ON THE MEASUREMENT OF GROWTH PROPERTIES OF ENTIRE AND MEROMORPHIC FUNCTIONS FOCUSING THEIR RELATIVE TYPE AND RELATIVE WEAK TYPE

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Abstract. The concepts of relative growth indicators such as relative order, relative type, relative weak type, etc. have widely been used to avoid comparing growths of entire and meromorphic functions just with exp functions. Using the notions of several relative growth indicators as mentioned earlier, in this paper we would like to find out the limits in terms of classical growth indicators (i.e. order, type, weak type etc.) in which the relative type, relative weak type, etc. of meromorphic functions with respect to entire functions should lie.

Keywords: Meromorphic function; Entire function; Relative order; Relative lower order; Relative type; Relative weak type

1. Introduction, Definitions and Notations

Let $\mathbb{C}$ be the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function corresponding to entire f is defined as $M_f(r) = \max\{|f(z)| : |z| = r\}$. The order (lower order) of an entire function f is defined in terms of the growth of f with respect to $\exp z$ function which is as follows:

$$\rho_f = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log \log M_{\exp z}(r)} = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log(r)}$$

$$\left(\lambda_f = \liminf_{r \to \infty} \frac{\log \log M_f(r)}{\log \log M_{\exp z}(r)} = \liminf_{r \to \infty} \frac{\log \log M_f(r)}{\log(r)} \right).$$

When f is a meromorphic function, $M_f(r)$ cannot be defined as f is not analytic. In this case one may define another function $T_f(r)$ known as Nevanlinna's

Characteristic function of f, playing the same role as maximum modulus function in the following manner:

$$T_f(r) = N_f(r) + m_f(r),$$

where the function $N_f(r,a)\left(\bar{N}_f(r,a)\right)$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a) \log r$$

$$\left(\bar{N}_f(r,a) = \int_0^r \frac{\bar{n}_f(t,a) - \bar{n}_f(0,a)}{t} dt + \bar{n}_f(0,a) \log r\right),$$

moreover we denote by $n_f(r,a)\left(\bar{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞-point is a pole of f. In many occasions $N_f(r,\infty)$ and $\bar{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\bar{N}_f(r)$ respectively.

Also the function $m_f(r,\infty)$ alternatively denoted by $m_f(r)$ known as the proximity function of f is defined as follows:

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta,$$

where
$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0 .$$

Also we may denote $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r,a)$.

If f is an entire function, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as
$$T_f(r) = m_f(r) .$$

Further, if f is a non-constant entire then $T_f(r)$ is strictly increasing and continuous function of r. Also its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exist and is such that $\lim_{s \to \infty} T_f^{-1}(s) = \infty$. However, in case of meromorphic functions, the growth indicators such as order and lower order which are classical in complex analysis are defined in terms of their growths with respect to $\exp z$ function as the following:

$$\rho_f = \limsup_{r \to \infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r \to \infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r \to \infty} \frac{\log T_f(r)}{\log(r) + O(1)}$$

$$\left(\lambda_f = \liminf_{r \to \infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \liminf_{r \to \infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \liminf_{r \to \infty} \frac{\log T_f(r)}{\log(r) + O(1)}\right).$$

Next, we give the definitions of *type* and *weak type* of meromorphic functions which are also another type of classical growth indicators used for comparing the relative growth of two meromorphic functions having same non zero finite *order* with respect to another meromorphic function:

Definition 1.1. The *type* σ_f and *lower type* $\overline{\sigma}_f$ of a meromorphic function f are defined as

$$\sigma_f = \limsup_{r \to \infty} \frac{T_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r \to \infty} \frac{T_f(r)}{r^{\rho_f}}, \ \ 0 < \rho_f < \infty \ .$$

Datta and Jha [4] introduced the definition of *weak type* of a meromorphic function of finite positive lower order in the following way:

Definition 1.2. [4] The *weak type* τ_f and the growth indicator $\overline{\tau}_f$ of a meromorphic function f of finite positive lower order λ_f are defined by

$$\overline{\tau}_f = \limsup_{r \to \infty} \frac{T_f(r)}{r^{\lambda_f}} \ \ \text{and} \ \ \tau_f = \liminf_{r \to \infty} \frac{T_f(r)}{r^{\lambda_f}}, \ 0 < \lambda_f < \infty \ .$$

Extending the notion of relative order as introduced by Bernal $\{[1], [2]\}$, Lahiri and Banerjee [7] gave the definition of *relative order* of a meromorphic function f with respect to an entire function g , denoted by $\rho_g(f)$ as follows:

$$
\begin{aligned}
\rho_g(f) &= \inf \left\{ \mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r \right\} \\
&= \limsup_{r \to \infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.
\end{aligned}
$$

The definition coincides with the classical one [7] if $g(z) = \exp z$.

In the same way, one can define the *relative lower order* of a meromorphic function f with respect to entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda_g(f) = \liminf_{r \to \infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

In the case of meromorphic functions, it therefore seems reasonable to define suitably the *relative type* and *relative weak type* of a meromorphic function with respect to an entire function to determine the relative growth of two meromorphic functions having same non zero finite *relative order* or *relative lower order* with respect to an entire function. Datta and Biswas [5] also gave such definitions of *relative type* and *relative weak type* of a meromorphic function f with respect to an entire function g which are as follows:

Definition 1.3. [5] The *relative type* $\sigma_g(f)$ of a meromorphic function f with respect to an entire function g are defined as

$$\sigma_g(f) = \limsup_{r \to \infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty.$$

Similarly, one can define the *lower relative type* $\overline{\sigma}_g(f)$ in the following way:

$$\overline{\sigma}_g(f) = \liminf_{r \to \infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty.$$

Definition 1.4. [5] The *relative weak type* $\tau_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ is defined by

$$\tau_g(f) = \liminf_{r \to \infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}}.$$

In a like manner, one can define the growth indicator $\overline{\tau}_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ as

$$\overline{\tau}_g(f) = \limsup_{r \to \infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}}.$$

Considering $g = \exp z$ one may easily verify that Definition 1.3 and Definition 1.4 coincide with the classical definition of type (lower type) and weak type of a meromorphic function.

For entire and meromorphic functions, the notion of the growth indicators of its such as *order*, *type* and *weak type* are classical in complex analysis and during the past decades, several researchers have already been continued their studies in the area of comparative growth properties of entire and meromorphic functions in different directions using the growth indicator such as *order*, *type* and *weak type*. But at that time, the concept of *relative order* and consequently *relative type* and *relative weak type* of meromorphic function with respect to another entire function which have already been discussed above was mostly unknown to complex analysts and was not known for the technical advantage given by such notion which gives an idea to avoid comparing growth just with exp function to calculate *order*, *type* and *weak type* respectively. In the paper, we investigate some relative growth properties of entire and meromorphic functions with respect to another entire function on the basis of relative type and relative weak type. We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [6] and [9]. Hence we do not explain those in details.

2.   Lemma

In this section we present a lemma due to Debnath et al. [3]:

Lemma 2.1. *[3] Let f be a meromorphic function $0 < \lambda_f \le \rho_f < \infty$ and g be an entire function with $0 < \lambda_g \le \rho_g < \infty$. Then*

$$\frac{\lambda_f}{\rho_g} \le \lambda_g(f) \le \min\left\{\frac{\lambda_f}{\lambda_g}, \frac{\rho_f}{\rho_g}\right\} \le \max\left\{\frac{\lambda_f}{\lambda_g}, \frac{\rho_f}{\rho_g}\right\} \le \rho_g(f) \le \frac{\rho_f}{\lambda_g}.$$

3. Main Results

In this section we state the main results of the paper.

Theorem 3.1. *Let f be a meromorphic function $0 < \lambda_f \le \rho_f < \infty$ and g be an entire function with $0 < \lambda_g \le \rho_g < \infty$. Then*

$$\max\left\{\left[\frac{\overline{\sigma}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\sigma_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \le \sigma_g(f) \le \min\left\{\left[\frac{\overline{\tau}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\overline{\tau}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}\right\}$$

and

$$\left[\frac{\overline{\sigma}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \le \overline{\sigma}_g(f) \le \min\left\{\left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\tau_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}\right\}.$$

Proof. From the definitions of σ_f and $\overline{\sigma}_f$, we have for all sufficiently large values of r that

(3.1) $$T_f(r) \le (\sigma_f + \varepsilon) \cdot r^{\rho_f},$$

(3.2) $$T_f(r) \ge (\overline{\sigma}_f - \varepsilon) \cdot r^{\rho_f}$$

and also for a sequence of values of r tending to infinity we get that

(3.3) $$T_f(r) \ge (\sigma_f - \varepsilon) \cdot r^{\rho_f},$$

(3.4) $$T_f(r) \le (\overline{\sigma}_f + \varepsilon) \cdot r^{\rho_f}.$$

Similarly from the definitions of σ_g and $\overline{\sigma}_g$, it follows for all sufficiently large values of r that

$$T_g(r) \le (\sigma_g + \varepsilon) \cdot r^{\rho_g} \, i.e., \ r \le T_g^{-1}\left[(\sigma_g + \varepsilon) \cdot r^{\rho_g}\right] i.e., \ T_g^{-1}(r) \ge \left[\left(\frac{r}{(\sigma_g + \varepsilon)}\right)^{\frac{1}{\rho_g}}\right],$$

$$T_g(r) \ge (\overline{\sigma}_g - \varepsilon) \cdot r^{\rho_g} \, i.e., \ r \ge T_g^{-1}\left[(\overline{\sigma}_g - \varepsilon) \cdot r^{\rho_g}\right] i.e., \ T_g^{-1}(r) \le \left[\left(\frac{r}{(\overline{\sigma}_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}\right]$$

and for a sequence of values of r tending to infinity, we obtain that

$$T_g(r) \geq (\sigma_g - \varepsilon) \cdot r^{\rho_g} \, i.e., \ r \geq T_g^{-1}\left[(\sigma_g - \varepsilon) \cdot r^{\rho_g}\right] i.e., \ T_g^{-1}(r) \leq \left[\left(\frac{r}{(\sigma_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}\right],$$

$$T_g(r) \leq (\overline{\sigma}_g + \varepsilon) \cdot r^{\rho_g} \, i.e., \ r \leq T_g^{-1}\left[(\overline{\sigma}_g + \varepsilon) \cdot r^{\rho_g}\right] i.e., \ T_g^{-1}(r) \geq \left[\left(\frac{r}{(\overline{\sigma}_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}\right].$$

Further from the definitions of $\overline{\tau}_f$ and τ_f, it follows for all sufficiently large values of r that

$$(3.5) \qquad\qquad\qquad T_f(r) \leq (\overline{\tau}_f + \varepsilon) \cdot r^{\lambda_f},$$

$$(3.6) \qquad\qquad\qquad T_f(r) \geq (\tau_f - \varepsilon) \cdot r^{\lambda_f}$$

and also for a sequence of values of r tending to infinity, we get that

$$(3.7) \qquad\qquad\qquad T_f(r) \geq (\overline{\tau}_f - \varepsilon) \cdot r^{\lambda_f},$$

$$(3.8) \qquad\qquad\qquad T_f(r) \leq (\tau_f + \varepsilon) \cdot r^{\lambda_f}.$$

Also from the definitions of $\overline{\tau}_g$ and τ_g, we obtain for all sufficiently large values of r that

$$T_g(r) \leq (\overline{\tau}_g + \varepsilon) \cdot r^{\lambda_g} \, i.e., \ r \leq T_g^{-1}\left[(\overline{\tau}_g + \varepsilon) \cdot r^{\lambda_g}\right] i.e., \ T_g^{-1}(r) \geq \left[\left(\frac{r}{(\overline{\tau}_g + \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right],$$

$$T_g(r) \geq (\tau_g - \varepsilon) \cdot r^{\lambda_g} \, i.e., \ r \geq T_g^{-1}\left[(\tau_g - \varepsilon) \cdot r^{\lambda_g}\right] i.e., \ T_g^{-1}(r) \leq \left[\left(\frac{r}{(\tau_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right]$$

and for a sequence of values of r tending to infinity, we obtain that

$$T_g(r) \geq (\overline{\tau} - \varepsilon) \cdot r^{\lambda_g} \, i.e., \ r \geq T_g^{-1}\left[(\overline{\tau}_g - \varepsilon) \cdot r^{\lambda_g}\right] i.e., \ T_g^{-1}(r) \leq \left[\left(\frac{r}{(\overline{\tau}_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right],$$

$$T_g(r) \leq (\tau_g + \varepsilon) \cdot r^{\lambda_g} \, i.e., \ r \leq T_g^{-1}\left[(\tau_g + \varepsilon) \cdot r^{\lambda_g}\right] i.e., \ T_g^{-1}(r) \geq \left[\left(\frac{r}{(\tau_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right].$$

Now from (3.3) and in view of (3.9), we get for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \geq T_g^{-1}\left[(\sigma_f - \varepsilon) \cdot r^{\rho_f}\right]$$

$$i.e., \ T_g^{-1}T_f(r) \geq \left(\frac{(\sigma_f - \varepsilon) \cdot r^{\rho_f}}{(\overline{\tau}_g + \varepsilon)}\right)^{\frac{1}{\lambda_g}}$$

$$i.e., \ T_g^{-1}T_f(r) \geq \left[\frac{(\sigma_f - \varepsilon)}{(\overline{\tau}_g + \varepsilon)}\right]^{\frac{1}{\lambda_g}} \cdot r^{\frac{\rho_f}{\lambda_g}} i.e., \ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\lambda_g}}} \geq \left[\frac{(\sigma_f - \varepsilon)}{(\overline{\tau}_g + \varepsilon)}\right]^{\frac{1}{\lambda_g}}.$$

Since in view of Lemma 2.1, $\frac{\rho_f}{\lambda_g} \geq \rho_g(f)$ and as $\varepsilon (> 0)$ is arbitrary, therefore it follows from above that

$$\limsup_{r \to \infty} \frac{T_g^{-1}T_f(r)}{(r)^{\rho_g(f)}} \geq \left[\frac{\sigma_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} i.e., \ \sigma_g(f) \geq \left[\frac{\sigma_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}.$$

Similarly from (3.2) and in view of (3.9), it follows for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \geq T_g^{-1}\left[(\overline{\sigma}_f - \varepsilon) \cdot r^{\rho_f}\right]$$

$$i.e., \ T_g^{-1}T_f(r) \geq \left(\frac{(\overline{\sigma}_f - \varepsilon) \cdot r^{\rho_f}}{(\tau_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}$$

$$i.e., \ T_g^{-1}T_f(r) \geq \left[\frac{(\overline{\sigma}_f - \varepsilon)}{(\tau_g + \varepsilon)}\right]^{\frac{1}{\lambda_g}} \cdot r^{\frac{\rho_f}{\lambda_g}} i.e., \ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\lambda_g}}} \geq \left[\frac{(\overline{\sigma}_f - \varepsilon)}{(\tau_g + \varepsilon)}\right]^{\frac{1}{\lambda_g}}.$$

Since in view of Lemma 2.1, it follows that $\frac{\rho_f}{\lambda_g} \geq \rho_g(f)$. Also $\varepsilon (> 0)$ is arbitrary. So we get from above that

$$\limsup_{r \to \infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \geq \left[\frac{\overline{\sigma}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}} i.e., \ \sigma_g(f) \geq \left[\frac{\overline{\sigma}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}.$$

Again in view of (3.9), we have from (3.5) for all sufficiently large values of r that

$$T_g^{-1}T_f(r) \leq T_g^{-1}\left[(\overline{\tau}_f + \varepsilon) \cdot r^{\lambda_f}\right]$$

$$i.e., \ T_g^{-1}T_f(r) \leq \left(\frac{(\overline{\tau}_f + \varepsilon) \cdot r^{\lambda_f}}{(\tau_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}$$

$$i.e., \ T_g^{-1}T_f(r) \leq \left[\frac{(\overline{\tau}_f + \varepsilon)}{(\tau_g - \varepsilon)}\right]^{\frac{1}{\lambda_g}} \cdot r^{\frac{\lambda_f}{\lambda_g}} i.e., \ \frac{T_g^{-1}T_f(r)}{r^{\frac{\lambda_f}{\lambda_g}}} \leq \left[\frac{(\overline{\tau}_f + \varepsilon)}{(\tau_g - \varepsilon)}\right]^{\frac{1}{\lambda_g}}.$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\lambda_g} \leq \rho_g(f)$ and as $\varepsilon (> 0)$ is arbitrary. Therefore it follows from above that

$$\limsup_{r \to \infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \leq \left[\frac{\overline{\tau}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}} i.e., \ \sigma_g(f) \leq \left[\frac{\overline{\tau}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}.$$

Again in view of (3.5), we have from (3.1) for all sufficiently large values of r that

$$T_g^{-1}T_f(r) \leq T_g^{-1}\left[(\sigma_f + \varepsilon) \cdot r^{\rho_f}\right]$$

$$i.e., T_g^{-1}T_f(r) \leq \left(\frac{(\sigma_f + \varepsilon) \cdot r^{\rho_f}}{(\overline{\sigma}_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e., \ T_g^{-1}T_f(r) \leq \left[\frac{(\sigma_f + \varepsilon)}{(\overline{\sigma}_g - \varepsilon)}\right]^{\frac{1}{\rho_g}} \cdot r^{\frac{\rho_f}{\rho_g}} i.e., \ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\rho_g}}} \leq \left[\frac{(\sigma_f + \varepsilon)}{(\overline{\sigma}_g - \varepsilon)}\right]^{\frac{1}{\rho_g}}.$$

As in view of Lemma 2.1, it follows that $\frac{\rho_f}{\rho_g} \leq \rho_g(f)$. Since $\varepsilon \, (> 0)$ is arbitrary, we get from (3.9) that

$$\limsup_{r \to \infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \quad \leq \left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \ \sigma_g(f) \quad \leq \left[\frac{\sigma_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}.$$

Further in view of (3.5), we have from (3.5) for all sufficiently large values of r that

$$T_g^{-1}T_f(r) \leq T_g^{-1}\left[(\overline{\tau}_f + \varepsilon) \cdot r^{\lambda_f}\right]$$

$$i.e., \ T_g^{-1}T_f(r) \leq \left(\frac{(\overline{\tau}_f + \varepsilon) \cdot r^{\lambda_f}}{(\overline{\sigma}_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e., \ T_g^{-1}T_f(r) \quad \leq \left[\frac{(\overline{\tau}_f + \varepsilon)}{(\overline{\sigma}_g - \varepsilon)}\right]^{\frac{1}{\rho_g}} \cdot r^{\frac{\lambda_f}{\rho_g}}$$

$$i.e., \ \frac{T_g^{-1}T_f(r)}{r^{\frac{\lambda_f}{\rho_g}}} \quad \leq \left[\frac{(\overline{\tau}_f + \varepsilon)}{(\overline{\sigma}_g - \varepsilon)}\right]^{\frac{1}{\rho_g}}.$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\rho_g} \leq \rho_g(f)$ and as $\varepsilon \, (> 0)$ is arbitrary. Therefore it follows from above that

$$\limsup_{r \to \infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \quad \leq \left[\frac{\overline{\tau}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \ \sigma_g(f) \quad \leq \left[\frac{\overline{\tau}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}.$$

Thus the first part of the theorem follows from $(3.9), (3.9), (3.9), (3.9)$ and (3.9).

Further from (3.2) and in view of (3.9), we get for all sufficiently large values of r that

$$T_g^{-1}T_f(r) \geq T_g^{-1}\left[(\overline{\sigma}_f - \varepsilon) \cdot r^{\rho_f}\right]$$

$$i.e., \ T_g^{-1}T_f(r) \geq \left(\frac{(\overline{\sigma}_f - \varepsilon) \cdot r^{\rho_f}}{(\overline{\tau}_g + \varepsilon)}\right)^{\frac{1}{\lambda_g}}$$

$$i.e., \ T_g^{-1}T_f(r) \quad \geq \left[\frac{(\overline{\sigma}_f - \varepsilon)}{(\overline{\tau}_g + \varepsilon)}\right]^{\frac{1}{\lambda_g}} \cdot r^{\frac{\rho_f}{\lambda_g}}$$

$$i.e., \ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\lambda_g}}} \quad \geq \left[\frac{(\overline{\sigma}_f - \varepsilon)}{(\overline{\tau}_g + \varepsilon)}\right]^{\frac{1}{\lambda_g}}.$$

Now in view of Lemma 2.1, it follows that $\frac{\rho_f}{\lambda_g} \geq \rho_g(f)$. Since $\varepsilon(> 0)$ is arbitrary, we get from above that

$$\liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \geq \left[\frac{\overline{\sigma}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e., \ \overline{\sigma}_g(f) \geq \left[\frac{\overline{\sigma}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \ .$$

Also in view of (3.5), we get from (3.1) for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \leq T_g^{-1}\left[(\sigma_f + \varepsilon) \cdot r^{\rho_f}\right]$$

$$i.e., \ T_g^{-1}T_f(r) \leq \left(\frac{(\sigma_f + \varepsilon) \cdot r^{\rho_f}}{(\sigma_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e., \ T_g^{-1}T_f(r) \leq \left[\frac{(\sigma_f+\varepsilon)}{(\sigma_g-\varepsilon)}\right]^{\frac{1}{\rho_g}} \cdot r^{\frac{\rho_f}{\rho_g}}$$

$$i.e., \ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\rho_g}}} \leq \left[\frac{(\sigma_f+\varepsilon)}{(\sigma_g-\varepsilon)}\right]^{\frac{1}{\rho_g(m,p)}} \ .$$

Again in view of Lemma 2.1, $\frac{\rho_f}{\rho_g} \leq \rho_g(f)$ and $\varepsilon(> 0)$ is arbitrary, so we get from (3.9) that

$$\liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \leq \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \ \overline{\sigma}_g(f) \leq \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} \ .$$

Likewise from (3.4) and in view of (3.5), it follows for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \leq T_g^{-1}\left[(\overline{\sigma}_f + \varepsilon) \cdot r^{\rho_f}\right]$$

$$i.e., \ T_g^{-1}T_f(r) \leq \left(\frac{(\overline{\sigma}_f + \varepsilon) \cdot r^{\rho_f}}{(\overline{\sigma}_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e., \ T_g^{-1}T_f(r) \leq \left[\frac{(\overline{\sigma}_f+\varepsilon)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}} \cdot r^{\frac{\rho_f}{\rho_g}}$$

$$i.e., \ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\rho_g}}} \leq \left[\frac{(\overline{\sigma}_f+\varepsilon)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}} \ .$$

Analogously, we get from (3.9) that

$$\liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \leq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \ \overline{\sigma}_g(f) \leq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}} ,$$

since in view of Lemma 2.1, $\frac{\rho_f}{\rho_g} \leq \rho_g(f)$ and $\varepsilon(>0)$ is arbitrary.

Further in view of (3.9), we get from (3.5) for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \leq T_g^{-1}\left[(\overline{\tau}_f + \varepsilon) \cdot r^{\lambda_f}\right]$$

$$i.e.,\ T_g^{-1}T_f(r) \leq \left(\frac{(\overline{\tau}_f + \varepsilon) \cdot r^{\lambda_f}}{(\overline{\tau}_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}$$

$$i.e.,\ T_g^{-1}T_f(r) \quad \leq \left[\frac{(\overline{\tau}_f + \varepsilon)}{(\overline{\tau}_g - \varepsilon)}\right]^{\frac{1}{\lambda_g}} \cdot r^{\frac{\lambda_f}{\lambda_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f(r)}{r^{\frac{\lambda_f}{\lambda_g}}} \quad \leq \left[\frac{(\overline{\tau}_f + \varepsilon)}{(\overline{\tau}_g - \varepsilon)}\right]^{\frac{1}{\lambda_g}}\ .$$

As in view of Lemma 2.1, we get that $\frac{\lambda_f}{\lambda_g} \leq \rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary. Therefore it follows from above that

$$\liminf_{r\to\infty}\frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \quad \leq \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e.,\ \overline{\sigma}_g(f) \quad \leq \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\ .$$

Similarly from (3.8) and in view of (3.9), it follows for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \leq T_g^{-1}\left[(\tau_f + \varepsilon) \cdot r^{\lambda_f}\right]$$

$$i.e.,\ T_g^{-1}T_f(r) \leq \left(\frac{(\tau_f + \varepsilon) \cdot r^{\lambda_f}}{(\tau_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}$$

$$i.e.,\ T_g^{-1}T_f(r) \quad \leq \left[\frac{(\tau_f + \varepsilon)}{(\tau_g - \varepsilon)}\right]^{\frac{1}{\lambda_g}} \cdot r^{\frac{\lambda_f}{\lambda_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f(r)}{r^{\frac{\lambda_f}{\lambda_g}}} \quad \leq \left[\frac{(\tau_f + \varepsilon)}{(\tau_g - \varepsilon)}\right]^{\frac{1}{\lambda_g}}\ .$$

Also in view of Lemma 2.1, we get that $\frac{\lambda_f}{\lambda_g} \leq \rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\liminf_{r\to\infty}\frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}} \quad \leq \left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e.,\ \overline{\sigma}_g(f) \quad \leq \left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}\ .$$

Again in view of (3.5), we get from (3.5) for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \leq T_g^{-1}\left[(\overline{\tau}_f + \varepsilon) \cdot r^{\lambda_f}\right]$$

$$i.e.,\ T_g^{-1}T_f\left(r\right) \leq \left(\frac{\left(\overline{\tau}_f+\varepsilon\right)\cdot r^{\lambda_f}}{\left(\sigma_g-\varepsilon\right)}\right)^{\frac{1}{\rho_g}}$$

$$i.e.,\ T_g^{-1}T_f\left(r\right)\ \leq \left[\frac{\left(\overline{\tau}_f+\varepsilon\right)}{\left(\sigma_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\lambda_f}{\rho_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f\left(r\right)}{r^{\frac{\lambda_f}{\rho_g}}}\ \leq \left[\frac{\left(\overline{\tau}_f+\varepsilon\right)}{\left(\sigma_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\ .$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\rho_g}\leq \rho_g\left(f\right)$ and as $\varepsilon\left(>0\right)$ is arbitrary, so it follows from above that

$$\liminf_{r\to\infty}\frac{T_g^{-1}T_f\left(r\right)}{r^{\rho_g\left(f\right)}}\ \leq \left[\frac{\overline{\tau}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \overline{\sigma}_g\left(f\right)\ \leq \left[\frac{\overline{\tau}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\ .$$

Similarly from (3.8) and in view of (3.5), it follows for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f\left(r\right)\leq T_g^{-1}\left[\left(\tau_f+\varepsilon\right)\cdot r^{\lambda_f}\right]$$

$$i.e.,\ T_g^{-1}T_f\left(r\right)\leq \left(\frac{\left(\tau_f+\varepsilon\right)\cdot r^{\lambda_f}}{\left(\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{\rho_g}}$$

$$i.e.,\ T_g^{-1}T_f\left(r\right)\ \leq \left[\frac{\left(\tau_f+\varepsilon\right)}{\left(\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\lambda_f}{\rho_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f\left(r\right)}{r^{\frac{\lambda_f}{\rho_g}}}\ \leq \left[\frac{\left(\tau_f+\varepsilon\right)}{\left(\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\ .$$

As in view of Lemma 2.1, we get that $\frac{\lambda_f}{\rho_g}\leq \rho_g\left(f\right)$. Also as $\varepsilon\left(>0\right)$ is arbitrary, therefore it follows from above that

$$\liminf_{r\to\infty}\frac{T_g^{-1}T_f\left(r\right)}{r^{\rho_g\left(f\right)}}\ \leq \left[\frac{\tau_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \overline{\sigma}_g\left(f\right)\ \leq \left[\frac{\tau_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}\ .$$

Hence the second part of the theorem follows from $\left(3.9\right),\left(3.9\right),\left(3.9\right),\left(3.9\right),\left(3.9\right),$ $\left(3.9\right)$ and $\left(3.9\right).$ $\square$

Theorem 3.2. *Let f be a meromorphic function $0<\lambda_f\leq \rho_f<\infty$ and g be an entire function with $0<\lambda_g\leq \rho_g<\infty$. Then*

$$\max\left\{\left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}},\left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}},\left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}},\left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}},\left[\frac{\sigma_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}},\left[\frac{\overline{\sigma}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}\right\}\leq \overline{\tau}_g\left(f\right)\leq \left[\frac{\overline{\tau}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

and

$$\max\left\{\left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\tau_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\sigma}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \leq \tau_g(f) \leq \min\left\{\left[\frac{\tau_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\overline{\tau}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\}.$$

Proof. We obtain from (3.7) and (3.9), for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \geq T_g^{-1}\left[(\overline{\tau}_f - \varepsilon)\cdot r^{\lambda_f}\right]$$

$$i.e., \ T_g^{-1}T_f(r) \geq \left(\frac{(\overline{\tau}_f - \varepsilon)\cdot r^{\lambda_f}}{(\overline{\tau}_g + \varepsilon)}\right)^{\frac{1}{\lambda_g}}$$

$$i.e., \ T_g^{-1}T_f(r) \geq \left[\frac{(\overline{\tau}_f - \varepsilon)}{(\overline{\tau}_g + \varepsilon)}\right]^{\frac{1}{\lambda_g}}\cdot r^{\frac{\lambda_f}{\lambda_g}}$$

$$i.e., \ \frac{T_g^{-1}T_f(r)}{r^{\frac{\lambda_f}{\lambda_g}}} \geq \left[\frac{(\overline{\tau}_f - \varepsilon)}{(\overline{\tau}_g + \varepsilon)}\right]^{\frac{1}{\lambda_g}}.$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\lambda_g} \geq \lambda_g(f)$ and as $\varepsilon\,(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{r\to\infty}\frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}} \geq \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e., \ \overline{\tau}_g(f) \geq \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}.$$

Further we obtain from (3.6) and (3.9), for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \geq T_g^{-1}\left[(\tau_f - \varepsilon)\cdot r^{\lambda_f}\right]$$

$$i.e., \ T_g^{-1}T_f(r) \geq \left(\frac{(\tau_f - \varepsilon)\cdot r^{\lambda_f}}{(\tau_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}$$

$$i.e., \ T_g^{-1}T_f(r) \geq \left[\frac{(\tau_f - \varepsilon)}{(\tau_g - \varepsilon)}\right]^{\frac{1}{\lambda_g}}\cdot r^{\frac{\lambda_f}{\lambda_g}}$$

$$i.e., \ \frac{T_g^{-1}T_f(r)}{r^{\frac{\lambda_f}{\lambda_g}}} \geq \left[\frac{(\tau_f - \varepsilon)}{(\tau_g - \varepsilon)}\right]^{\frac{1}{\lambda_g}}.$$

As in view of Lemma 2.1, we get that $\frac{\lambda_f}{\lambda_g} \geq \lambda_g(f)$ and as $\varepsilon\,(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{r\to\infty}\frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}} \geq \left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e., \ \overline{\tau}_g(f) \geq \left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}.$$

Now from (3.3) and in view of (3.5), we get for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \geq T_g^{-1}\left[(\sigma_f - \varepsilon) \cdot r^{\rho_f}\right]$$

$$i.e., \; T_g^{-1}T_f(r) \geq \left(\frac{(\sigma_f - \varepsilon) \cdot r^{\rho_f}}{(\sigma_g + \varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e., \; T_g^{-1}T_f(r) \; \geq \left[\frac{(\sigma_f - \varepsilon)}{(\sigma_g + \varepsilon)}\right]^{\frac{1}{\rho_g}} \cdot r^{\frac{\rho_f}{\rho_g}}$$

$$i.e., \; \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\rho_g}}} \; \geq \left[\frac{(\sigma_f - \varepsilon)}{(\sigma_g + \varepsilon)}\right]^{\frac{1}{\rho_g}} \; .$$

Analogously from (3.2) and in view of (3.5), it follows for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \geq T_g^{-1}\left[(\overline{\sigma}_f - \varepsilon) \cdot r^{\rho_f}\right]$$

$$i.e., \; T_g^{-1}T_f(r) \geq \left(\frac{(\overline{\sigma}_f - \varepsilon) \cdot r^{\rho_f}}{(\overline{\sigma}_g + \varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e., \; T_g^{-1}T_f(r) \; \geq \left[\frac{(\overline{\sigma}_f - \varepsilon)}{(\overline{\sigma}_g + \varepsilon)}\right]^{\frac{1}{\rho_g}} \cdot r^{\frac{\rho_f}{\rho_g}}$$

$$i.e., \; \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\rho_g}}} \; \geq \left[\frac{(\overline{\sigma}_f - \varepsilon)}{(\overline{\sigma}_g + \varepsilon)}\right]^{\frac{1}{\rho_g}} \; .$$

As in view of Lemma 2.1, $\frac{\rho_f}{\rho_g} \geq \lambda_g(f)$ and as $\varepsilon \, (> 0)$ is arbitrary, we get from (3.9) that

$$\limsup_{r \to \infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}} \; \geq \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \; \overline{\tau}_g(f) \; \geq \left[\frac{\sigma_f}{\sigma_g}\right]^{\frac{1}{\rho_g}} \; .$$

Simialrly, we get from (3.9) that

$$\limsup_{r \to \infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}} \; \geq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \; \overline{\tau}_g(f) \; \geq \left[\frac{\overline{\sigma}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}} \; ,$$

since in view of Lemma 2.1, $\frac{\rho_f}{\rho_g} \leq \lambda_g(f)$ and $\varepsilon \, (> 0)$ is arbitrary.

Likewise from (3.3) and in view of (3.9), we get for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \geq T_g^{-1}\left[(\sigma_f - \varepsilon) \cdot r^{\rho_f}\right]$$

$$i.e.,\ T_g^{-1}T_f\left(r\right) \geq \left(\frac{\left(\sigma_f - \varepsilon\right)\cdot r^{\rho_f}}{\left(\overline{\tau}_g + \varepsilon\right)}\right)^{\frac{1}{\lambda_g}}$$

$$i.e.,\ T_g^{-1}T_f\left(r\right)\ \geq \left[\frac{\left(\sigma_f - \varepsilon\right)}{\left(\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{\lambda_g}}\cdot r^{\frac{\rho_f}{\lambda_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f\left(r\right)}{r^{\frac{\rho_f}{\lambda_g}}}\ \geq \left[\frac{\left(\sigma_f - \varepsilon\right)}{\left(\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{\lambda_g}}\ .$$

Since in view of Lemma 2.1, we get that $\frac{\rho_f}{\lambda_g} \geq \lambda_g\left(f\right)$ and as $\varepsilon\left(> 0\right)$ is arbitrary, therefore it follows from above that

$$\limsup_{r\to\infty}\frac{T_g^{-1}T_f\left(r\right)}{r^{\lambda_g\left(f\right)}}\ \geq \left[\frac{\sigma_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e.,\ \overline{\tau}_g\left(f\right)\ \geq \left[\frac{\sigma_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\ .$$

Further from (3.2) and in view of (3.9), it follows for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f\left(r\right) \geq T_g^{-1}\left[\left(\overline{\sigma}_f - \varepsilon\right)\cdot r^{\rho_f}\right]$$

$$i.e.,\ T_g^{-1}T_f\left(r\right) \geq \left(\frac{\left(\overline{\sigma}_f - \varepsilon\right)\cdot r^{\rho_f}}{\left(\tau_g - \varepsilon\right)}\right)^{\frac{1}{\lambda_g}}$$

$$i.e.,\ T_g^{-1}T_f\left(r\right)\ \geq \left[\frac{\left(\overline{\sigma}_f - \varepsilon\right)}{\left(\tau_g - \varepsilon\right)}\right]^{\frac{1}{\lambda_g}}\cdot r^{\frac{\rho_f}{\lambda_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f\left(r\right)}{r^{\frac{\rho_f}{\lambda_g}}}\ \geq \left[\frac{\left(\overline{\sigma}_f - \varepsilon\right)}{\left(\tau_g - \varepsilon\right)}\right]^{\frac{1}{\lambda_g}}\ .$$

As in view of Lemma 2.1, we get that $\frac{\rho_f}{\lambda_g} \geq \lambda_g\left(f\right)$ and also as $\varepsilon\left(> 0\right)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{T_g^{-1}T_f\left(r\right)}{r^{\lambda_g\left(f\right)}}\ \geq \left[\frac{\overline{\sigma}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e.,\ \overline{\tau}_g\left(f\right)\ \geq \left[\frac{\overline{\sigma}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}\ .$$

Again from (3.5) and (3.5), we have for all sufficiently large values of r that

$$T_g^{-1}T_f\left(r\right) \leq T_g^{-1}\left[\left(\overline{\tau}_f + \varepsilon\right)\cdot r^{\lambda_f}\right]$$

$$i.e.,\ T_g^{-1}T_f\left(r\right) \leq \left(\frac{\left(\overline{\tau}_f + \varepsilon\right)\cdot r^{\lambda_f}}{\left(\overline{\sigma}_g - \varepsilon\right)}\right)^{\frac{1}{\rho_g}}$$

$$i.e.,\ T_g^{-1}T_f\left(r\right)\ \leq\ \left[\frac{(\overline{\tau}_f+\varepsilon)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\lambda_f}{\rho_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f\left(r\right)}{r^{\frac{\lambda_f}{\rho_g}}}\ \leq\ \left[\frac{(\overline{\tau}_f+\varepsilon)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\ .$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\rho_g}\leq\lambda_g\left(f\right)$ and as $\varepsilon\left(>0\right)$ is arbitrary, therefore it follows from above that

$$\limsup_{r\to\infty}\frac{T_g^{-1}T_f\left(r\right)}{r^{\lambda_g(f)}}\ \leq\ \left[\frac{\overline{\tau}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \overline{\tau}_g\left(f\right)\ \leq\ \left[\frac{\overline{\tau}_f}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}\ .$$

Thus the first part of the theorem follows from $(3.9),(3.9),(3.9),(3.9),(3.9),(3.9)$ and $(3.9).$

Further from (3.6) and in view of (3.9), we get for all sufficiently large values of r that

$$T_g^{-1}T_f\left(r\right)\geq T_g^{-1}\left[\left(\tau_f-\varepsilon\right)\cdot r^{\lambda_f}\right]$$

$$i.e.,\ T_g^{-1}T_f\left(r\right)\geq\left(\frac{\left(\tau_f-\varepsilon\right)\cdot r^{\lambda_f}}{\left(\overline{\tau}_g+\varepsilon\right)}\right)^{\frac{1}{\lambda_g}}$$

$$i.e.,\ T_g^{-1}T_f\left(r\right)\ \geq\ \left[\frac{(\tau_f-\varepsilon)}{(\overline{\tau}_g+\varepsilon)}\right]^{\frac{1}{\lambda_g}}\cdot r^{\frac{\lambda_f}{\lambda_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f\left(r\right)}{r^{\frac{\lambda_f}{\lambda_g}}}\ \geq\ \left[\frac{(\tau_f-\varepsilon)}{(\overline{\tau}_g+\varepsilon)}\right]^{\frac{1}{\lambda_g}}\ .$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\lambda_g}\geq\lambda_g\left(f\right)$ and as $\varepsilon\left(>0\right)$ is arbitrary, therefore it follows from above that

$$\liminf_{r\to\infty}\frac{T_g^{-1}T_f\left(r\right)}{r^{\lambda_g(f)}}\ \geq\ \left[\frac{\tau_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e.,\ \tau_g\left(f\right)\ \geq\ \left[\frac{\tau_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\ .$$

Again from (3.2) and in view of (3.5), we get for all sufficiently large values of r that

$$T_g^{-1}T_f\left(r\right)\geq T_g^{-1}\left[\left(\overline{\sigma}_f-\varepsilon\right)\cdot r^{\rho_f}\right]$$

$$i.e.,\ T_g^{-1}T_f\left(r\right)\geq\left(\frac{\left\{\left(\overline{\sigma}_f-\varepsilon\right)\cdot r^{\rho_f}\right\}}{\left(\sigma_g+\varepsilon\right)}\right)^{\frac{1}{\rho_g}}$$

$$i.e.,\ T_g^{-1}T_f\left(r\right)\ \geq\ \left[\frac{(\overline{\sigma}_f-\varepsilon)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\rho_f}{\rho_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f\left(r\right)}{r^{\frac{\rho_f}{\rho_g}}}\ \geq\ \left[\frac{(\overline{\sigma}_f-\varepsilon)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}\ .$$

As in view of Lemma 2.1, $\frac{\rho_f}{\rho_g} \geq \lambda_g(f)$ and $\varepsilon(>0)$ is arbitrary, we get from (3.9) that

$$\liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}} \geq \left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \tau_g(f) \geq \left[\frac{\overline{\sigma}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}.$$

Again from (3.2) and in view of (3.9), we get for all sufficiently large values of r that

$$T_g^{-1}T_f(r) \geq T_g^{-1}\left[(\overline{\sigma}_f - \varepsilon)\cdot r^{\rho_f}\right]$$

$$i.e.,\ T_g^{-1}T_f(r) \geq \left(\frac{(\overline{\sigma}_f - \varepsilon)\cdot r^{\rho_f}}{(\overline{\tau}_g + \varepsilon)}\right)^{\frac{1}{\lambda_g}}$$

$$i.e.,\ T_g^{-1}T_f(r) \geq \left[\frac{(\overline{\sigma}_f - \varepsilon)}{(\overline{\tau}_g + \varepsilon)}\right]^{\frac{1}{\lambda_g}}\cdot r^{\frac{\rho_f}{\lambda_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f(r)}{r^{\frac{\rho_f}{\lambda_g}}} \geq \left[\frac{(\overline{\sigma}_f - \varepsilon)}{(\overline{\tau}_g + \varepsilon)}\right]^{\frac{1}{\lambda_g}}.$$

Since in view of Lemma 2.1, we get that $\frac{\rho_f}{\lambda_g} \geq \lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}} \geq \left[\frac{\overline{\sigma}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e.,\ \tau_g(f) \geq \left[\frac{\overline{\sigma}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}.$$

Moreover, we get from (3.5) and (3.5) for a sequence of values of r tending to infinity that

$$T_g^{-1}T_f(r) \leq T_g^{-1}\left[(\overline{\tau}_f + \varepsilon)\cdot r^{\lambda_f}\right]$$

$$i.e.,\ T_g^{-1}T_f(r) \leq \left(\frac{(\overline{\tau}_f + \varepsilon)\cdot r^{\lambda_f}}{(\sigma_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}$$

$$i.e.,\ T_g^{-1}T_f(r) \leq \left[\frac{(\overline{\tau}_f + \varepsilon)}{(\sigma_g - \varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot r^{\frac{\lambda_f}{\rho_g}}$$

$$i.e.,\ \frac{T_g^{-1}T_f(r)}{r^{\frac{\lambda_f}{\rho_g}}} \leq \left[\frac{(\overline{\tau}_f + \varepsilon)}{(\sigma_g - \varepsilon)}\right]^{\frac{1}{\rho_g}}.$$

As in view of Lemma 2.1, we get that $\frac{\lambda_f}{\rho_g} \leq \lambda_g(f)$ and also as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}} \leq \left[\frac{\overline{\tau}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \tau_g(f) \leq \left[\frac{\overline{\tau}_f}{\sigma_g}\right]^{\frac{1}{\rho_g}}.$$

Similarly, from (3.8) and in view of (3.5), it follows for a sequence of values of r tending to infinity that

$$T_g^{-1} T_f(r) \leq T_g^{-1} \left[(\tau_f + \varepsilon) \cdot r^{\lambda_f} \right]$$

$$i.e., \; T_g^{-1} T_f(r) \leq \left(\frac{(\tau_f + \varepsilon) \cdot r^{\lambda_f}}{(\overline{\sigma}_g - \varepsilon)} \right)^{\frac{1}{\rho_g}}$$

$$i.e., \; T_g^{-1} T_f(r) \; \leq \left[\frac{(\tau_f + \varepsilon)}{(\overline{\sigma}_g - \varepsilon)} \right]^{\frac{1}{\rho_g}} \cdot r^{\frac{\lambda_f}{\rho_g}}$$

$$i.e., \; \frac{T_g^{-1} T_f(r)}{r^{\frac{\lambda_f}{\rho_g}}} \; \leq \left[\frac{(\tau_f + \varepsilon)}{(\overline{\sigma}_g - \varepsilon)} \right]^{\frac{1}{\rho_g}} .$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\rho_g} \leq \lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\liminf_{r \to \infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}} \; \leq \left[\frac{\tau_f}{\overline{\sigma}_g} \right]^{\frac{1}{\rho_g}}$$

$$i.e., \; \tau_g(f) \; \leq \left[\frac{\tau_f}{\overline{\sigma}_g} \right]^{\frac{1}{\rho_g}} .$$

Hence the second part of the theorem follows from $(3.9), (3.9), (3.9), (3.9)$ and (3.9). $\square$

Acknowledgment

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