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WEAKLY α_γ-REGULAR AND WEAKLY α_γ-NORMAL SPACES

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Abstract. The aim of this paper is to introduce and study two new classes of spaces, called weakly α_γ-regular and weakly α_γ-normal spaces. Some basic properties of these separation axioms are studied by utilizing α_γ-open and α_γ-closed sets.

Keywords: operation; α_γ-open set; weakly α_γ-regular; weakly α_γ-normal.

1. Introduction

Topology is the branch of mathematics, whose concepts exist not only in all branches of mathematics, but also in many real life applications. Several researchers are working on different structures of topological spaces. Njastad [6] initiated and explored the notion of α-open sets. Ibrahim [5] introduced and discussed an operation of a topology $\alpha O(X)$ into the power set $P(X)$ of a space X and also he introduced the concept of α_γ-open sets and investigated the related topological properties of the associated topology $\alpha O(X,\tau)_\gamma$ and $\alpha O(X,\tau)$ by using operation γ.

2. Preliminaries

Let (X,τ) be a topological space and A a subset of X. The closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a topological space (X,τ) is said to be α-open [6] if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is said to be α-closed. The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha Cl(A)$. The family of all α-open (resp., α-closed) sets in a topological space (X,τ) is denoted by $\alpha O(X,\tau)$ (resp., $\alpha C(X,\tau)$). An operation $\gamma : \alpha O(X,\tau) \to P(X)$ [5] is a mapping satisfying the condition, $V \subseteq V^\gamma$ for each $V \in \alpha O(X,\tau)$. We call the mapping γ an operation on $\alpha O(X,\tau)$. A subset A of X is called an α_γ-open set [5] if for each point $x \in A$,

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there exists an α-open set U of X containing x such that $U^\gamma \subseteq A$. The complement of an α_γ-open set is said to be α_γ-closed. We denote the set of all α_γ-open (resp., α_γ-closed) sets of (X,τ) by $\alpha O(X,\tau)_\gamma$ (resp., $\alpha C(X,\tau)_\gamma$). The α_γ-closure [5] of a subset A of X with an operation γ on $\alpha O(X)$ is denoted by $\alpha_\gamma Cl(A)$ and is defined to be the intersection of all α_γ-closed sets containing A. A point $x \in X$ is in αCl_γ-closure [5] of a set $A \subseteq X$, if $U^\gamma \cap A \neq \phi$ for each α-open set U containing x. The αCl_γ-closure of A is denoted by $\alpha Cl_\gamma(A)$. The union of all α_γ-open sets contained in A is called the α_γ-interior of A and denoted by $\alpha_\gamma Int(A)$ [3]. An operation γ on $\alpha O(X,\tau)$ is said to be α-regular [5] if for every α-open sets U and V of each $x \in X$, there exists an α-open set W of x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$. A subset A of the space X is said to be α_γ-generalized closed (Briefly. α_γ-g.closed) [5] if $\alpha_\gamma Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an α_γ-open set in X. The complement of an α_γ-g.closed set is called an α_γ-g.open set [5].

Throughout this paper (X,τ) and (Y,σ) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let $\gamma : \alpha O(X,\tau) \to P(X)$ and $\beta : \alpha O(Y,\sigma) \to P(Y)$ be two operations on $\alpha O(X,\tau)$ and $\alpha O(Y,\sigma)$, respectively.

Remark 2.1. [2] For an operation $\gamma : \alpha O(X) \to P(X)$ and a subset H of X. we have $\alpha O(X)_\gamma|H = \{V \cap H \in P(H) : V \text{ is } \alpha_\gamma\text{-open in } X\}$.

Definition 2.1. [3] Let (X,τ) be a topological space and $x \in X$, then a subset N of X is said to be α_γ-neighbourhood of x, if there exists an α_γ-open set U in X such that $x \in U \subseteq N$.

Remark 2.2. [3] Let (X,τ) be a topological space and γ an α-regular operation on $\alpha O(X)$. If A is a subset of X, then

1. For every α_γ-open set G of X, we have that $\alpha_\gamma Cl(A) \cap G \subseteq \alpha_\gamma Cl(A \cap G)$.
2. For every α_γ-closed set F of X, we have that $\alpha_\gamma Int(A \cup F) \subseteq \alpha_\gamma Int(A) \cup F$.

Proposition 2.1. [3] *A subset A of X is α_γ-g.open if and only if $F \subseteq \alpha_\gamma Int(A)$ whenever $F \subseteq A$ and F is α_γ-closed in X.*

Theorem 2.1. [4] *Let (X,τ) be a topological space and γ an operation on $\alpha O(X)$. Then, the following properties are equivalent:*

1. *(X,τ) is α-γ-T_1.*
2. *For every point $x \in X$, $\{x\}$ is an α_γ-closed set.*
3. *(X,τ) is $\alpha_\gamma T_1$.*

Definition 2.2. [1] A topological space (X,τ) is said to be α_γ-regular if for each $x \in X$ and for each α-open set V in X containing x, there exists an α-open set U in X containing x such that $U^\gamma \subseteq V$.

Theorem 2.2. [3] *Let (X, τ) be a topological space and γ an operation on $\alpha O(X)$. Then the following statements are equivalent.*

1. $\alpha O(X, \tau) = \alpha O(X, \tau)_\gamma$.
2. (X, τ) *is an α_γ-regular space.*
3. *For every $x \in X$ and every α-open set U of X containing x there exists an α_γ-open set W of X such that $x \in W$ and $W \subseteq U$.*

Definition 2.3. [5] A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be $\alpha_{(\gamma,\beta)}$-continuous if for each x of X and each α_β-open set V containing $f(x)$, there exists an α_γ-open set U such that $x \in U$ and $f(U) \subseteq V$.

Definition 2.4. [5] A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be $\alpha_{(\gamma,\beta)}$-closed if for any α_γ-closed set A of (X, τ), $f(A)$ is α_β-closed in (Y, σ).

3. Weakly α_γ-Regular and Weakly α_γ-Normal Spaces

Definition 3.1. A space X is said to be weakly α_γ-regular space, if for any α_γ-closed set A and $x \notin A$, there exist α_γ-open sets U, V such that $x \in U$, $A \subseteq V$ and $U \cap V = \phi$.

From the following examples, It can be easily seen that weakly α_γ-regular and regular spaces are incomparable in general.

Example 3.1. Consider $X = \{a, b, c\}$ with the discrete topology τ on X. For a non-empty set A, we define an operation γ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \\ X & \text{otherwise.} \end{cases}$$

Then, the space X is a regular but it is not weakly α_γ-regular.

Example 3.2. Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$. For a non-empty set A, we define an operation γ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{c\} \text{ or } \{a, b\} \\ X & \text{otherwise.} \end{cases}$$

Then, X is a weakly α_γ-regular space but it is not regular.

Remark 3.1. The indiscrete space is a weakly α_γ-regular space, for if any α_γ-closed set A and any point $x \notin A$, there are α_γ-open sets U and V such that $x \in U$, $A \subseteq V$ and $U \cap V = \phi$.

In the next result, we give some characterizations of weakly α_γ-regular spaces.

Theorem 3.1. *Consider the following statements on a space X.*

1. *X is weakly α_γ-regular.*
2. *For any α_γ-open set U in X and $x \in U$, there is an α_γ-open set V containing x such that $\alpha Cl_\gamma(V) \subseteq U$.*
3. *For every point $x \in X$ and every α_γ-neighbourhood N of x, there exists an α-closed B such that $x \in B \subseteq N$.*

Then (1) *implies* (2) *and* (2) *implies* (3) .

Proof. (1) $\Rightarrow$ (2): Let U be α_γ-open set and $x \in U$. Then $X \setminus U$ is an α_γ-closed set such that $x \notin X \setminus U$. By the weakly α_γ-regularity of X, there are α_γ-open sets V and G such that $x \in V$, $X \setminus U \subseteq G$ and $V \cap G = \phi$. Clearly $X \setminus G$ is an α_γ-closed set contained in U. Then $V \subseteq X \setminus G \subseteq U$. This gives $\alpha Cl_\gamma(V) \subseteq X \setminus G \subseteq U$. Consequently, $x \in V$ and $\alpha Cl_\gamma(V) \subseteq U$.

(2) $\Rightarrow$ (3): Let $x \in X$ be any element and N be any α_γ-neighbourhood of x. By Definition 2.1, there exists an α_γ-open set U such that $x \in U \subseteq N$. By (2) there is an α_γ-open set V containing x such that $\alpha Cl_\gamma(V) \subseteq U$. Then, $x \in \alpha Cl_\gamma(V) \subseteq N$ and since $\alpha Cl_\gamma(V)$ is α-closed set by [[5], Theorem 2.26 (1)]. Thus for each $x \in X$, the set N form an α_γ-neighbourhood consisting of α-closed set of X. This proves (3). $\square$

In general, the statements in above theorem are not equivalent as it is shown in the following example.

Example 3.3. Consider $X = \{a, b, c\}$ with the discrete topology τ on X. For a non-empty set A, we define an operation γ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{b, c\} \\ X & \text{otherwise.} \end{cases}$$

Then, the statement (3) is holds but X is not weakly α_γ-regular.

Remark 3.2. In Theorem 3.1, the statements (1), (2) and (3) are equivalent to each other if one of the following conditions is satisfied:

1. γ is identity operation and $\alpha O(X, \tau) = \alpha C(X, \tau)$.
2. γ is identity operation and every subsets of X is α_γ-g.closed.
3. X is α_γ-regular and $\alpha O(X, \tau) = \alpha C(X, \tau)$.
4. X is α_γ-regular and every subsets of X is α_γ-g.closed.

Proof. It follows from the proof of Theorem 3.1, that we know the following implications, (1) $\Rightarrow$ (2) $\Rightarrow$ (3), so it is sufficient to prove (3) $\Rightarrow$ (1). Let A be an α_γ-closed set such that $x \notin A$. Then, $X \setminus A$ is α_γ-open and α_γ-neighbourhood of x. By (3) there is an α-closed set B such that $x \in B$ and $B \subseteq X \setminus A$, implies that $A \subseteq X \setminus B$. Then, $x \in B$, $A \subseteq X \setminus B$ and $B \cap X \setminus B = \phi$. Therefore,

1. by [[5], Remark 2.3] and $\alpha O(X, \tau) = \alpha C(X, \tau)$, or
2. by [[5], Remark 2.3] and [[5], Theorem 2.34], or
3. by Theorem 2.2 and $\alpha O(X, \tau) = \alpha C(X, \tau)$, or
4. by Theorem 2.2 and [[5], Theorem 2.34],

B and $X \setminus B$ are α_γ-open sets. Hence, X is weakly α_γ-regular. □

Theorem 3.2. *The following statements are equivalent for a topological space (X, τ) with an operation γ on $\alpha O(X, \tau)$:*

1. *X is weakly α_γ-regular.*
2. *For each $x \in X$ and each α_γ-open set U containing x, there exists an α_γ-open set V such that $x \in V \subseteq \alpha_\gamma Cl(V) \subseteq U$.*
3. *For each α_γ-closed subset F of X,*
   $F = \bigcap\{\alpha_\gamma Cl(V) : F \subseteq V \text{ and } V \in \alpha O(X, \tau)_\gamma\}$.
4. *For each A subset of X and each $U \in \alpha O(X, \tau)_\gamma$ with $A \cap U \neq \phi$, there exists $V \in \alpha O(X, \tau)_\gamma$ such that $A \cap V \neq \phi$ and $\alpha_\gamma Cl(V) \subseteq U$.*
5. *For each nonempty subset A of X and each α_γ-closed subset F of X with $A \cap F = \phi$, there exists $V, W \in \alpha O(X, \tau)_\gamma$ such that $A \cap V \neq \phi$, $F \subseteq W$ and $W \cap V = \phi$.*
6. *For each α_γ-closed set F and $x \notin F$, there exists $U \in \alpha O(X, \tau)_\gamma$ and an α_γ-g.open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$.*
7. *For each $A \subseteq X$ and each α_γ-closed set F with $A \cap F = \phi$, there exists $U \in \alpha O(X, \tau)_\gamma$ and an α_γ-g.open set V such that $A \cap U \neq \phi$, $F \subseteq V$ and $U \cap V = \phi$.*
8. *For each α_γ-closed set F of X,*
   $F = \bigcap\{\alpha_\gamma Cl(V)\text{: } F \subseteq V \text{ and } V \text{ is } \alpha_\gamma\text{-g.open}\}$.

Proof. (1) ⇒ (2): Let $x \notin X \setminus U$, where U is any α_γ-open set containing x. Then by (1), there exist $G, V \in \alpha O(X, \tau)_\gamma$ such that $(X \setminus U) \subseteq G$, $x \in V$ and $G \cap V = \phi$. Therefore $V \subseteq (X \setminus G)$ and so $x \in V \subseteq \alpha_\gamma Cl(V) \subseteq (X \setminus G) \subseteq U$.

(2) ⇒ (3): Let $X \setminus F$ be any α_γ-open set containing x. Then by (2), there exists an α_γ-open set U containing x such that $x \in U \subseteq \alpha_\gamma Cl(U) \subseteq (X \setminus F)$. So, $F \subseteq X \setminus \alpha_\gamma Cl(U) = V$, $V \in \alpha O(X, \tau)_\gamma$ and $V \cap U = \phi$. Then by [[5], Theorem 2.23], $x \notin \alpha_\gamma Cl(V)$. Hence, we obtain that
$F \supseteq \bigcap\{\alpha_\gamma Cl(V)\text{: } F \subseteq V, V \in \alpha O(X, \tau)_\gamma\}$.

(3) ⇒ (4): Let $U \in \alpha O(X, \tau)_\gamma$ with $x \in U \cap A$. Then $x \notin X \setminus U$ and hence

by (3), there exists an α_γ-open set W such that $X \setminus U \subseteq W$ and $x \notin \alpha_\gamma Cl(W)$. We put $V = X \setminus \alpha_\gamma Cl(W)$, which is an α_γ-open set containing x and hence $V \cap A \neq \phi$. Now $V \subseteq (X \setminus W)$ and so $\alpha_\gamma Cl(V) \subseteq (X \setminus W) \subseteq U$.

(4) $\Rightarrow$ (5): Let F be a set as in the hypothesis of (5). Then, $X \setminus F$ is α_γ-open and $(X \setminus F) \cap A \neq \phi$. Then, there exists $V \in \alpha O(X, \tau)_\gamma$ such that $A \cap V \neq \phi$ and $\alpha_\gamma Cl(V) \subseteq X \setminus F$. If we put $W = X \setminus \alpha_\gamma Cl(V)$, then $F \subseteq W$ and $W \cap V = \phi$.

(5) $\Rightarrow$ (1): Let F be an α_γ-closed set not containing x. Then by (5), there exist $W, V \in \alpha O(X, \tau)_\gamma$ such that $F \subseteq W$ and $x \in V$ and $W \cap V = \phi$.

(1) $\Rightarrow$ (6): Obvious.

(6) $\Rightarrow$ (7): For $a \in A$, $a \notin F$ and hence by (6), there exist $U \in \alpha O(X, \tau)_\gamma$ and an α_γ-g.open set V such that $a \in U$, $F \subseteq V$ and $U \cap V = \phi$. So, $A \cap U \neq \phi$.

(7) $\Rightarrow$ (1): Let $x \notin F$, where F is α_γ-closed. Since $\{x\} \cap F = \phi$, by (7), there exist $U \in \alpha O(X, \tau)_\gamma$ and an α_γ-g.open set W such that $x \in U$, $F \subseteq W$ and $U \cap W = \phi$. Now put $V = \alpha_\gamma Int(W)$. Using Proposition 2.1 of α_γ-g.open sets we get $F \subseteq V$ and $V \cap U = \phi$.

(3) $\Rightarrow$ (8): We have $F \subseteq \bigcap\{\alpha_\gamma Cl(V)\colon F \subseteq V$ and V is α_γ-g.open $\} \subseteq \bigcap\{\alpha_\gamma Cl(V)\colon F \subseteq V$ and V is α_γ-open $\} = F$.

(8) $\Rightarrow$ (1): Let F be an α_γ-closed set in X not containing x. Then by (8), there exists an α_γ-g.open set W such that $F \subseteq W$ and $x \in X \setminus \alpha_\gamma Cl(W)$. Since F is α_γ-closed and W is α_γ-g.open, then $F \subseteq \alpha_\gamma Int(W)$. Take $V = \alpha_\gamma Int(W)$. Then $F \subseteq V$, $x \in U = X \setminus \alpha_\gamma Cl(V)$ and $U \cap V = \phi$. $\square$

The following theorem shows that weakly α_γ-regularity is a hereditary property.

Theorem 3.3. *Let γ be an operation on $\alpha O(X)$ and $(H, \alpha O(X)_\gamma|H)$ be a subspace of a topological space (X, τ). If X is weakly α_γ-regular, then H is also weakly α_γ-regular.*

Proof. Suppose that A is α_γ-closed set in H and $y \in H$ such that $y \notin A$. Then, there exists an α_γ-open U in X such that $H \setminus A = U \cap H$. This implies that $A = B \cap H$, where $B = X \setminus U$ is α_γ-closed in X. Then $y \notin B$. Since X is weakly α_γ-regular, there exist disjoint α_γ-open sets U, V in X such that $y \in U$, $B \subseteq V$. Then, $U \cap H$ and $V \cap H$ are disjoint α_γ-open sets in H containing y and A respectively. $\square$

Theorem 3.4. *Let (X, τ) be a topological space and γ an α-regular operation on $\alpha O(X)$. Then, X is weakly α_γ-regular if and only if for each $x \in X$ and an α_γ-closed set A such that $x \notin A$, there exist α_γ-open sets U, V in X such that $x \in U$ and $A \subseteq V$ and $\alpha_\gamma Cl(U) \cap \alpha_\gamma Cl(V) = \phi$.*

Proof. Let $x \in X$ and A be an α_γ-closed set such that $x \notin A$, then by Theorem 3.1, there is an α_γ-open set W such that $x \in W$, $\alpha_\gamma Cl(W) \subseteq X \setminus A$. Again by Theorem 3.1, there is an α_γ-open set U containing x such that $\alpha_\gamma Cl(U) \subseteq W$. Let $V = X \setminus \alpha_\gamma Cl(W)$. Then $\alpha_\gamma Cl(U) \subseteq W \subseteq \alpha_\gamma Cl(W) \subseteq X \setminus A$ implies that $A \subseteq X \setminus \alpha_\gamma Cl(W) = V$. Also $\alpha_\gamma Cl(U) \cap \alpha_\gamma Cl(V) = \alpha_\gamma Cl(U) \cap \alpha_\gamma Cl(X \setminus \alpha_\gamma Cl(W)) \subseteq W \cap \alpha_\gamma Cl(X \setminus \alpha_\gamma Cl(W)) \subseteq \alpha_\gamma Cl(W \cap (X \setminus \alpha_\gamma Cl(W)) = \alpha_\gamma Cl(\phi) = \phi$ (by Remark 2.2). Thus U, V are the required α_γ-open sets in X. This proves the necessity. The sufficiency is immediate. □

Definition 3.2. A space X is said to be weakly α_γ-normal space, if for any disjoint α_γ-closed sets A, B of X, there exist α_γ-open sets U, V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$.

From the following examples, we can observe that weakly α_γ-normal and normal spaces are incomparable in general.

Example 3.4. Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. For any non-empty subset A of X, we define an operation γ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \\ X & \text{otherwise.} \end{cases}$$

Then, the space X is a normal but it is not weakly α_γ-normal.

Example 3.5. Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. For any non-empty subset A of X, we define an operation γ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{a, b\} \\ X & \text{otherwise.} \end{cases}$$

Then, X is a weakly α_γ-normal space but it is not normal.

In the next results, we give several characterizations of weakly α_γ-normal space.

Theorem 3.5. *Let (X, τ) be a topological space and γ an operation on $\alpha O(X)$. Then, X is weakly α_γ-normal if and only if for any α_γ-closed set A and α_γ-open set U containing A, there is an α_γ-open set V such that $A \subseteq V \subseteq \alpha_\gamma Cl(V) \subseteq U$.*

Proof. Since U is α_γ-open set containing A, then $X \setminus U$ is α_γ-closed and $A \cap (X \setminus U) = \phi$. Since X is weakly α_γ-normal, there exist α_γ-open sets V and V_1 such that $A \subseteq V$, $X \setminus U \subseteq V_1$ and $V \cap V_1 = \phi$. Hence, $A \subseteq V \subseteq \alpha_\gamma Cl(V) \subseteq \alpha_\gamma Cl(X \setminus V_1) = X \setminus V_1 \subseteq U$, or $A \subseteq V \subseteq \alpha_\gamma Cl(V) \subseteq U$.

Conversely, let A and B be two disjoint α_γ-closed sets in X. Then $A \subseteq X \setminus B$, where $X \setminus B$ is α_γ-open in X. By hypothesis, there is an α_γ-open set V such that $A \subseteq V \subseteq \alpha_\gamma Cl(V) \subseteq X \setminus B$, implies that $B \subseteq X \setminus \alpha_\gamma Cl(V)$ and $V \cap X \setminus \alpha_\gamma Cl(V) = \phi$. Consequently, $A \subseteq V$, $B \subseteq X \setminus \alpha_\gamma Cl(V)$. This proves that X is weakly α_γ-normal. □

Theorem 3.6. *For a topological space (X, τ) with an operation γ on $\alpha O(X, \tau)$, the following statements are equivalent:*

1. *X is weakly α_γ-normal.*
2. *For each pair of disjoint α_γ-closed sets A, B of X, there exist disjoint α_γ-g.open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
3. *For each α_γ-closed A and any α_γ-open set V containing A, there exists an α_γ-g.open set U such that $A \subseteq U \subseteq \alpha_\gamma Cl(U) \subseteq V$.*
4. *For each α_γ-closed set A and any α_γ-g.open set B containing A, there exists an α_γ-g.open set U such that $A \subseteq U \subseteq \alpha_\gamma Cl(U) \subseteq \alpha_\gamma Int(B)$.*
5. *For each α_γ-closed set A and any α_γ-g.open set B containing A, there exists an α_γ-open set G such that $A \subseteq G \subseteq \alpha_\gamma Cl(G) \subseteq \alpha_\gamma Int(B)$.*
6. *For each α_γ-g.closed set A and any α_γ-open set B containing A, there exists an α_γ-open set U such that $\alpha_\gamma Cl(A) \subseteq U \subseteq \alpha_\gamma Cl(U) \subseteq B$.*
7. *For each α_γ-g.closed set A and any α_γ-open set B containing A, there exists an α_γ-g.open set G such that $\alpha_\gamma Cl(A) \subseteq G \subseteq \alpha_\gamma Cl(G) \subseteq B$.*

Proof. (1) $\Rightarrow$ (2): Follows from the fact that every α_γ-open set is α_γ-g.open.

(2) $\Rightarrow$ (3): Let A be any α_γ-closed set and V any α_γ-open set containing A. Since A and $X \setminus V$ are disjoint α_γ-closed sets, there exist α_γ-g.open sets U and W such that $A \subseteq U$, $X \setminus V \subseteq W$ and $U \cap W = \phi$. By Proposition 2.1, we get $X \setminus V \subseteq \alpha_\gamma Int(W)$. Since $U \cap \alpha_\gamma Int(W) = \phi$, we have $\alpha_\gamma Cl(U) \cap \alpha_\gamma Int(W) = \phi$, and hence $\alpha_\gamma Cl(U) \subseteq X \setminus \alpha_\gamma Int(W) \subseteq V$. Therefore $A \subseteq U \subseteq \alpha_\gamma Cl(U) \subseteq V$.

(3) $\Rightarrow$ (1): Let A and B be any two disjoint α_γ-closed sets of X. Since $X \setminus B$ is an α_γ-open set containing A, there exists an α_γ-g.open set G such that $A \subseteq G \subseteq \alpha_\gamma Cl(G) \subseteq X \setminus B$. Since G is an α_γ-g.open set, using Proposition 2.1, we have $A \subseteq \alpha_\gamma Int(G)$. Taking $U = \alpha_\gamma Int(G)$ and $V = X \setminus \alpha_\gamma Cl(G)$, we have two disjoint α_γ-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Hence, X is weakly α_γ-normal.

(5) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3): Obvious.

(5) $\Rightarrow$ (3): Let A be any α_γ-closed set and V any α_γ-open set containing A. Since every α_γ-open set is α_γ-g.open, there exists an α_γ-open set G such that $A \subseteq G \subseteq \alpha_\gamma Cl(G) \subseteq \alpha_\gamma Int(V)$. Hence, we have an α_γ-g.open set G such that $A \subseteq G \subseteq \alpha_\gamma Cl(G) \subseteq \alpha_\gamma Int(V) \subseteq V$.

(3) $\Rightarrow$ (5): Let A be any α_γ-closed set and B any α_γ-g.open set containing A. Using Proposition 2.1, of an α_γ-g.open set we get $A \subseteq \alpha_\gamma Int(B) = V$, say. Then applying (3), we get an α_γ-g.open set U such that $A = \alpha_\gamma Cl(A) \subseteq U \subseteq \alpha_\gamma Cl(U) \subseteq V$. Again, using the same Proposition 2.1, we get $A \subseteq \alpha_\gamma Int(U)$, and hence $A \subseteq \alpha_\gamma Int(U) \subseteq$

$U \subseteq \alpha_\gamma Cl(U) \subseteq V$, which implies $A \subseteq \alpha_\gamma Int(U) \subseteq \alpha_\gamma Cl(\alpha_\gamma Int(U)) \subseteq \alpha_\gamma Cl(U) \subseteq V$, that is $A \subseteq G \subseteq \alpha_\gamma Cl(G) \subseteq \alpha_\gamma Int(B)$, where $G = \alpha_\gamma Int(U)$.

(6) $\Rightarrow$ (7) and (7) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (7) : Let A be any α_γ-g.closed set and B any α_γ-open set containing A. Since A is an α_γ-g.closed set, we have $\alpha_\gamma Cl(A) \subseteq B$, therefore by (3) we can find an α_γ-g.open set U such that $\alpha_\gamma Cl(A) \subseteq U \subseteq \alpha_\gamma Cl(U) \subseteq B$.

(7) $\Rightarrow$ (6): Let A be any α_γ-g.closed set and B any α_γ-open set containing A, then by (7), there exists an α_γ-g.open set G such that $\alpha_\gamma Cl(A) \subseteq G \subseteq \alpha_\gamma Cl(G) \subseteq B$. Since G is an α_γ-g.open set, then by Proposition 2.1, we get $\alpha_\gamma Cl(A) \subseteq \alpha_\gamma Int(G)$. If we take $U = \alpha_\gamma Int(G)$, the proof follows. □

Remark 3.3. The space in Example 3.5, is a weakly α_γ-normal but it is not weakly α_γ-regular.

Theorem 3.7. *Every weakly α_γ-normal α-γ-T_1 space is weakly α_γ-regular.*

Proof. Suppose that A is an α_γ-closed set and $x \notin A$. Since X is an α-γ-T_1 space, then by Theorem 2.1, $\{x\}$ is α_γ-closed in X. Also since X is weakly α_γ-normal, then there exist α_γ-open sets U and V such that $\{x\} \subseteq U$, $A \subseteq V$ and $U \cap V = \phi$, or $x \in U$, $A \subseteq V$ and $U \cap V = \phi$ give X is weakly α_γ-regular. □

Theorem 3.8. *Let γ be an operation on $\alpha O(X)$ and $(A, \alpha O(X)_\gamma|A)$ be a subspace of a topological space (X, τ). If A is α_γ-closed in X and X is weakly α_γ-normal, then A is weakly α_γ-normal.*

Proof. Let A_1 and A_2 be two disjoint α_γ-closed sets of A. Then, there are α_γ-closed sets B_1 and B_2 in X such that $A_1 = B_1 \cap A$, $A_2 = B_2 \cap A$. Since A is α_γ-closed in X, then A_1 and A_2 are α_γ-closed in X. Also since X is weakly α_γ-normal, there exist α_γ-open sets U_1 and U_2 in X such that $A_1 \subseteq U_1$, $A_2 \subseteq U_2$ and $U_1 \cap U_2 = \phi$. But $A_1 \subseteq A \cap U_1$ and $A_2 \subseteq A \cap U_2$, where $A \cap U_1$, $A \cap U_2$ are disjoint α_γ-open sets in A. This proves that A is weakly α_γ-normal. □

Theorem 3.9. *Suppose that $f : (X, \tau) \to (Y, \sigma)$ is a bijective, $\alpha_{(\gamma,\beta)}$-continuous and $\alpha_{(\gamma,\beta)}$-closed mapping. If X is weakly α_γ-normal, then Y is α_β-normal.*

Proof. Let A_1 and B_1 be two disjoint α_β-closed subsets of Y. Then by $\alpha_{(\gamma,\beta)}$-continuity of f, $A = f^{-1}(A_1)$, $B = f^{-1}(B_1)$ are disjoint α_γ-closed subsets of X. Since X is weakly α_γ-normal, then there exist α_γ-open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$. Since f is $\alpha_{(\gamma,\beta)}$-closed, then $f(X \setminus U)$ and $f(X \setminus V)$ are α_β-closed in Y. Then easy calculations give that $U_1 = Y \setminus f(X \setminus U)$, $U_2 = Y \setminus f(X \setminus V)$ are disjoint α_β-open sets in Y containing A_1, B_1 respectively. This proves that Y is α_β-normal. □

Theorem 3.10. *Let (X, τ) be a topological space and γ an α-regular operation on $\alpha O(X)$. Then, X is weakly α_γ-normal if and only if for each disjoint α_γ-closed sets A and B in X, there exist α_γ-open sets U and V in X such that $A \subseteq U$, $B \subseteq V$ and $\alpha_\gamma Cl(U) \cap \alpha_\gamma Cl(V) = \phi$.*

Proof. The sufficiency is clear. Let A and B be any two disjoint α_γ-closed sets in X. Then $X \setminus B$ is α_γ-open and $A \subseteq X \setminus B$. Then by Theorem 3.5, there is an α_γ-open set C such that $A \subseteq C \subseteq \alpha_\gamma Cl(C) \subseteq X \setminus B$. Since $A \subseteq C$, again by Theorem 3.5, there is an α_γ-open set U containing A such that $\alpha_\gamma Cl(U) \subseteq C$. Consequently, $A \subseteq U \subseteq \alpha_\gamma Cl(U) \subseteq C$ and $\alpha_\gamma Cl(C) \subseteq X \setminus B$ implies that $B \subseteq X \setminus \alpha_\gamma Cl(C)$. Put $V = X \setminus \alpha_\gamma Cl(C)$. Then V is α_γ-open containing B and moreover $\alpha_\gamma Cl(U) \cap \alpha_\gamma Cl(V) = \alpha_\gamma Cl(U) \cap \alpha_\gamma Cl(X \setminus \alpha_\gamma Cl(C)) \subseteq C \cap \alpha_\gamma Cl(X \setminus \alpha_\gamma Cl(C)) \subseteq \alpha_\gamma Cl(C \cap X \setminus \alpha_\gamma Cl(C)) = \alpha_\gamma Cl(\phi) = \phi$ (by Remark 2.2). Thus, U and V are the required α_γ-open sets in X. This proves the necessity. Hence the proof. $\square$

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