^{n-2}$ in $E^{n+1}, n \geq 4$. The principal curvatures of M are $\tilde{\lambda}$, $\tilde{\mu}$, 0,..., 0, where 0 occures (n-2)-times. Hence the manifold is a 2-quasi umbilical hypersurface and so it is generalized quasi-Einstein.

Example 2.5. [22] Özgür and Sular assumed a sphere S^2 in E^{k+2} given by the immersion $f : S^2 \to E^{k+2}$ and BS^2 be the bundle of unit normal to S^2. The hypersurface M defined by the map $\varphi_t : BS^2 \to E^{k+2}$, $\varphi_t(x, \xi) = F(x, t\xi) = f(x) + t\xi$ is called the tube of radius t over S^2. It was proved in [4] that if (λ, λ) are the principal curvature of S^2 then the principal curvatures of M are $(\frac{\lambda}{1-t\lambda}, \frac{\lambda}{1-t\lambda}, -\frac{1}{t}, ..., -\frac{1}{t})$, where $-\frac{1}{t}$ occures $(k-1)$-times. So M is 2-quasi umbilical and hence it is generalized quasi-Einstein.

3. Decomposable $G(QE)_n$

A Riemannian manifold (M^n, g) is said to be decomposable or a product manifold [23] if it can be expressed as $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq (n-2)$, that is, in some coordinate neighbourhood of the Riemannian space (M^n, g), the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + g^*_{\alpha\beta}dx^\alpha dx^\beta, \tag{3.1}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, ..., x^p$ denoted by $\bar{x}$ and $g^*_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, ..., x^n$ denoted by x^*; $a, b, c, ...$ run from 1 to p and $\alpha, \beta, \gamma, ...$ run from p+1 to n.

The two parts of (3.1) are the metrics of $M_1^p (p \geq 2)$ and $M_2^{n-p} (n-p \geq 2)$ which are called the components of the decomposable manifold $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$.

Let (M^n, g) be a Riemannian manifold such that $M_1^p (p \geq 2)$ and $M_2^{n-p} (n-p \geq 2)$ are components of this manifold. Here throughout this section each object denoted by a 'bar' is assumed to be from M_1 and each object denoted by 'star' is assumed to be from M_2.

Then in a decomposable Riemannian manifold $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$, the following relations hold [26]:

$R_{ab} = \bar{R}_{ab}$; $R_{\alpha\beta} = R^*_{\alpha\beta}$; $R_{a\alpha} = 0$; $r = \bar{r} + r^*$,

where $r, \bar{r}$ and r^* are scalar curvatures of M, M_1 and M_2 respectively.

Let us consider a Riemannian manifold (M^n, g), which is a decomposable $G(QE)_n$.

Then $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$. Now from (1.3) we get

$$\bar{R}_{ab} = \lambda \bar{g}_{ab} + \mu \bar{A}_a \bar{A}_b + \nu \bar{B}_a \bar{B}_b, \tag{3.2}$$

and

(3.3) $$R^*_{\alpha\beta} = \lambda g^*_{\alpha\beta} + \mu A^*_\alpha A^*_\beta + \nu B^*_\alpha B^*_\beta,$$

where

(3.4) $$A_i(x) = \begin{cases} \bar{A}_i & \text{for i=1,2, ..., p} \\ A^*_i & \text{for i=p+1, ..., n.} \end{cases}$$

Also we have

(3.5) $$R_{a\alpha} = \lambda g_{a\alpha} + \mu \bar{A}_a A^*_\alpha + \nu \bar{B}_a B^*_\alpha.$$

which implies that

(3.6) $$\mu \bar{A}_a A^*_\alpha + \nu \bar{B}_a B^*_\alpha = 0.$$

If possible, let

(3.7) $$\mu \bar{A}_a A^*_\alpha = 0,$$

which implies

(3.8) $$\bar{A}_a A^*_\alpha = 0,$$

since $\mu \neq 0$. Hence

(3.9) $$\textit{either } \bar{A}_a = 0 \textit{ or } A^*_\alpha = 0$$

(but not both, since A_i is no more a unit vector).

Using (3.7) in (3.6) we get

(3.10) $$\nu \bar{B}_a B^*_\alpha = 0,$$

which implies

(3.11) $$\bar{B}_a B^*_\alpha = 0,$$

since $\nu \neq 0$. Therefore

(3.12) $$\textit{either } \bar{B}_a = 0 \textit{ or } B^*_\alpha = 0,$$

From (3.9) and (3.12) we have four cases as follows:

Case I: $\bar{A}_a = 0$ and $\bar{B}_a = 0$,

Case II: $A^*_\alpha = 0$ and $B^*_\alpha = 0$,

Case III: $\bar{A}_a = 0$ and $B^*_\alpha = 0$,

Case IV: $A^*_\alpha = 0$ and $\bar{B}_a = 0$.

Now if possible let $\bar{A}_a = 0$ and $\bar{B}_a = 0$, then (3.2) reduces to

(3.13) $$\bar{R}_{ab} = \lambda \bar{g}_{ab}.$$

This shows that the manifold M^p_1 is an Einstein manifold. On the other hand, if possible let $A^*_\alpha = 0$ and $B^*_\alpha = 0$, then (3.3) reduces to

(3.14) $$R^*_{\alpha\beta} = \lambda g^*_{\alpha\beta}.$$

As above (3.14) shows that the manifold M^{n-p}_2 is an Einstein manifold.

Obviously the other cases are trivial. We get the similar results if we assume that (3.10) holds.

Thus we have the follwing:

Theorem 3.1. *If a $G(QE)_n$ is a decomposable Riemannian manifold (M^n, g) such that $M = M_1^p \times M_2^{n-p}, (2 \le p \le n-2)$, and either (3.7) or (3.10) holds, then one component of the decomposable manifold is an Einstein manifold and the other is a generalized quasi Einstein manifold.*

4. $G(QE)_n$ warped product manifolds

The study of warped product manifold was initiated by Kručkovič [18] in 1957. Again in 1969 Bishop and O'Neill [3] also obtained the same notion of the warped product manifolds while they were constructing a large class of manifolds of negative curvature. Warped product are generalizations of the Cartesian product of Riemannian manifolds. Let $(\bar{M}, \bar{g})$ and (M^*, g^*) be two Riemannian manifolds. Let $\bar{M}$ and M^* be covered with coordinate charts $(U; x^1, x^2, ..., x^p)$ and $(V; y^{p+1}, y^{p+2}, ..., y^n)$ respectively.

Then the warped product $M = \bar{M} \times_f M^*$ is the product manifold of dimension n furnished with the metric

$$g = \pi^*(\bar{g}) + (f \circ \pi)\sigma^*(g^*), \tag{4.1}$$

where $\pi : M \to \bar{M}$ and $\sigma : M \to M^*$ are natural projections such that the warped product manifold $\bar{M} \times_f M^*$ is covered with the coordinate chart

$$(U \times V; x^1, x^2, ..., x^p, x^{p+1} = y^{p+1}, x^{p+2} = y^{p+2}, ..., x^n = y^n).$$

Then the local components of the metric g with respect to this coordinate chart are given by

$$g_{ij} = \begin{cases} \bar{g}_{ij} & \text{for i=a and j=b,} \\ f g^*_{ij} & \text{for } i = \alpha \text{ and } j = \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

Here $a, b, c, ... \in \{1, 2, ..., p\}$ and $\alpha, \beta, \gamma, ... \in \{p+1, p+2, ..., n\}$ and $i, j, k, ... \in \{1, 2, ..., n\}$. Here $\bar{M}$ is called the base, M^* is called the fiber and f is called warping function of the warped product $M = \bar{M} \times_f M^*$. We denote by Γ^i_{jk}, R_{ijkl}, R_{ij} and r as the components of Levi-Civita connection ∇, the Riemann-Christoffel curvature tensor R, Ricci tensor S and the scalar curvature of (M, g) respectively. Moreover we consider that, when Ω is a quantity formed with respect to g, we denote by $\bar{\Omega}$ and Ω^*, the similar quantities formed with respect to $\bar{g}$ and g^* respectively. Then the non-zero local components of Levi-Civita connection ∇ of (M, g) are given by

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc},\ \ \Gamma^\alpha_{\beta\gamma} = \Gamma^{*\alpha}_{\beta\gamma},\ \ \Gamma^a_{\beta\gamma} = -\frac{1}{2}\bar{g}^{ab} f_b g^*_{\beta\gamma},\ \ \Gamma^\alpha_{a\beta} = \frac{1}{2f} f_a \delta^\alpha_\beta, \tag{4.3}$$

where $f_a = \partial_a f = \frac{\partial f}{\partial x^a}$. The local components $R_{hijk} = g_{hl} R^l_{ijk} = g_{hl}(\partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^m_{ij}\Gamma^l_{mk} - \Gamma^m_{ik}\Gamma^l_{mj}), \partial_k = \frac{\partial}{\partial x^k}$, of the Riemann-Christoffel curvature tensor R of (M, g) which may not vanish identically are the following:

$$R_{abcd} = \bar{R}_{abcd}, R_{a\alpha b\beta} = -f T_{ab} g^*_{\alpha\beta}, R_{\alpha\beta\gamma\delta} = f R^*_{\alpha\beta\gamma\delta} - f^2 P G^*_{\alpha\beta\gamma\delta}, \tag{4.4}$$

where $G_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}$ and

$T_{ab} = -\frac{1}{2f}(\nabla_b f_a - \frac{1}{2f} f_a f_b)$, $tr(T) = g^{ab}T_{ab}$, $P = \frac{1}{4f^2} g^{ab} f_a f_b$. Again the non-zero local components of the Ricci tensor $R_{jk} = g^{il}R_{ijkl}$ of (M, g) are given by

$$R_{ab} = \bar{R}_{ab} + (n-p)T_{ab}, \quad R_{\alpha\beta} = R^*_{\alpha\beta} - Qg^*_{\alpha\beta}, \tag{4.5}$$

where $Q = f((n-p-1)P - tr(T))$. The scalar curvature r of (M, g) is given by

$$r = \bar{r} + \frac{r^*}{f} - (n-p)[(n-p-1)P - 2tr(T)]. \tag{4.6}$$

Let $M = \bar{M} \times_f M^*$ be a non-flat warped product manifold and also let M be a $G(QE)_n$. That is,

$$R_{ab} = \lambda g_{ab} + \mu A_a A_b + \nu B_a B_b. \tag{4.7}$$

From (4.7), using (4.5) we get

$$\bar{R}_{ab} + (n-p)T_{ab} = \lambda \bar{g}_{ab} + \mu \bar{A}_a \bar{A}_b + \nu \bar{B}_a \bar{B}_b, \tag{4.8}$$

where

$$A_i(x) = \begin{cases} \bar{A}_i & \text{for i=1, ..., p} \\ A^*_i & \text{otherwise,} \end{cases} \tag{4.9}$$

and

$$B_i(x) = \begin{cases} \bar{B}_i & \text{for i=1, ..., p} \\ B^*_i & \text{otherwise,} \end{cases} \tag{4.10}$$

Then from (4.8) we get

$$\bar{R}_{ab} = \lambda \bar{g}_{ab} + \mu \bar{A}_a \bar{A}_b + \nu \bar{B}_a \bar{B}_b - (n-p)T_{ab}, \tag{4.11}$$

If possible, we assume that $\bar{M}$ is also $G(QE)_p$, then from (4.11) we get

$$T_{ab} = 0. \tag{4.12}$$

Conversely, if (4.12) holds, then from (4.11) we can conclude that $\bar{M}$ is a $G(QE)_p$. Thus we have the following:

Theorem 4.1. *$M = \bar{M} \times_f M^*$ is a $G(QE)_n$ warped product manifold, if and only if $\bar{M}$ is a $G(QE)_p$ provided $T_{ab} = 0$.*

Now if in particular

$$T_{ab} = k\bar{g}_{ab}, \tag{4.13}$$

where $k \neq 0$ is some constant. Then (4.11) takes the form

$$\bar{R}_{ab} = \{\lambda - k(n-p)\}\bar{g}_{ab} + \mu \bar{A}_a \bar{A}_b + \nu \bar{B}_a \bar{B}_b, \tag{4.14}$$

where A_i and B_i are defined by (4.9) and (4.10) from which it follwos that $\bar{M}$ is a $G(QE)_p$. Conversely, if $\bar{M}$ is a $G(QE)_n$ then using (4.14) in (4.11) we get (4.13). Thus we have the following:

Theorem 4.2. *$M = \bar{M} \times_f M^*$ is a $G(QE)_n$ warped product manifold, then $\bar{M}$ is a $G(QE)_p$ if and only if (4.13) holds.*

Similarly, we get from (4.7)

$$R_{\alpha\beta} = \lambda g_{\alpha\beta} + \mu A_\alpha A_\beta + \nu B_\alpha B_\beta. \tag{4.15}$$

Using (4.5), (4.15) yields

$$R^*_{\alpha\beta} = (\lambda f + Q) g^*_{\alpha\beta} + \mu A^*_\alpha A^*_\beta + \nu B^*_\alpha B^*_\beta. \tag{4.16}$$

Hence M^* is a $G(QE)_{n-p}$.

Converse is trivial. Thus we have the following:

Theorem 4.3. *$M = \bar{M} \times_f M^*$ is a $G(QE)_n$ warped product manifold, if and only if M^* is a generalized quasi-Einstein manifold of dimension (n-p).*

5. $G(QE)_n$ warped product manifolds with unit dimensional base

In this section, we consider $G(QE)_n$ warped product manifolds $M = I \times_f M^*$, $dimI = 1$, $dimM^* = n - 1 (n \geq 3)$, $f = exp\{\frac{q}{2}\}$. We take the metric on I as $(dt)^2$. Using the above consideration and (4.5), we get

$$R_{tt} = \bar{R}_{tt} - \frac{(n-1)}{16}[4q'' + (q')^2]. \tag{5.1}$$

which implies

$$R_{tt} = -\frac{(n-1)}{16}[4q'' + (q')^2], \tag{5.2}$$

since $\bar{R}_{tt}$ of I is zero. Also

$$R_{\alpha\beta} = R^*_{\alpha\beta} - \frac{e^{\frac{q}{2}}}{16}[4(n-1)q'' + (2n-3)(q')^2] g^*_{\alpha\beta}, \tag{5.3}$$

where '$'$' and '$''$' denote the 1st order and 2nd order partial derivative respectively, with respect to 't'. Since M is a generalized quasi Einstein manifold, from (1.3) we have

$$R_{tt} = \lambda g_{tt} + \mu A_t A_t + \nu B_t B_t. \tag{5.4}$$

and

$$R_{\alpha\beta} = \lambda g_{\alpha\beta} + \mu A_\alpha A_\beta + \nu B_\alpha B_\beta, \tag{5.5}$$

where we take A_i and B_i as defined in (4.9) and (4.10). Now since $dimI = 1$, we can take

$$\bar{A}_t = l \ \ and \ \ \bar{B}_t = m, \tag{5.6}$$

where l and m are functions on M. Using (4.1), (4.2), (4.9), (4.10) and (5.6), the equations (5.4) and (5.5) reduce to

$$R_{tt} = \lambda + \mu l^2 + \nu m^2, \tag{5.7}$$

and

$$R_{\alpha\beta} = \lambda e^{\frac{q}{2}} g^*_{\alpha\beta} + \mu A^*_\alpha A^*_\beta + \nu B^*_\alpha B^*_\beta. \tag{5.8}$$

From (5.2) and (5.7) we get

$$\lambda + \mu l^2 + \nu m^2 = -\frac{(n-1)}{16}[4q'' + (q')^2]. \tag{5.9}$$

Again from (5.3) and (5.8) we obtain

$$R^*_{\alpha\beta} = \frac{e^{\frac{q}{2}}}{16}[4(n-1)q'' + (2n-3)(q')^2 + 16\lambda]g^*_{\alpha\beta} + \mu A^*_\alpha A^*_\beta + \nu B^*_\alpha B^*_\beta, \tag{5.10}$$

where λ, μ and ν are related by (5.9). Thus (5.10) implies that M^* is a generalized quasi Einstein manifold. Hence we have the following:

Theorem 5.1. *If $M = I \times_f M^*$, is a $G(QE)_n$ warped product manifold and $dimI = 1$, $dimM^* = n-1 (n \geq 3)$, then M^* is a generalized quasi Einstein manifold.*

Now, we consider warped product manifolds $M = I \times_f M^*$, $dimI = 1$, $dimM^* = n-1 (n \geq 3)$, $f = exp\{\frac{q}{2}\}$ and M^* is a $(QE)_n$. We take the metric on I as $(dt)^2$. In this case, (5.2) and (5.3) can also be obtained using the above consideration and (4.5). Since M^* is $(QE)_n$, from (1.2) we have

$$R^*_{\alpha\beta} = \lambda g^*_{\alpha\beta} + \mu A^*_\alpha A^*_\beta, \tag{5.11}$$

where λ and μ are certain non-zero scalars and A^*_i is an unit covariant vector such that $g^*_{ij} A^*_i A^*_j = 1$ and

$$A_i(x) = \begin{cases} \bar{A}_i & \text{for i=1} \\ A^*_i & \text{otherwise.} \end{cases} \tag{5.12}$$

Using (5.11) in (5.3) we get

$$R_{\alpha\beta} = \lambda g^*_{\alpha\beta} + \mu A^*_\alpha A^*_\beta - \frac{e^{\frac{q}{2}}}{16}[4(n-1)q'' + (2n-3)(q')^2]g^*_{\alpha\beta}. \tag{5.13}$$

which implies

$$R_{\alpha\beta} = -\frac{e^{\frac{q}{2}}}{16}[4(n-1)q'' + (2n-3)(q')^2]g^*_{\alpha\beta} + \lambda g^*_{\alpha\beta} + \mu A^*_\alpha A^*_\beta. \tag{5.14}$$

Now using (4.2) and (5.12) in (5.14) we obtain

$$R_{\alpha\beta} = -\frac{1}{16}[4(n-1)q'' + (2n-3)(q')^2]g_{\alpha\beta} + \frac{\lambda}{e^{\frac{q}{2}}} g_{\alpha\beta} + \mu A_\alpha A_\beta. \tag{5.15}$$

Now if we choose $g_{\alpha\beta} = e^{\frac{q}{2}} B_\alpha B_\beta$, where

$$B_i(x) = \begin{cases} \bar{B}_i & \text{for i=1} \\ B^*_i & \text{otherwise.} \end{cases} \tag{5.16}$$

Then (5.15) yields

$$R_{\alpha\beta} = -\frac{1}{16}[4(n-1)q'' + (2n-3)(q')^2]g_{\alpha\beta} + \mu A_\alpha A_\beta + \lambda B_\alpha B_\beta. \tag{5.17}$$

Again from (5.2) we get

$$\begin{aligned} R_{tt} = &\frac{1}{16}[4(n-1)q'' + (2n-3)(q')^2]g_{tt} - \frac{1}{16}[4(n-1)q'' \\ &+(2n-3)(q')^2] - \frac{(n-1)}{16}[(q')^2 + 4q''], \end{aligned} \tag{5.18}$$

since $\bar{g}_{tt} = 1$ and $g_{tt} = \bar{g}_{tt}$ in I. Thus (5.18) can be written as

$$R_{tt} = \frac{1}{16}[4(n-1)q'' + (2n-3)(q')^2]g_{tt} - \frac{(3n-4)}{16}(q')^2 + \frac{2(n-1)}{4}q''. \tag{5.19}$$

Since $dimI = 1$, we can take

$$\bar{A}_t = q' \ \ and \ \ \bar{B}_t = \sqrt{q''}, \tag{5.20}$$

where $q'$and q'' are functions on M. Then using (5.12),(5.16) and (5.20) we can write (5.19) as follows:

$$R_{tt} = \frac{1}{16}[4(n-1)q'' + (2n-3)(q')^2]g_{tt} - \frac{(3n-4)}{16}A_tA_t + \frac{2(n-1)}{4}B_tB_t. \tag{5.21}$$

Thus from (5.17) and (5.21) we caconclude that $M = I \times_f M^*$ is a generalized quasi Einstein manifold if M^* is a quasi Einstein manifold. Hence we have the following:

Theorem 5.2. *If $M = I \times_f M^*$, is a warped product manifold and $dimI = 1$, $dimM^* = n-1 (n \geq 3)$ and M^* is a quasi Einstein manifold, then M is a generalized quasi Einstein manifold.*

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