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ESSENTIALLY SEMI-REGULAR LINEAR RELATIONS

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Abstract. In this paper, we study the essentially semi-regular linear relation operators everywhere defined in Hilbert space. We establish a Kato-type decomposition of essentially semi-regular relations in Hilbert spaces. The result is then applied to study and give some properties of the Samuel-multiplicity.
Keywords: linear relation, semi regular relation, essentially semi regular relation

1. Introduction

The notion of essentially semi-regularity operators amongst the various concepts of regularity originated by the classical treatment of perturbation theory is owed to Kato and has been studied by many authors, for instance, we cite [1], [7], [11], [12] and [18]. We remark that all the above authors considered only the case of bounded linear operators. It is the purpose of this paper to consider the class of essentially semi-regularity in the more general setting of linear relations in Hilbert spaces. Many properties of essentially semi-regularity for the case of linear operators remain to be valid in the context of linear relations, sometimes under supplementary conditions.
The purpose of this paper is to extend the results of the type mentioned above to multi-valued linear relations in Hilbert spaces.
In section 2 we make the paper easily accessible, the presentation is more or less self-contained. Some basic notations and results from the theory of linear relations in linear and Hilbert space are recalled in [4], [9] and [17]. Section 3 contains the main results of the paper. We begin to give a definition of semi-regular and essentially semi-regular linear relations of a closed linear relation A everywhere defined in Hilbert space. In the second part of Section 3, we collect several results concerning the adjoint and the structure of essentially semi-regular linear relations. Finally, we establish some results on Samuel multiplicity of essentially semi-regular linear relations.

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We will denote the set of nonnegative integers by $\mathbb{N}$. Let X denote an infinite dimensional separable Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A multi-valued linear operator in X or simply a linear relation in X, $A : X \to X$ is a mapping from a subspace $D(A) \subset X$, called the domain of A, into the collection of nonempty subsets of X such that $A(\alpha x_1 + \beta x_2) = \alpha A x_1 + \beta A x_2$ for all nonzero α, β scalars and x_1, $x_2 \in D(A)$. For $x \in X \setminus D(A)$ we define $Ax = \emptyset$. With this convention, we have $D(A) = \{x \in X : Ax \neq \emptyset\}$. If A maps the points of its domain to singletons, then A is said to be a single-valued or simply an operator. We denote the class of linear relations in X by $\mathcal{LR}(X,Y)$. $A \in \mathcal{LR}(X,Y)$ is uniquely determined by its graph $G(A)$, which is defined by:

$$G(A) = \{(x, y) \in X \times Y \text{ such that } x \in D(A) \text{ and } y \in Ax\}.$$

Let $A \in \mathcal{LR}(X,Y)$. The inverse of A is a linear relation A^{-1} given by:

$$G(A^{-1}) = \{(y, x) \in Y \times X \text{ such that } (x, y) \in G(A)\}.$$

The subspaces $A(0)$, $A^{-1}(0) = N(A) = \{x \in X : (x, 0) \in G(A)\}$ and $R(A) = A(D(A))$ are called the multi-valued part, the null space and the range of A, respectively. Furthermore, we define the nullity and the defect of A by

$$\alpha(A) = \dim N(A) \text{ and } \beta(A) = \textit{ codim } R(A)$$

respectively. A closed linear relation A from a Banach space X into a Banach space Y is said to be upper semi-Fredholm relation, which we abbreviate as Φ_+ if A has closed range and $\alpha(A) < \infty$, we denoted by $A \in \Phi_+(X,Y)$. A is called lower semi-Fredholm relation, which we abbreviate as Φ_- if $R(A)$ is a closed finite codimensional subspace of Y and we denoted by $A \in \Phi_-(X,Y)$. A is said to be semi-Fredholm (resp. Fredholm) relation if $A \in \Phi_+(X,Y) \cup \Phi_-(X,Y)$ (resp. $\Phi_+(X,Y) \cap \Phi_-(X,Y)$). If A is semi-Fredholm, then the index of A is defined as follows:

$$i(A) = \alpha(A) - \beta(A)$$

where the value of the difference is computed as $i(A) = +\infty$ if $\alpha(A)$ is infinite and $\beta(A) < \infty$ and $i(A) = -\infty$ if $\beta(A)$ is infinite and $\alpha(A) < \infty$.
Let M and N are subspaces of X and of the dual space X^*, respectively, then

$$M^{\perp} := \{x^{'} \in X^* \text{ such that } x^{'}x = 0 \text{ for all } x \in M\}$$

and

$$N^{\top} := \{x \in X \text{ such that } x^{'}x = 0 \text{ for all } x^{'} \in N\}.$$

Let $A \in \mathcal{LR}(X,Y)$. Then the adjoint relation A^* of A is defined by

$$G(A^*) = \{(y^{'}, x^{'}) \in Y^* \times X^* : \ x^{'}x = y^{'}y \text{ for all } (x, y) \in G(A)\}.$$

For A and $B \in \mathcal{LR}(X,Y)$, the linear relations $A + B$ and AB are defined by

$$G(A + B) = \{(x, y + z) \text{ such that } (x, y) \in G(A) \text{ and } (x, z) \in G(B)\},$$

$G(AB) = \{(x, y) \in X \times Y$ such that $(x, y) \in G(A)$ and $(y, z) \in G(B)$ for some $y \in Y\}$.
Since the composition of linear relations is clearly associative, for all $n \in \mathbb{Z}$, A^n is defined as usual with $A^0 = I$ and $A^1 = A$. It is easy to see that $(A^{-1})^n = (A^n)^{-1}$.
Let $A \in \mathcal{LR}(X)$, if $\lambda \in \mathbb{K}$, then $\lambda - A$ stands $\lambda I - A$ where I is the identity operator in X.
Let $A \in \mathcal{LR}(X)$, if M is a subspace of X then $A|_M$ is defined by $G(A|_M) = G(A) \cap (M \times X)$ and A_M is defined by $G(A_M) = G(A) \cap (M \times M)$. Assume that X_1 and X_2 are two subspaces of X such that $X = X_1 \oplus X_2$. We say that A is completely reduced by the pair (X_1, X_2) if $A = A_{X_1} \oplus A_{X_2}$. In such a case, we have $D(A) = D(A_{X_1}) \oplus D(A_{X_2})$, $N(A) = N(A_{X_1}) \oplus N(A_{X_2})$, $R(A) = R(A_{X_1}) \oplus R(A_{X_2})$, $A(0) = A_{X_1}(0) \oplus A_{X_2}(0)$ and $A^n = (A_{X_1})^n \oplus (A_{X_2})^n$ for all $n \in \mathbb{N}$, (see [17],[6]).
In order to characterize these classes of linear relations, one introduces the following notations. Let Q_A denote the quotient map from X onto $X/\overline{A(0)}$. It is easy to see that $Q_A A$ is single-valued and so we can define $\|Ax\| = \|Q_A Ax\|$, $x \in D(A)$ and $\|A\| = \|Q_A A\|$ called the norm of Ax and A respectively, and the minimum modulus of A is the quantity

$$\gamma(A) := \begin{cases} +\infty, & \text{if } D(A) \subset \overline{N(A)}; \\ \inf\left\{\dfrac{\|Ax\|}{d(x, N(A))} : x \in D(A) \backslash \overline{N(A)}\right\}, & \text{otherwise.} \end{cases}$$

Definition 1.1. *We say that A is closed if its graph is a closed subspace of $X \times X$, continuous if for each neighborhood V in $R(A)$, A^{-1} is neighborhood in $D(A)$ (equivalently $\|A\| < \infty$), bounded if it is continuous and its domain is whole X and open if its inverse is continuous* (*i.e.*, $\gamma(A) > 0$).

We denote by $\mathcal{CR}(X)$ (resp. by $\mathcal{BR}(X)$) the set of all closed (resp. bounded) linear relations on X. The class of all bounded and closed linear relations on X is denoted by $\mathcal{BCR}(X)$.
For $A \in \mathcal{LR}(X)$, the kernels and the ranges of the iterates A^n, $n \in \mathbb{N}$, form two increasing and decreasing chains, respectively, i.e., the chain of kernels

$$N(A^0) = 0 \subset N(A) \subset N(A^2) \subset \ldots$$

and the chain of ranges

$$R(A^0) = X \supset R(A) \supset R(A^2) \supset \ldots$$

We define the generalized kernel and the generalized range of A by

$$N^\infty(A) = \bigcup_{n \in \mathbb{N}} N(A^n) \text{ and } R^\infty(A) = \bigcap_{n \in \mathbb{N}} R(A^n).$$

We define the singular chain manifold of $A \in \mathcal{LR}(X)$ by

$$\mathfrak{R}_c(A) = \left(\bigcup_{n=1}^{\infty} N(A^n)\right) \bigcap \left(\bigcup_{n=1}^{\infty} A^n(0)\right).$$

We say that A has trivial singular chain if $\mathfrak{R}_c(A) = 0$. The resolvent set of A is the set

$$\rho(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \text{ is bijective, open with dense range}\}.$$

When A is closed, $\rho(A)$ coincides with the set

$$\rho(A) = \left\{\lambda \in \mathbb{C} \text{ such that } (\lambda - A)^{-1} \text{ is everywhere defined and single valued}\right\}.$$

We denote the set of all closed linear relations on X by $\mathcal{CR}(X)$ and by $\mathcal{BR}(X)$ the set of all bounded linear relations on X.

1.1. Preliminary and auxiliary result

In this section we collect some results of the theory of multi-valued linear operators which are used to prove the main results in the next.

Lemma 1.1. [9, Theorems I.3.1, IV.5.2] *Let $A \in \mathcal{LR}(X)$. Then*
(i) $AA^{-1}(M) = M \cap R(A) + A(0)$ for all $M \subset X$.
(ii) $A^{-1}A(M) = M \cap D(A) + N(A)$ for all $M \subset X$.
(iii) $A(M + N) = A(M) + A(N)$, for all $M \subset X$ and $N \subset D(A)$.
(iv) If M, N are closed and $N \subset M$, then $\dfrac{X/N}{M/N} = X/M$.
(v) If $N \subset M$ is closed, then M/N is closed in X/N if and only if M is closed in X.

Lemma 1.2. [9, Chapter III] *Let $A \in \mathcal{LR}(X)$. Then*
(i) A^* is a bounded linear relation if A is a bounded linear relation.
(ii) If B is continuous with $D(A) \subset D(B)$ then $(A + B)^* = A^* + B^*$.
(iii) $N(A^*) = R(A)^\perp$.
(iv) If A is closed, then $R(A)$ is closed if and only if $R(A^*)$ is closed if and only if A is open if and only if A^* is open.

The proof of the next result can be found in [17].

Lemma 1.3. [17, Lemma 7.2] *Let A be a linear relation in Banach space. Then $N(A - \lambda)^n \subset R(A^m)$ for all $m, n \in \mathbb{N}$ and for each $\lambda \neq 0$.*

Lemma 1.4. [13, Lemma 9] *Let X an Hilbert space and $A \in \mathcal{LR}(X)$.*
If A is bounded then, for all $n \in \mathbb{N}^*$, $A^{n*} = A^{*n}$.

2. Some properties of essentially semi-regular linear relations

The goal of this section is to introduce, study and develop some properties of essential semi-regular linear relations. We begin with the following Lemma.

Lemma 2.1. [14, Lemma 3.7] *Let $A \in \mathcal{LR}(X)$. Then the following statements are equivalent*

(i) $N(A) \subset R(A^n)$ for every $n \in \mathbb{N}$,

(ii) $N(A^m) \subset R(A)$ for every $m \in \mathbb{N}$,

(iii) $N(A^m) \subset R(A^n)$ for every $n, m \in \mathbb{N}$.

Definition 2.1. *We say that a linear relation $A \in \mathcal{LR}(X)$ is semi regular if $R(A)$ is closed and A verifies one of the equivalent conditions of Lemma 2.1.*

Trivial examples of regular linear relations are surjective multi-valued operators as well as injective multi-valued operators with closed range. For an essential version of semi regular linear relation we use the following notations. For subspaces $M, N \subset X$ we write $M \subseteq_e N$ if there exists a finite-dimensional subspace $F \subset X$ such that $M \subseteq N + F$.
In [12], V. Muller proved that $M \subseteq_e N$ if and only if $\dim \left(\frac{M}{M \cap N} \right) < \infty$ if and only if $\dim \left(\frac{M+N}{M} \right) < \infty$ if and only if there is a finite-dimensional subspace $F \subset X$ such that $M \subseteq N + F$.

Definition 2.2. *Let $A \in \mathcal{LR}(X)$ is said essentially semi-regular if $R(A)$ is closed and $N(A) \subseteq_e R^\infty(A)$.*

It is clear that if A is semi-regular, then A is essentially semi-regular.

Lemma 2.2. *Let X an Hilbert space and $A \in \mathcal{CR}(X)$ be a closed range, everywhere defined and $\rho(A) \neq \emptyset$. Suppose that for every $n \in \mathbb{N}$ there is a finite-dimensional subspace $F_n \subset X$ such that $N(A) \subseteq R(A^n) + F_n$ for every $n \in \mathbb{N}$ (i.e., A is essentially semi-regular). Then $R(A^n)$ is closed for all $n \in \mathbb{N}$.*

Proof. Observe that A^n is closed and everywhere defined for all $n \in \mathbb{N}$, since A is closed and everywhere defined and $\rho(A) \neq \emptyset$ by [10, Lemma 3.1]. We show by induction that $R(A^n)$ is closed. For $n = 1$, it is clear by the assumption. We may assume that $F_n \subseteq N(A)$. Since $R(A^n)$ is closed and $\dim F_n < \infty$, then $R(A^n) + F_n$ is closed. However, A is closed, then $A_0 = A|_{R(A^n)+F_n}$ is also closed. Since $F_n \subseteq N(A)$, then using [9, II.6.1], we have $0 < \gamma(A) < \gamma(A_0)$, then A_0 is open. Consequently, by Lemma 1.2 (iv), we infer that $R(A_0)$ is closed.
Moreover, since $F_n \subseteq N(A)$ and $N(A) \subseteq R(A^n) + F_n$, then

$$A(F_n) + R(A^{n+1}) \subseteq A(N(A)) + R(A^{n+1}) \subseteq A(F_n) + R(A^{n+1}).$$

Therefore,

$$A(N(A)) + R(A^{n+1}) = R(A^{n+1}) + A(F_n).$$

Hence, by Lemma 1.1 (iii), we have

$$\begin{array}{rcl} R(A_0) & = & A(F_n + R(A^n)) \\ & = & A(F_n) + A(R(A^n)) \\ & = & A(N(A)) + R(A^{n+1}) \\ & = & AA^{-1}(0) + R(A^{n+1}) \\ & = & A(0) + R(A^{n+1}) \\ & = & R(A^{n+1}), \ (A(0) \subset R(A^{n+1})). \end{array}$$

Then $R(A^{n+1})$ is closed and hence $R(A^n)$ is closed for all $n \in \mathbb{N}$. $\square$

Theorem 2.1. *Let X be a Hilbert space and A is a bounded linear relation. If A is essentially semi-regular then A^* is essentially semi-regular.*

Proof. Since $A \in \mathcal{BR}(X)$ then from (i) of Lemma 1.2, we have $A^* \in \mathcal{BR}(X^*)$. Let A is essentially semi-regular, then $R(A)$ is closed. It follows from Lemma 1.2 (iv) that $R(A^*)$ is closed.
By Lemma 2.2, we have $R(A^n)$ is closed. Let $n \in \mathbb{N}$, then from Lemma 1.2 and Lemma 1.4, we infer that

$$\begin{array}{rcl} R^\infty(A^*) & = & \bigcap\limits_{n\in\mathbb{N}} R(A^{*n}) \\ & = & \bigcap\limits_{n\in\mathbb{N}} R(A^{n*}) \\ & = & \bigcap\limits_{n\in\mathbb{N}} N(A^n)^\perp \\ & = & \Big(\bigcup\limits_{n\in\mathbb{N}} N(A^n)\Big)^\perp \\ & = & \Big(N^\infty(A)\Big)^\perp \supset (R(A)+F)^\perp \\ & = & R(A)^\perp \cap F^\perp \\ & = & N(A^*) \cap F^\perp. \end{array}$$

Since $\operatorname{codim}(F^\perp) < \infty$, then $N(A^*) \subseteq_e R^\infty(A^*)$, this implies that A^* is essentially semi regular. $\square$

Theorem 2.2. *Let $A,\ B \in \mathcal{BCR}(X)$ and commute $(AB = BA)$ such that $N(B) \subset R(A)$ and $N(A) \subset R(B)$.
If AB is essentially semi-regular then A and B are essentially semi-regular.*

Proof. We show that B is essentially semi-regular. Clearly

$$N(B) \subset N(AB) \subseteq N(BA) \subseteq_e R^\infty(BA) \subseteq_e R^\infty(B).$$

To prove that $R(B)$ is closed. Let y_n is a sequence in $R(B)$ such that $y_n \to y$. Since A is closed and everywhere defined then $Q_A A$ is a bounded operator with $A(0)$ that is closed and by [9, Lemma IV.5.2], we infer that $R(Q_A AB) = R(AB)/A(0)$ is closed and $Q_A Ay_n$ converging to $Q_A Ay \in R(Q_A AB)$. Thus $Ay \subset R(AB)$, hence $y \in A^{-1}(R(AB)) = N(A) + R(B) = R(B)$ since $N(A) \subset R(B)$, which implies that $R(B)$ is closed. The same reasoning shows that A is essentially semi-regular. $\square$

If X is a Hilbert space, then the class of semi-regular coincides with the class of all quasi-Fredholm linear relations of degree 0 introduced in [16], in order to study the Kato decomposable linear relations in Hilbert spaces. We shall establish an analogous result when A is essentially a semi-regular relation in a Hilbert space which will be crucial to prove the main theorems of the next.

Theorem 2.3. *Let X be an infinite dimensional Hilbert space and $A \in \mathcal{BCR}(X)$. Then the following statements are equivalent.*

(i) $N^\infty(A) \subseteq_e R^\infty(A)$ *and* $R(A)$ *is closed,*

(ii) $N^\infty(A) \subseteq_e R(A)$ *and* $R(A)$ *is closed,*

(iii) $N(A) \subseteq_e R^\infty(A)$ *and* $R(A)$ *is closed,*

(iv) there exists a decomposition $X = X_1 \oplus X_2$ *with the properties that* $A_{X_1} \subset X_1$, $A_{X_2} \subset X_2$, $\dim X_1 < \infty$, A_{X_2} *a semi regular linear relation and* A_{X_1} *is a bounded operator nilpotent of degree* d.

Proof. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear because $N(A) \subset N^\infty(A)$ and $R^\infty(A) \subset R(A)$.
$(iii) \Rightarrow (iv)$ To prove this, we first show that: there exists $d \in \mathbb{N} \cup \{0\}$ such that

a) $N(A) \cap R(A^d) = N(A) \cap R(A^n), \quad \forall\, n \geq d.$

b) $N(A) \cap R(A^d)$ is closed.

c) $R(A) + N(A^d)$ is closed in X.

Since $N(A) \subseteq_e R^\infty(A)$ then

$$\left(\dim \frac{N(A)}{N(A) \cap R^\infty(A)}\right) = \dim \left(\frac{R(A) + N^\infty(A)}{R(A)}\right) < \infty,$$

so we can deduce the codimension of $N(A) \cap R(A^m)$ in $N(A)$ is bounded independent of m, therefore $(\operatorname{codim}(N(A) \cap R(A^n)))_n$ is an increasing and bounded sequence and therefore it has a limit. Hence, there is some smallest $d \in \mathbb{N} \cup \{0\}$ for which $N(A) \cap R(A^d) = N(A) \cap R(A^{d+m})$ for all nonnegative integer m, and thus (a) is satisfied. From Lemma 2.2, we infer that $N(A) \cap R(A^d)$ is closed and by Lemma 1.1 (v), $N(A^d) + R(A)$ is also closed and thus (b) and (c) are satisfied.
Now, proceeding exactly as in the proof of [15, Theorem 5.2], we can construct two closed subspaces X_1 and X_2 of X verifying the following properties:

a. $X = X_1 \oplus X_2$ with $\dim X_1 < \infty$.

b. $A_X = A_{X_1} \oplus A_{X_2}$.

c. A_{X_2} is a regular linear relation.

d. A_{X_1} is a bounded operator nilpotent of degree d.

$(iv) \Rightarrow (i)$ Since $A = A_{X_1} \oplus A_{X_2}$, then $N(A^n) = N((A_{X_1})^n) \oplus N((A_{X_2})^n)$ and

$R^\infty(A) = R^\infty(A_{X_1}) \oplus R^\infty(A_{X_2})$. Since A_{X_1} is nilpotent of degree d, we obtain that $R^\infty(A) = R^\infty(A_{X_2})$. The semi-regularity of A_{X_2} and the fact that $R(A) = R(A_{X_1}) \oplus R(A_{X_2})$ entails that $R(A)$ is closed and $N(A^n_{X_2}) \subset R^\infty(A_{X_2}) = R^\infty(A)$ for every $n \in \mathbb{N}$. By assumption there is $d \in \mathbb{N}$ such that $A^d_{X_1} = 0$ and thus for every $n \geq d$, we obtain

$$N(A^n) = N \oplus N((A_{X_2})^n) \subset R^\infty(A) \oplus N.$$

Hence $N^\infty(A) \subseteq_e R^\infty(A)$, since N is finite-dimensional. □

Remark 2.1. Let X be n Hilbert space and $A \in \mathcal{BCR}(X)$ is everywhere defined. Clearly, using the previous theorem that A is essentially semi-regular if it satisfies any of the equivalent conditions of Theorem 2.3.

Lemma 2.3. [5, Proposition 2.5] *Let $A \in \mathcal{CR}(X)$ verify $X = X_1 \oplus X_2$ such that A_{X_2} is a semi-regular linear relation and A_{X_1} is a bounded operator nilpotent of degree d. Then*
(i) $N(A_{X_2}) = N(A) \cap R^\infty(A) = N(A) \cap R(A^n)$, for every $n \in \mathbb{N}$
(ii) For every nonnegative integer $n \geq d$, we have $R(A) + N(A^n) = A(X_2) \oplus X_1$ is closed.

Theorem 2.4. *Let X be a Hilbert space such that $A \in \mathcal{CR}(X)$ is essentially semi-regular and $\lambda \in \mathbb{D}(0,\delta)\backslash\{0\}$. Then*

(i) $A \in \Phi_+(X)$ if and only if $\lambda - A \in \Phi_+(X)$

(ii) $A \in \Phi_-(X)$ if and only if $\lambda - A \in \Phi_-(X)$

(iii) $A \in \Phi(X)$ if and only if $\lambda - A \in \Phi(X)$.

Proof. (i) Let $\lambda - A \in \Phi_+(X)$ for all $\lambda \in \mathbb{D}(0,\delta)\backslash\{0\}$. Write for short $A_0 = A_{R^\infty(A)}$ is the restriction of A to $R^\infty(A)$ and $\lambda_0 = \lambda_{R^\infty(A)}$. Since A is essentially semi-regular, then by Lemma 2.2, $R(A^n)$ is closed for every $n \in \mathbb{N}$ and thus $R^\infty(A)$ is closed and A_0 is closed. It follows from [3, Lemma 20] that A_0 is surjective and therefore $A_0 \in \Phi_-(X)$. Applying [3, Proposition 8], we get $\lambda_0 - A_0 \in \Phi_-(X)$ with $\beta(\lambda_0 - A_0) \leq \beta(A_0) = 0$ and $i(\lambda_0 - A_0) = i(A_0)$ for all $\mid \lambda \mid < \delta$. Moreover, by Lemma 1.3, $N(\lambda_0 - A_0) = N(\lambda - A) \cap R^\infty(A) = N(\lambda - A)$. Hence,

$$\begin{aligned} \alpha(\lambda - A) &= i(\lambda_0 - A_0) \\ &= i(A_0) = \alpha(A_0) \\ &= \dim(N(A) \cap R^\infty(A)) \end{aligned}$$

for all $\mid \lambda \mid < \delta$. Since A is essentially semi-regular and $\lambda - A \in \Phi_+(X)$, then $\dim N(\lambda - A) = \dim(N(A) \cap R^\infty(A)) < \infty$. On the other hand, by Theorem 2.3, there exists a decomposition $X = X_1 \oplus X_2$ such that A_{X_2} is a semi-regular linear relation, A_{X_1} is a bounded operator nilpotent and $\dim X_1 < \infty$. Then by Lemma 2.3, we have

$$\begin{aligned} N(A) &= N(A_{X_1}) + N(A_{X_2}) \\ &= N(A_{X_1}) + N(A) \cap R^\infty(A). \end{aligned}$$

By assumption, $N(A_{X_1})$ is finite-dimensional and since $\dim N(A) \cap R^{\infty}(A) < \infty$, then $N(A)$ is finite-dimensional. Furthermore, we have $R(A) = R(A_{X_1}) + R(A_{X_2})$ is closed since A_{X_2} is semi-regular and $R(A_{X_1})$ is finite dimensional. Hence, $A \in \Phi_+(X)$.
The opposite implication from [3, Theorem 25].
(ii) Assume that $\lambda - A \in \Phi_-(X)$, then $R(\lambda - A)$ is closed, and by [9, Proposition III.1.4], $R(\lambda - A)^{\perp} = N(\lambda - A^*)$ for all $\lambda \in \mathbb{D}(0,\delta)\backslash\{0\}$. Thus, by part (i)

$$\begin{aligned} \beta(\lambda - A) = \alpha(\lambda - A^*) &= \dim(N(A^*) \cap R(A^{*d})) \\ &= \operatorname{codim}[N(A^*) \cap R(A^{*d})]^{\perp} \\ &= \operatorname{codim}[N(A^*)^{\perp} + R(A^{*d})^{\perp}] \\ &= \operatorname{codim}[N(A^d) + R(A)] \end{aligned}$$

for all $0 <\mid \lambda \mid< \delta$. Then $\operatorname{codim}[R(A) + N(A^d)]$ is finite. From Lemma 2.3, we infer that $R(A) + N(A^d) = R(A_{X_2}) \oplus X_1$, which implies that $R(A_{X_2}) + X_1$ is finitely codimensional. Since $\dim X_1 < \infty$ then $R(A_{X_2})$ is finitely codimensional and thus $\beta(A) = \operatorname{codim} R(A) = \operatorname{codim} R(A_{X_2}) < \infty$. Applying the same reasoning to the proof of the part (i), we have that $R(A)$ is closed and hence $A \in \Phi_-(X)$. The opposite implication from [3, Theorem 25].
The proof of (iii) is an immediate of (i) and (ii). $\square$

Samuel multiplicity operators have been studied by several authors. Particularly, in [18], the authors studied the Samuel multiplicity of essentially semi-regular operators. In what follows, we extend this study to the general case of multi-valued linear operators.

Definition 2.3. *For any essentially semi-regular linear relation operator A in a Hilbert space X, define its shift (Samuel) multiplicity by*

$$s.mult(A) = \lim_{k\to\infty}\left(\frac{\beta(A^k)}{k}\right)$$

Similarly, define its backward shift (Samuel) multiplicity by

$$b.s.mult(A) = \lim_{k\to\infty}\left(\frac{\alpha(A^k)}{k}\right).$$

Lemma 2.4. *Let X be a Banach space and $A \in \mathcal{CR}(X)$ then*
(i) *If A is semi-regular with a trivial singular chain and $\alpha(A) < \infty$ then $\alpha(A^n) = n\alpha(A)$.*
(ii) *If A is semi-regular with finite codimensional range then $\beta(A^n) = n\beta(A)$.*
(iii) *If A is semi regular then there exists a positive constant $\delta > 0$, such that $\lambda - A$ is semi regular for all $0 <\mid \lambda \mid< \delta$.*

Proof. (i) Since A is a trivial singular chain, then by [17, Lemma 4.4],

$$\frac{N(A^n)}{N(A^{n-1})} \simeq N(A) \cap R(A^{n-1}).$$

Since A is semi-regular, it follows that $\dim N(A^n) = \dim N(A) + \dim N(A^{n-1})$. Thus, a successive repetition of this argument leads to $\alpha(A^n) = n\alpha(A)$.
(ii) The same technique used in [8, Lemma 3.4] gives the result. In fact, let n be a positive integer. [17, Lemma 4.1] implies that

$$\frac{R(A^{n-1})}{R(A^n)} \simeq \frac{X}{N(A^{n-1}) + R(A)}$$

for every $n \in \mathbb{N}$. Since A is semi regular then for every $n \in \mathbb{N}$, $N(A) \subset R(A^n)$, or equivalently, $N(A^n) \subset R(A)$, which implies

$$\frac{R(A^{n-1})}{R(A^n)} \simeq \frac{X}{R(A)}.$$

On the other hand, it follows from Lemma 1.1(iv) that

$$\left(\frac{X}{R(A^{n-1})} \times \frac{R(A^{n-1})}{R(A^n)}\right) \simeq \frac{X}{R(A^n)}$$

and hence

$$\text{codim}R(A^n) = \text{codim}R(A^{n-1}) + \text{codim}R(A).$$

By induction, $\beta(A^n) = n\beta(A)$ for all $n \in \mathbb{N}$.
(iii) See [3, Theorem 23]. $\square$

Lemma 2.5. *Let X be a Hilbert space and $A \in \mathcal{CR}(X)$ is essentially semi-regular.*
(i) $\alpha(\lambda - A) = \alpha(A_{X_2}) \leq \alpha(A)$ for all $0 <| \lambda |< \delta$.
(ii) $\beta(\lambda - A) = \beta(A_{X_2}) \leq \beta(A)$ for all $0 <| \lambda |< \delta$.

Proof. (i) Since A is essentially semi-regular, there exists a decomposition $X = X_1 \oplus X_2$ with the properties that $\dim X_1 < \infty$, A_{X_2} a semi-regular linear relation and A_{X_1} is a bounded operator nilpotent of degree d ($A^d = 0$). Then $\lambda - A = (\lambda - A)_{X_1} \oplus (\lambda - A)_{X_2}$ for all $\lambda \in \mathbb{C}$. Therefore,

$$\alpha(\lambda - A) = \alpha((\lambda - A)_{X_1}) + \alpha((\lambda - A)_{X_2}).$$

Since A_{X_1} is a bounded operator nilpotent of degree d, then $(\lambda - A)_{X_1}$ is invertible for all $\lambda \neq 0$ which implies that $\alpha((\lambda - A)_{X_1}) = 0$. Since A_{X_2} a semi-regular linear relation, then by Lemma 2.4, there exists a positive constant $\delta > 0$ such that $(\lambda - A)_{X_2}$ is semi regular for all $0 <| \lambda |< \delta$. It follows from [3, Theorem 27] that $\alpha((\lambda - A)_{X_2}) = \alpha(A_{X_2})$ for all $0 <| \lambda |< \delta$. Hence, $\alpha(\lambda - A) = \alpha((\lambda - A)_{X_2}) = \alpha(A_{X_2}) \leq \alpha(A)$ for all $0 <| \lambda |< \delta$.
(ii) Let A be essentially semi-regular, then there exists a decomposition $X = X_1 \oplus X_2$ with the properties that $A_{X_1} \subset X_1$, $A_{X_2} \subset X_2$, $\dim X_1 < \infty$, A_{X_2} a semi-regular linear relation and A_{X_1} is a bounded operator nilpotent of degree d. Then $\lambda - A = (\lambda - A)_{X_1} \oplus (\lambda - A)_{X_2}$. Therefore,

$$\beta(\lambda - A) = \beta((\lambda - A)_{X_1}) + \beta((\lambda - A)_{X_2}).$$

Now, reasoning as in the proof of the part (i), we obtain the result. $\square$

Theorem 2.5. *Suppose X is a Hilbert space and $A \in \mathcal{CR}(X)$ is an essentially semi-regular relation operator, then*
(i) *If $\alpha(A) < \infty$ and $\mathfrak{R}_c(A) = 0$, then there exists a positive constant $\delta > 0$ such that*

$$\alpha(\lambda - A) = b.s.mult(A)$$

and if $\beta(A) < \infty$, then there exists a positive constant $\delta > 0$ such that

$$\beta(\lambda - A) = s.mult(A)$$

for all $0 <\mid \lambda \mid< \delta$.
In particular, the following two functions

$$\lambda \longrightarrow s.mult(\lambda - A)$$

and

$$\lambda \longrightarrow b.s.mult(\lambda - A)$$

are constant on a neighborhood of the origin $\mathcal{O}(0, \delta) = \{\lambda : \ \mid \lambda \mid< \delta\}$.
(ii) *When k is large enough, if $\alpha(A) < \infty$ and $\mathfrak{R}_c(A) = 0$*

$$s.mult(A) = \dim\left(\frac{X}{R(A) + N(A^k)}\right) = \dim\left(\frac{X}{R(A) + \overline{N^{\infty}(A)}}\right)$$

and if $\beta(A) < \infty$ then

$$b.s.mult(A) = \dim(N(A) \cap R(A^k)) = \dim(N(A) \cap R^{\infty}(A)).$$

(iii) *A is upper semi-Fredholm (resp. lower semi-Fredholm), if and only if, $b.s.mult(A) < \infty$ (resp. $s.mult(A) < \infty$). Moreover, if $b.s.mult(A) < \infty$ and $s.mult(A) < \infty$, then A is semi-Fredholm and $i(A) = b.s.mult(A) - s.mult(A)$.*

Proof. (i) Since A is essentially semi-regular, applying Theorem 2.3, there exists a decomposition $X = X_1 \oplus X_2$ with the properties that $A_{X_1} \subset X_1$, $A_{X_2} \subset X_2$, $\dim X_1 < \infty$, A_{X_2} a semi regular linear relation and A_{X_1} is a bounded operator nilpotent of degree d. Let $\dim X_1 = n_0$, then by Lemma 2.4,

$$\begin{aligned} b.s.mult(A) &= \lim_{k\to\infty}\left(\frac{\alpha(A^k)}{k}\right) \\ &= \lim_{k\to\infty}\left(\frac{\alpha(A^k_{X_2})}{k}\right) + \lim_{k\to\infty}\left(\frac{\alpha(A^k_{X_1})}{k}\right) \\ &= \lim_{k\to\infty}\left(\frac{\alpha(A^k_{X_2})}{k}\right) + \left(\lim_{k\to\infty}\frac{n_0}{k}\right) \\ &= \lim_{k\to\infty}\left(\frac{k\alpha(A_{X_2})}{k}\right), \\ &= \alpha(A_{X_2}), \end{aligned}$$

and

$$\begin{aligned}
s.mult(A) &= \lim_{k\to\infty}\frac{\beta(A^k)}{k},\\
&= \lim_{k\to\infty}\left(\frac{\beta(A^k_{X_1})}{k}\right)+\lim_{k\to\infty}\left(\frac{\beta(A^k_{X_2})}{k}\right)\\
&= \lim_{k\to\infty}\left(\frac{\beta(A^k_{X_2})}{k}\right)+\lim_{k\to\infty}\left(\frac{n_0}{k}\right)\\
&= \lim_{k\to\infty}\left(\frac{k\beta(A_{X_2})}{k}\right)\\
&= \beta(A_{X_2}).
\end{aligned}$$

By Lemma 2.5, there exists a positive constant $\delta > 0$, such that

$$\alpha(\lambda - A) = \alpha(A_{X_2}) = b.s.mult(A)$$

and

$$\beta(\lambda - A) = \beta(A_{X_2}) = s.mult(A)$$

for all $0 <\mid \lambda \mid< \delta$.

(ii) Since A is essentially semi-regular, applying Theorem 2.3, there exists a decomposition $X = X_1 \oplus X_2$ with the properties that $A_{X_1} \subset X_1$, $A_{X_2} \subset X_2$, $\dim X_1 < \infty$, A_{X_2} a semi-regular linear relation and A_{X_1} is a bounded operator nilpotent of degree d. By Lemma 2.3 and the proof of part (i), we have

$$b.s.mult(A) = \alpha(A_{X_2}) = \dim(N(A)\cap R(A^k)) = \dim(N(A)\cap R^\infty(A)).$$

We begin to prove the following equality: for all $k \geq \dim X_1$

$$\begin{aligned}
R(A)+\overline{N^\infty(A)} &= R(A_{X_1})+R(A_{X_2})+\overline{N^\infty(A_{X_1})+N^\infty(A_{X_2})}\\
&= R(A_{X_1})+R(A_{X_2})+\overline{N^\infty(A_{X_2})+X_1}\\
&= R(A_{X_1})+R(A_{X_2})+\overline{N^\infty(A_{X_2})}+X_1\\
&= R(A_{X_1})+R(A_{X_2})+X_1\\
&= R(A_{X_1})+R(A_{X_2})+N(A^k_{X_1})+N(A^k_{X_2})\\
&= R(A)+N(A^k).
\end{aligned}$$

Thus, $\dfrac{X}{R(A)+N(A^k)} = \dfrac{X}{R(A)+\overline{N^\infty(A)}}$ for all $k \geq \dim X_1$, and consequently

$$\begin{aligned}
s.mult(A) &= \beta(A_{X_2}) = \dim\left(\frac{X_2}{A(X_2)}\right) = \dim\left(\frac{X_2\oplus X_1}{A(X_2)\oplus X_1}\right)\\
&= \dim\left(\frac{X}{R(A)+N(A^k)}\right) = \dim\left(\frac{X}{R(A)+\overline{N^\infty(A)}}\right)
\end{aligned}$$

for all $k \geq \dim X_1$.
(iii) If $A \in \Phi_+(X)$, then $\dim N(A) < \infty$ which implies that $\dim(N(A) \cap R(A)) < \infty$ and by (ii), $b.s.mult(A) < \infty$. If $A \in \Phi_-(X)$, then $\beta(A) < \infty$ which implies that codim $(N^\infty(A) + R(A)) < \infty$ and then by (ii), $s.mult(A) < \infty$. To prove the converse, let A be essentially semi-regular, then $R(A)$ is closed and applying Theorem 2.3, there exists a decomposition $X = X_1 \oplus X_2$ such that A_{X_2} is a semi-regular linear relation, A_{X_1} is a bounded operator nilpotent and $\dim X_1 < \infty$. Suppose that $b.s.mult(A) < \infty$ then $\alpha(A_{X_2}) < \infty$ and consequently, $\alpha(A) = \alpha(A_{X_1}) + \alpha(A_{X_2}) < \infty$. Since $R(A)$ is closed then $A \in \Phi_+(X)$. Let $s.mult(A) < \infty$ then $\beta(A_{X_2}) < \infty$ and consequently, $\beta(A) = \beta(A_{X_1}) + \beta(A_{X_2}) < \infty$ and since $R(A)$ is closed, then $A \in \Phi_-(X)$. On the other hand, if $\alpha(A)$ and $\beta(A)$ are finite and $\mathfrak{R}_c(A) = 0$, then by Lemma 2.4, $i(A^k) = \alpha(A^k) - \beta(A^k) = k.i(A)$ and

$$i(A) = \lim_{k\to\infty} \left(\frac{\alpha(A^k)}{k} \right) - \lim_{k\to\infty} \left(\frac{\beta(A^k)}{k} \right) = b.s.mult(A) - s.mult(A).$$

□

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