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APPROXIMATION THEOREMS FOR LIMIT $(p, q)-$BERNSTEIN-DURRMEYER OPERATOR

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Abstract. In the present paper, using the method developed in [6], we prove the existence of the limit operator of the slight modification of the sequence of (p, q)-Bernstein-Durrmeyer operators introduced recently in [10]. We also establish the rate of convergence of this limit operator.
Keywords: Approximation theorems; $(p, q)-$Bernstein-Durrmeyer operator; Rate of convergence

1. Introduction

The applications of q-calculus in the field of approximation theory have led to the discovery of new generalizations of Bernstein operators. The first generalization involving q-integers was obtained by Lupaş [13] in 1987. Ten years later Phillips [18] gave another generalization of the Bernstein operators introducing the so-called q-Bernstein operators. After that, several well-known positive linear operators and other new operators have been generalized to their q-variants and their approximation behavior have been studied (see e.g. [3] and [11]). The concept of the limit q-Bernstein operator was introduced by Il'inskii and Ostrovska [12], and its rate of convergence was established by Wang and Meng [23], and Finta [7], respectively. Nowadays, the (p, q)-calculus renders to find new generalizations of q-Bernstein operators possible (see [1], [2], [4], [16], [22], [15], [8], [14]). Some basic definitions and theorems of (p, q)-calculus may be found in the papers [9], [21], [19] and [20].

The (p, q)-integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad q \neq p.$$

For $p = 1$, we recover the well-known q-integers $[n]_q = (1 - q^n)/(1 - q)$. Obviously

$$[n]_{p,q} = p^{n-1}[n]_{q/p}. \tag{1.1}$$

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The (p,q)-factorials $[n]_{p,q}!$ are defined by

$$[n]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q}\dots[n]_{p,q}, & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n = 0, \end{cases}$$

and the (p,q)-binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \qquad 0 \leq k \leq n.$$

Further, the (p,q)-power basis is defined by

$$(x \ominus a)^n_{p,q} = \begin{cases} (x-a)(px-qa)(p^2x-q^2a)\dots(p^{n-1}x-q^{n-1}a), & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n = 0, \end{cases}$$

and the (p,q)-integral of f over the interval $[0,a]$ is defined as

$$\int_0^a f(t)\, d_{p,q}t = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{aq^k}{p^{k+1}}\right) \quad \text{for} \quad 0 < q < p \leq 1.$$

In [10] the (p,q)-analogue of the Bernstein-Durrmeyer operators was introduced in the following way:

$$\begin{aligned} & D_n^{p,q}(f;x) \\ (1.2) \qquad = \quad & [n+1]_{p,q} \sum_{k=0}^{n} p^{-(n^2+3n-k^2-k)/2} b_{n,k}^{p,q}(1,x) \int_0^1 \widetilde{b}_{n,k}^{p,q}(p,pqt) f(t)\, d_{p,q}t, \end{aligned}$$

where $f \in C[0,1]$, $x \in [0,1]$, $0 < q < p \leq 1$,

$$b_{n,k}^{p,q}(1,x) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(k(k-1)-n(n-1))/2} x^k (1 \ominus x)^{n-k}_{p,q}$$

and

$$\widetilde{b}_{n,k}^{p,q}(p,pqt) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (pt)^k (p \ominus pqt)^{n-k}_{p,q}.$$

In what follows we propose the following slight modification of (1.2):

$$\begin{aligned} & \tilde{D}_n^{p,q}(f;x) \\ (1.3) \qquad = \quad & [n+1]_{p,q} \sum_{k=0}^{n} p^{-(n^2+3n-k^2-k)/2} b_{n,k}^{p,q}(1,x) \int_0^1 \widetilde{b}_{n,k}^{p,q}(p,pqt) f(pt)\, d_{p,q}t, \end{aligned}$$

where $b_{n,k}^{p,q}(1,x)$ and $\widetilde{b}_{n,k}^{p,q}(p,pqt)$ have got the same expressions as above. For $p = q = 1$, we recover the Durrmeyer operators (see [5]). The goal of the paper is to study the limit (p,q)-Bernstein-Durrmeyer operator $\tilde{D}_{\infty}^{p,q} : C[0,1] \to C[0,1]$ defined by $\tilde{D}^{p,q}_{\infty}(f;x) = \lim_{n\to\infty} \tilde{D}^{p,q}_n(f;x)$, where $f \in C[0,1]$ is arbitrary. Throughout the paper we fix the parameters p,q such that $0 < q < p \leq 1$. We establish the rate of convergence of $\tilde{D}^{p,q}_{\infty}(f;x)$ using the modulus of continuity of $f \in C[0,1]$ given by

$$\omega(f,\delta) = \sup\{|f(x) - f(y)| : x, y \in [0,1], |x-y| \leq \delta\}, \quad \delta > 0. \tag{1.4}$$

The existence of $\tilde{D}^{p,q}_{\infty}$ is proven with the aid of the method developed in [6]. More precisely, we shall apply the following result (see [6, p. 393, Theorem 2.1] and [6, p. 394, Corollary 2.1]):

Theorem 1.1. *Let Λ be a set of parameters and for $\lambda \in \Lambda$ let $(L^{\lambda}_n)_{n\geq 1}$ be a sequence of positive linear operators on $C[0,1]$. If there exist the positive sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ such that*

a) $\alpha_n \to 0$ *as* $n \to \infty$,

b) there exists $C_1 > 0$ *with* $\beta_n + \beta_{n+1} + \ldots + \beta_{n+m-1} \leq C_1\alpha_n$ *for all* $n, m \geq 1$,

c) there exists $C_2 > 0$ *with* $\|L^{\lambda}_n g - L^{\lambda}_{n+1} g\| \leq C_2 \beta_n \|g'\|$ *for all* $n \geq 1$ *and* $g \in C^1[0,1]$,

then there exists $C_3 = C_3(\|L^{\lambda}_1 e_0\|) > 0$ *and a positive linear operator* $L^{\lambda}_{\infty} : C[0,1] \to C[0,1]$ *such that*

$$\|L^{\lambda}_n f - L^{\lambda}_{\infty} f\| \leq C_3 \omega(f, \alpha_n)$$

for all $f \in C[0,1]$ *and* $n = 1, 2, \ldots$

We mention that $\|\cdot\|$ denotes the uniform norm on $C[0,1]$, $e_0(x) = 1$ for $x \in [0,1]$, and the sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ may depend on λ.

2. Main results

First, we establish some auxiliary results.

Lemma 2.1. *With the notation*

$$\lambda^{p,q}_{n,k}(f) = [n+1]_{p,q}\, p^{-(n^2+3n-k^2-k)/2} \int_0^1 \tilde{b}^{p,q}_{n,k}(p, pqt) f(pt)\, d_{p,q}t,$$

where $k = 0, 1, \ldots, n$ *and* $f \in C[0,1]$, *we have for* $x \in [0,1]$ *that*

$$\begin{aligned}
&\tilde{D}^{p,q}_n(f;x) - \tilde{D}^{p,q}_{n+1}(f;x) \\
&= b^{p,q}_{n+1,0}(1,x)\{\lambda^{p,q}_{n,0}(f) - \lambda^{p,q}_{n+1,0}(f)\} + \sum_{k=1}^{n} b^{p,q}_{n+1,k}(1,x) \left\{ \lambda^{p,q}_{n,k}(f) \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} p^k \right. \\
&\quad \left. + \lambda^{p,q}_{n,k-1}(f) \frac{[k]_{p,q}}{[n+1]_{p,q}} q^{n+1-k} - \lambda^{p,q}_{n+1,k}(f) \right\} + b^{p,q}_{n+1,n+1}(1,x)\{\lambda^{p,q}_{n,n}(f) - \lambda^{p,q}_{n+1,n+1}(f)\}.
\end{aligned}$$

Proof. We express the difference $\tilde{D}_n^{p,q}(f;x) - \tilde{D}_{n+1}^{p,q}(f;x)$ as follows (see also [17, pp. 411-412]):

$$\prod_{l=0}^{n}(p^l-q^lx)^{-1}\{\tilde{D}_n^{p,q}(f;x)-\tilde{D}_{n+1}^{p,q}(f;x)\}$$

$$= \sum_{k=0}^{n}\lambda_{n,k}^{p,q}(f)\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(k(k-1)-n(n-1))/2}x^k \prod_{l=n-k}^{n}(p^l-q^lx)^{-1}$$

$$-\sum_{k=0}^{n+1}\lambda_{n+1,k}^{p,q}(f)\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{(k(k-1)-(n+1)n)/2}x^k \prod_{l=n+1-k}^{n}(p^l-q^lx)^{-1}. \tag{2.1}$$

Because

$$x^k\prod_{l=n-k}^{n}(p^l-q^lx)^{-1} = x^kp^{-n+k}(p^{n-k}-q^{n-k}x+q^{n-k}x)\prod_{l=n-k}^{n}(p^l-q^lx)^{-1}$$

$$= p^{-n+k}x^k\prod_{l=n+1-k}^{n}(p^l-q^lx)^{-1} + \left(\frac{q}{p}\right)^{n-k}x^{k+1}\prod_{l=n-k}^{n}(p^l-q^lx)^{-1},$$

we get from (2.1), that

$$\prod_{l=0}^{n}(p^l-q^lx)^{-1}\{\tilde{D}_n^{p,q}(f;x)-\tilde{D}_{n+1}^{p,q}(f;x)\}$$

$$= \sum_{k=0}^{n}\lambda_{n,k}^{p,q}(f)\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(k(k-1)-n(n-1))/2}p^{-n+k}x^k \prod_{l=n+1-k}^{n}(p^l-q^lx)^{-1}$$

$$+\sum_{k=0}^{n}\lambda_{n,k}^{p,q}(f)\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(k(k-1)-n(n-1))/2}\left(\frac{q}{p}\right)^{n-k}x^{k+1} \prod_{l=n-k}^{n}(p^l-q^lx)^{-1}$$

$$-\sum_{k=0}^{n+1}\lambda_{n+1,k}^{p,q}(f)\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{(k(k-1)-(n+1)n)/2}x^k \prod_{l=n+1-k}^{n}(p^l-q^lx)^{-1}$$

$$= \{\lambda_{n,0}^{p,q}(f)p^{-n(n-1)/2}p^{-n} - \lambda_{n+1,0}^{p,q}(f)p^{-(n+1)n/2}\}$$

$$+\sum_{k=1}^{n}\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{(k(k-1)-(n+1)n)/2}x^k \prod_{l=n+1-k}^{n}(p^l-q^lx)^{-1}$$

$$\times\left\{\lambda_{n,k}^{p,q}(f)\frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}}{\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q}}p^k + \lambda_{n,k-1}^{p,q}(f)\frac{\begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}}{\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q}}p^{n+1-k}\left(\frac{q}{p}\right)^{n+1-k} - \lambda_{n+1,k}^{p,q}(f)\right\}$$

$$+\{\lambda_{n,n}^{p,q}(f)-\lambda_{n+1,n+1}^{p,q}(f)\}x^{n+1}\prod_{l=0}^{n}(p^l-q^lx)^{-1}$$

$$
\begin{aligned}
= \quad & \{\lambda_{n,0}^{p,q}(f) - \lambda_{n+1,0}^{p,q}(f)\}p^{-(n+1)n/2} \\
& + \sum_{k=1}^{n} \left[\begin{array}{c} n+1 \\ k \end{array} \right]_{p,q} p^{(k(k-1)-(n+1)n)/2} x^k \prod_{l=n+1-k}^{n} (p^l - q^l x)^{-1} \\
& \times \left\{ \lambda_{n,k}^{p,q}(f) \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} p^k + \lambda_{n,k-1}^{p,q}(f) \frac{[k]_{p,q}}{[n+1]_{p,q}} q^{n+1-k} - \lambda_{n+1,k}^{p,q}(f) \right\} \\
& + \{\lambda_{n,n}^{p,q}(f) - \lambda_{n+1,n+1}^{p,q}(f)\} x^{n+1} \prod_{l=0}^{n} (p^l - q^l x)^{-1}.
\end{aligned}
$$

Multiplying with $\prod_{l=0}^{n} (p^l - q^l x)$, we get the assertion of the lemma. $\square$

Lemma 2.2. *For*

$$
\widetilde{b}_{n,k}^{p,q}(p, pqt) = \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} (pt)^k (p \ominus pqt)_{p,q}^{n-k}, \quad k = 0, 1, \ldots, n,
$$

we have

$$
\begin{aligned}
[n+1]_{p,q}\, p^{-(n^2+3n-k^2-k)/2} \int_0^1 \widetilde{b}_{n,k}^{p,q}(p, pqt)\, d_{p,q}t &= 1, \\
[n+1]_{p,q}\, p^{-(n^2+3n-k^2-k)/2} \int_0^1 \widetilde{b}_{n,k}^{p,q}(p, pqt) t\, d_{p,q}t &= p^{n-k} \frac{[k+1]_{p,q}}{[n+2]_{p,q}}, \\
[n+1]_{p,q}\, p^{-(n^2+3n-k^2-k)/2} \int_0^1 \widetilde{b}_{n,k}^{p,q}(p, pqt) t^2\, d_{p,q}t &= p^{2(n-k)} \frac{[k+1]_{p,q}[k+2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}}.
\end{aligned}
$$

Proof. The equalities follow from the computations used in [10, Lemma 3.1]. $\square$

Lemma 2.3. *For $x \in [0,1]$, we have*

$$
\begin{aligned}
\tilde{D}_n^{p,q}(1;x) &= 1, \quad \tilde{D}_n^{p,q}(t;x) = \frac{p^{n+1} + pq[n]_{p,q}x}{[n+2]_{p,q}}, \\
\tilde{D}_n^{p,q}(t^2;x) &= \frac{p^{2n+2}[2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{(2q^2+qp)p^{n+2}[n]_{p,q}x}{[n+2]_{p,q}[n+3]_{p,q}} \\
& \quad + \frac{q^3[n]_{p,q}[p^2[n]_{p,q}x^2 + p^{n+1}x(1-x)]}{[n+2]_{p,q}[n+3]_{p,q}}.
\end{aligned}
$$

Proof. Analogously to the proof of [10, Lemma 3.1], we get the statements of the lemma. $\square$

Remark 2.1. If $p = p(n)$ and $q = q(n)$ such that $0 < q(n) < p(n) \leq 1$ and $q(n) \to 1$ as $n \to \infty$, then, by Korovkin's theorem, $\tilde{D}_n^{p,q}(f;x)$ converges uniformly to $f(x)$ for $x \in [0,1]$, as $n \to \infty$. Indeed, for each n the estimates

$$
\begin{aligned}
|\tilde{D}_n^{p,q}(t;x) - x| &\leq \frac{1}{[n+2]_{q/p}} + \left| \frac{q}{p} \frac{[n]_{q/p}}{[n+2]_{q/p}} - 1 \right|, \\
|\tilde{D}_n^{p,q}(t^2;x) - x^2| &\leq \frac{[2]_{q/p}}{[n+2]_{q/p}[n+3]_{q/p}} + \left(2\left(\frac{q}{p}\right)^3 + \frac{q}{p} \right) \frac{[n]_{q/p}}{[n+2]_{q/p}} \frac{1}{[n+3]_{q/p}} \\
&\quad + \left(\frac{q}{p}\right)^3 \frac{[n]_{q/p}}{[n+2]_{q/p}} \frac{1}{4[n+3]_{q/p}} + \left| \left(\frac{q}{p}\right)^3 \frac{[n]_{q/p}}{[n+2]_{q/p}} \frac{[n]_{q/p}}{[n+3]_{q/p}} - 1 \right|,
\end{aligned}
$$

and the facts that $[n]_{q_n/p_n} \to \infty$ and $\frac{[n]_{q_n/p_n}}{[n+2]_{q_n/p_n}} \to 1$ as $n \to \infty$, imply our statement.

In the next theorem we prove the existence of the limit (p,q)-Bernstein-Durrmeyer operator.

Theorem 2.1. *Let $\tilde{D}_n^{p,q}(f;x)$ be defined by (1.3), where p and q are fixed with $0 < q < p \leq 1$. Then there exist an absolute constant $C > 0$ and a positive linear operator $\tilde{D}_\infty^{p,q} : C[0,1] \to C[0,1]$ such that*

$$\|\tilde{D}_n^{p,q}(f;x) - \tilde{D}_\infty^{p,q}(f;x)\| \leq C\omega \left(f, \left(\frac{q}{p}\right)^{n/2} \right)$$

for all $f \in C[0,1]$ and $n = 1, 2, \ldots$

Proof. We have $[n+1]_{p,q} = p^k[n+1-k]_{p,q} + q^{n+1-k}[k]_{p,q}$ for $k = 0, 1, \ldots, n+1$. Using the notation of Lemma 2.1, we obtain for $f \in C[0,1]$, that

$$
\begin{aligned}
&\lambda_{n,k}^{p,q}(f) \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} p^k + \lambda_{n,k-1}^{p,q}(f) \frac{[k]_{p,q}}{[n+1]_{p,q}} q^{n+1-k} - \lambda_{n+1,k}^{p,q}(f) \\
&= \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} p^k \{\lambda_{n,k}^{p,q}(f) - \lambda_{n+1,k}^{p,q}(f)\} \\
&\quad + \frac{[k]_{p,q}}{[n+1]_{p,q}} q^{n+1-k} \{\lambda_{n,k-1}^{p,q}(f) - \lambda_{n+1,k}^{p,q}(f)\}. \qquad (2.2)
\end{aligned}
$$

Let $g \in C^1[0,1]$ and $x_k = p^{n+2-k} \frac{[k+1]_{p,q}}{[n+3]_{p,q}}$ for $k = 0, 1, \ldots, n$. Obviously, by (1.1), we have

$$x_k = p^{n+2-k} \frac{p^k[k+1]_{q/p}}{p^{n+2}[n+3]_{q/p}} = \frac{[k+1]_{q/p}}{[n+3]_{q/p}} \in [0,1],$$

where $k = 0, 1, \ldots, n$. Further $g(pt) = g(x_k) + \int_{x_k}^{pt} g'(u)\, du$, where $t \in [0,1]$ is arbitrary. Hence, by definition of $\lambda_{n,k}^{p,q}(g)$ and Lemma 2.2, we obtain

$$\lambda_{n,k}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)$$

$$= [n+1]_{p,q} p^{-(n^2+3n-k^2-k)/2} \int_0^1 \widetilde{b}^{p,q}_{n,k}(p,pqt) \left[g(x_k) + \int_{x_k}^{pt} g'(u)\,du\right] d_{p,q}t$$
$$-[n+2]_{p,q} p^{-((n+1)^2+3(n+1)-k^2-k)/2}$$
$$\int_0^1 \widetilde{b}^{p,q}_{n+1,k}(p,pqt) \left[g(x_k) + \int_{x_k}^{pt} g'(u)\,du\right] d_{p,q}t$$
$$= [n+2]_{p,q} p^{-((n+1)^2+3(n+1)-k^2-k)/2} \int_0^1 \widetilde{b}^{p,q}_{n+1,k}(p,pqt) \left(\int_{x_k}^{pt} g'(u)\,du\right)$$
$$\left\{ \frac{[n+1]_{p,q}}{[n+2]_{p,q}} p^{n+2} \frac{\widetilde{b}^{p,q}_{n,k}(p,pqt)}{\widetilde{b}^{p,q}_{n+1,k}(p,pqt)} - 1 \right\} d_{p,q}t \tag{2.3}$$

for $k = 0,1,\ldots,n$. On the other hand, using $[n+2]_{p,q} = p^{k+1}[n+1-k]_{p,q} + q^{n+1-k}[k+1]_{p,q}$, we have

$$\frac{[n+1]_{p,q}}{[n+2]_{p,q}} p^{n+2} \frac{b^{p,q}_{n,k}(p,pqt)}{b^{p,q}_{n+1,k}(p,pqt)} - 1$$
$$= \frac{[n+1]_{p,q}}{[n+2]_{p,q}} p^{n+2} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (pt)^k (p \ominus pqt)^{n-k}_{p,q}}{\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} (pt)^k (p \ominus pqt)^{n+1-k}_{p,q}} - 1$$
$$= \frac{[n+1]_{p,q}}{[n+2]_{p,q}} p^{n+2} \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} \frac{1}{p^{n+1-k} - pq^{n+1-k}t} - 1$$
$$= \frac{[n+1-k]_{p,q}}{[n+2]_{p,q}} \frac{p^{n+2}}{p^{n+1-k} - pq^{n+1-k}t} - 1$$
$$= q^{n+1-k} \left(\frac{[n+1-k]_{p,q}}{[n+2]_{p,q}} \frac{p^{k+2}t}{p^{n+1-k} - pq^{n+1-k}t} - \frac{[k+1]_{p,q}}{[n+2]_{p,q}} \right). \tag{2.4}$$

The equality $[n+2]_{p,q} = p^{k+1}[n+1-k]_{p,q} + q^{n+1-k}[k+1]_{p,q}$ implies, that

$$\frac{[n+1-k]_{p,q}}{[n+2]_{p,q}} \le p^{-(k+1)}. \tag{2.5}$$

Analogously, the equality $[n+2]_{p,q} = q^{k+1}[n+1-k]_{p,q} + p^{n+1-k}[k+1]_{p,q}$ implies, that

$$\frac{[k+1]_{p,q}}{[n+2]_{p,q}} \le p^{-(n+1-k)}. \tag{2.6}$$

Finally, the function $t \to p^{k+2}t/(p^{n+1-k} - pq^{n+1-k}t)$ is increasing on $[0,1]$, therefore

$$\frac{p^{k+2}t}{p^{n+1-k} - pq^{n+1-k}t} \le \frac{p^{k+2}}{p^{n+1-k} - pq^{n+1-k}} = \frac{p^{k+2}}{p^{n+1-k}\left(1 - p(\frac{q}{p})^{n+1-k}\right)}$$
$$\le \frac{p^{k+2}}{p^{n+1-k}\left(1 - p\frac{q}{p}\right)} = \frac{p^{k+2}}{p^{n+1-k}(1-q)}. \tag{2.7}$$

Combining (2.3)-(2.7), applying Lemma 2.2 and Hölder's inequality, we find

$$\begin{aligned}
& |\lambda_{n,k}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)| \\
\leq\ & \|g'\| [n+2]_{p,q} p^{-[(n+1)^2+3(n+1)-k^2-k]/2} \\
& \times \int_0^1 \widetilde{b}_{n+1,k}^{p,q}(p,pqt)|pt-x_k| q^{n+1-k} \left(p^{-(k+1)} \frac{p^{k+2}}{p^{n+1-k}(1-q)} + p^{-(n+1-k)} \right) d_{p,q}t \\
\leq\ & \frac{1+p-q}{1-q} \|g'\| \left(\frac{q}{p}\right)^{n+1-k} \Big\{ [n+2]_{p,q} p^{-((n+1)^2+3(n+1)-k^2-k)/2} \\
(2.8)\ & \int_0^1 \widetilde{b}_{n+1,k}^{p,q}(p,pqt)(pt-x_k)^2\, d_{p,q}t \Big\}^{1/2}.
\end{aligned}$$

Using Lemma 2.2, we get

$$\begin{aligned}
& [n+2]_{p,q} p^{-((n+1)^2+3(n+1)-k^2-k)/2} \int_0^1 \widetilde{b}_{n+1,k}^{p,q}(p,pqt)(pt-x_k)^2\, d_{p,q}t \\
=\ & p^{2(n+2-k)} \frac{[k+1]_{p,q}[k+2]_{p,q}}{[n+3]_{p,q}[n+4]_{p,q}} - 2p^{n+2-k} \frac{[k+1]_{p,q}}{[n+3]_{p,q}} p^{n+2-k} \frac{[k+1]_{p,q}}{[n+3]_{p,q}} \\
& + \left(p^{n+2-k} \frac{[k+1]_{p,q}}{[n+3]_{p,q}} \right)^2 \\
=\ & p^{2(n+2-k)} \frac{[k+1]_{p,q}}{[n+3]_{p,q}} \left(\frac{[k+2]_{p,q}}{[n+4]_{p,q}} - \frac{[k+1]_{p,q}}{[n+3]_{p,q}} \right) \\
(2.9) \quad =\ & p^{2(n+2-k)} \frac{[k+1]_{p,q}}{[n+3]_{p,q}} (pq)^{k+1} \frac{[n+2-k]_{p,q}}{[n+3]_{p,q}[n+4]_{p,q}}.
\end{aligned}$$

Analogously to (2.6) and (2.5), we find that

$$(2.10) \qquad \frac{[k+1]_{p,q}}{[n+3]_{p,q}} \leq p^{-(n+2-k)} \quad \text{and} \quad \frac{[n+2-k]_{p,q}}{[n+3]_{p,q}} \leq p^{-(k+1)}.$$

Further, using (1.1), we get $[n+4]_{p,q} = p^{n+3}[n+4]_{q/p} \geq p^{n+3}$. Hence, by (2.8)-(2.10), we have for $k = 0, 1, \ldots, n$, that

$$\begin{aligned}
|\lambda_{n,k}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)| \leq\ & \frac{1+p-q}{1-q} \|g'\| \\
& \left(\frac{q}{p}\right)^{n+1-k} \left\{ p^{2(n+2-k)} p^{-(n+2-k)} (pq)^{k+1} p^{-(k+1)} p^{-(n+3)} \right\}^{1/2} \\
=\ & \frac{1+p-q}{1-q} \|g'\| \left(\frac{q}{p}\right)^{n+1-k} \left(\frac{q}{p}\right)^{(k+1)/2} \\
=\ & \frac{1+p-q}{1-q} \|g'\| \left(\frac{q}{p}\right)^{(2n-k+3)/2} \\
(2.11) \qquad \leq\ & \frac{1+p-q}{1-q} \|g'\| \left(\frac{q}{p}\right)^{n/2} \left(\frac{q}{p}\right)^{3/2}.
\end{aligned}$$

Analogously to (2.3), we obtain for $g \in C^1[0,1]$ and $y_k = p^{n+2-k}\frac{[k]_{p,q}}{[n+2]_{p,q}} \in [0,1]$, where $k = 1, 2, \ldots, n+1$, that

$$\begin{aligned}
&\lambda_{n,k-1}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g) \\
&= [n+1]_{p,q} p^{-(n^2+3n-(k-1)^2-(k-1))/2} \int_0^1 \widetilde{b}_{n,k-1}^{p,q}(p,pqt) \left(\int_{y_k}^{pt} g'(u)\,du \right) \\
&\times \left\{ 1 - \frac{[n+2]_{p,q}}{[n+1]_{p,q}} p^{-(n+2-k)} \frac{\widetilde{b}_{n+1,k}^{p,q}(p,pqt)}{\widetilde{b}_{n,k-1}^{p,q}(p,pqt)} \right\} d_{p,q}t. \qquad (2.12)
\end{aligned}$$

But

$$\widetilde{b}_{n+1,k}^{p,q}(p,pqt) = \frac{[n+1]_{p,q}}{[k]_{p,q}} pt \widetilde{b}_{n,k-1}^{p,q}(p,pqt)$$

for $k = 1, 2, \ldots, n+1$, hence, in view of (2.12), Lemma 2.2 and Hölder's inequality, we find that

$$\begin{aligned}
&|\lambda_{n,k-1}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)| \\
&\leq \|g'\| [n+1]_{p,q} p^{-(n^2+3n-(k-1)^2-(k-1))/2} \int_0^1 \widetilde{b}_{n,k-1}^{p,q}(p,pqt) |pt - y_k| \\
&\quad \times \left(1 + \frac{[n+2]_{p,q}}{[n+1]_{p,q}} p^{-(n+2-k)} \frac{[n+1]_{p,q}}{[k]_{p,q}} pt \right) d_{p,q}t \\
&\leq \|g'\| \left(1 + p^{-(n+1-k)} \frac{[n+2]_{p,q}}{[k]_{p,q}} \right) \\
&\quad \times \left\{ [n+1]_{p,q} p^{-(n^2+3n-(k-1)^2-(k-1))/2} \int_0^1 \widetilde{b}_{n,k-1}^{p,q}(p,pqt)(pt - y_k)^2\, d_{p,q}t \right\}^{1/2} \\
&= \|g'\| \left(1 + p^{-(n+1-k)} \frac{[n+2]_{p,q}}{[k]_{p,q}} \right) \\
&\quad \times \left\{ [n+1]_{p,q} p^{-(n^2+3n-(k-1)^2-(k-1))/2} \int_0^1 \widetilde{b}_{n,k-1}^{p,q}(p,pqt)(pt - y_k)^2\, d_{p,q}t \right\}^{1/2} \\
&= \|g'\| \left(1 + p^{-(n+1-k)} \frac{[n+2]_{p,q}}{[k]_{p,q}} \right) \left\{ p^{2(n+2-k)} \frac{[k]_{p,q}[k+1]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \right. \\
&\quad \left. -2p^{n+2-k} \frac{[k]_{p,q}}{[n+2]_{p,q}} p^{n+2-k} \frac{[k]_{p,q}}{[n+2]_{p,q}} + \left(p^{n+2-k} \frac{[k]_{p,q}}{[n+2]_{p,q}} \right)^2 \right\} \\
&= \|g'\| \left(1 + p^{-(n+1-k)} \frac{[n+2]_{p,q}}{[k]_{p,q}} \right) p^{n+2-k} \left(\frac{[k]_{p,q}}{[n+2]_{p,q}} \left(\frac{[k+1]_{p,q}}{[n+3]_{p,q}} - \frac{[k]_{p,q}}{[n+2]_{p,q}} \right) \right)^{1/2}
\end{aligned}$$

Thus, we have

$$|\lambda_{n,k-1}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)| = \|g'\| \left(p^{n+2-k} \frac{[k]_{p,q}}{[n+2]_{p,q}} + p \right)$$

$$\left(\frac{[n+2]_{p,q}}{[k]_{p,q}}(pq)^k\frac{[n+2-k]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}}\right)^{1/2}. \tag{2.13}$$

On the other hand, by (1.1), we have $[n+2]_{p,q} = p^{n+1}[n+2]_{q/p} \geq p^{n+1}[k]_{q/p} = p^{n+2-k}[k]_{p,q}$ for $k = 1, 2, \ldots, n+1$. Further $[n+3]_{p,q} = p^{k+1}[n+2-k]_{p,q}+q^{n+2-k}[k+1]_{p,q}$, $k = 1, 2, \ldots, n+1$, thus

$$\frac{[n+2-k]_{p,q}}{[n+3]_{p,q}} \leq p^{-(k+1)};$$

moreover $[k]_{p,q} = p^{k-1}[k]_{q/p} \geq p^{k-1}[1]_{q/p} = p^{k-1}$ for $k = 1, 2, \ldots, n+1$. Hence, by (2.13), we have

$$\begin{aligned} |\lambda^{p,q}_{n,k-1}(g) - \lambda^{p,q}_{n+1,k}(g)| &\leq (1+p)\|g'\|\left((pq)^k p^{-(k-1)}p^{-(k+1)}\right)^{1/2} \\ &= (1+p)\|g'\|\left(\frac{q}{p}\right)^{k/2}. \end{aligned} \tag{2.14}$$

Because

$$\frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} = p^{-k}\frac{[n+1-k]_{q/p}}{[n+1]_{q/p}} \leq p^{-k}$$

for $k = 0, 1, \ldots, n$, and

$$\frac{[k]_{p,q}}{[n+1]_{p,q}} = p^{-(n+1-k)}\frac{[k]_{q/p}}{[n+1]_{q/p}} \leq p^{-(n+1-k)}$$

for $k = 1, 2, \ldots, n+1$, we obtain, in view of (2.2), (2.11) and (2.14), that

$$\begin{aligned} &\left| \lambda^{p,q}_{n,k}(g)\frac{[n+1-k]_{p,q}}{[n+1]_{p,q}}p^k + \lambda^{p,q}_{n,k-1}(g)\frac{[k]_{p,q}}{[n+1]_{p,q}}q^{n+1-k} - \lambda^{p,q}_{n+1,k}(g) \right| \\ \leq\ & \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}}p^k|\lambda^{p,q}_{n,k}(g) - \lambda^{p,q}_{n+1,k}(g)| + \frac{[k]_{p,q}}{[n+1]_{p,q}}q^{n+1-k}|\lambda^{p,q}_{n,k-1}(g) - \lambda^{p,q}_{n+1,k}(g)| \\ \leq\ & \frac{1+p-q}{1-q}\|g'\|\left(\frac{q}{p}\right)^{n/2}\left(\frac{q}{p}\right)^{3/2} + (1+p)\|g'\|\left(\frac{q}{p}\right)^{n+1-k}\left(\frac{q}{p}\right)^{k/2} \\ \leq\ & \frac{1+p-q}{1-q}\left(\frac{q}{p}\right)^{n/2}\left(\frac{q}{p}\right)^{3/2} + (1+p)\|g'\|\left(\frac{q}{p}\right)^{(2n-k+2)/2} \\ \leq\ & \|g'\|\left(\frac{q}{p}\right)^{n/2}\left\{\frac{1+p-q}{1-q}\left(\frac{q}{p}\right)^{3/2} + \sqrt{\frac{q}{p}}(1+p)\right\}. \end{aligned}$$

This means that we may choose $\beta_n = (q/p)^{n/2}$, $n \geq 1$ (see Theorem 1.1). Then for all $n, m \geq 1$, we have

$$\beta_n + \beta_{n+1} + \ldots + \beta_{n+m-1} = \left(\frac{q}{p}\right)^{n/2} + \left(\frac{q}{p}\right)^{(n+1)/2} + \ldots + \left(\frac{q}{p}\right)^{(n+m-1)/2}$$

$$= \left(\frac{q}{p}\right)^{n/2} \frac{1-\left(\frac{q}{p}\right)^{m/2}}{1-\left(\frac{q}{p}\right)^{1/2}} < \frac{\sqrt{p}}{\sqrt{p}-\sqrt{q}} \left(\frac{q}{p}\right)^{n/2}.$$

Thus we may choose $\alpha_n = (q/p)^{n/2}$, $n \geq 1$. Applying Theorem 1.1, we get the statement of our theorem.

In the next theorem we shall estimate the error $|\tilde{D}_{\infty}^{p,q}(f;x) - f(x)|$ with the aid of the modulus of continuity (1.4). □

Theorem 2.2. *For the limit (p, q)-Bernstein-Durrmeyer operator $\tilde{D}_{\infty}^{p,q}$, we have*

$$|\tilde{D}_{\infty}^{p,q}(f;x) - f(x)| \leq 2\omega\left(f, \sqrt{\delta_{p,q}(x)}\right)$$

for all $f \in C[0,1]$ and $x \in [0,1]$, where

$$\delta_{p,q}(x) = \frac{1}{p^4}(p-q)\{2p^2 + (3p+1)x + (p^3-1)x^2\}.$$

Proof. In view of Lemma 2.3, we have

$$\tilde{D}_{\infty}^{p,q}(1,x) = 1. \tag{2.15}$$

Further, by Lemma 2.3 and (1.1),

$$\tilde{D}_{n}^{p,q}(t;x) = \frac{p^{n+1} + p^n q[n]_{q/p}x}{p^{n+1}[n+2]_{q/p}} = \frac{p + q[n]_{q/p}x}{p[n+2]_{q/p}} \to \frac{p + \frac{q}{1-\frac{q}{p}}x}{p\frac{1}{1-\frac{q}{p}}} = \frac{p-q+qx}{p} =: \tilde{D}_{\infty}^{p,q}(t;x)$$

as $n \to \infty$, and analogously, we have

$$\begin{aligned}
\tilde{D}_{n}^{p,q}(t^2;x) &= \frac{p^{2n+2}[2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{(2q^2+qp)p^{n+2}[n]_{p,q}x}{[n+2]_{p,q}[n+3]_{p,q}} \\
&\quad + \frac{q^3[n]_{p,q}[p^2[n]_{p,q}x^2 + p^{n+1}x(1-x)]}{[n+2]_{p,q}[n+3]_{p,q}} \\
&\to \frac{(p+q)(p-q)^2}{p^3} + \frac{q(p+2q)(p-q)x}{p^3} + \frac{q^3}{p^3}\left\{x^2 + \frac{p-q}{p}x(1-x)\right\} \\
&=: \tilde{D}_{\infty}^{p,q}(t^2;x)
\end{aligned}$$

as $n \to \infty$. Then

$$\begin{aligned}
&\tilde{D}_{\infty}^{p,q}((t-x)^2;x) \\
&= \frac{1}{p^3}(p+q)(p-q)^2 + \frac{q^3}{p^4}(p-q)x(1-x) + \frac{1}{p^3}(pq+2q^2-2p^2)(p-q)x \\
&\quad + \frac{1}{p^3}(p^2-pq-q^2)(p-q)x^2 \\
&\leq \frac{2}{p^2}(p-q) + \frac{1}{p^4}(p-q)x(1-x) + \frac{3}{p^3}(p-q)x + \frac{1}{p}(p-q)x^2 \\
&= \delta_{p,q}(x).
\end{aligned} \tag{2.16}$$

For the modulus of continuity (1.4) we have $\omega(f,\lambda\delta) \leq (1+\lambda)\omega(f,\delta)$, $\lambda \geq 0$. Then

$$|f(t)-f(x)| \leq \omega(f,|t-x|) \leq (1+\delta^{-1}|t-x|)\omega(f,\delta)$$

for $t,x \in [0,1]$. Hence, by (2.15), Hölder's inequality and (2.16), we obtain

$$\begin{aligned} &|\tilde{D}_\infty^{p;q}(f;x)-f(x)| \\ \leq\ & \tilde{D}_\infty^{p;q}(|f(t)-f(x)|;x) \leq \omega(f,\delta)\left(1+\delta^{-1}\tilde{D}_\infty^{p;q}(|t-x|;x)\right) \\ \leq\ & \omega(f,\delta)\left\{1+\delta^{-1}(\tilde{D}_\infty^{p;q}((t-x)^2;x))^{1/2}\right\} \leq \omega(f,\delta)\left\{1+\delta^{-1}\sqrt{\delta_{p,q}(x)}\right\}. \end{aligned}$$

Choosing $\delta = \sqrt{\delta_{p,q}(x)}$, we get the assertion of the theorem. □

Remark 2.2. If $p=p(q)$ and $q \to 1$, then Theorem 2.2 implies that $\tilde{D}_\infty^{p;q}(f;x)$ converges uniformly to $f(x)$ for $x \in [0,1]$.

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