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BIFURCATION OF NONTRIVIAL PERIODIC SOLUTIONS FOR LEISHMANIASIS DISEASE MODEL *

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Abstract. We develop an impulsive model for zoonotic visceral leishmaniasis disease on a population of dogs. The disease infects a population D of dogs. We determine the basic reproduction number $\mathcal{R}_0$, which depends on the vectorial capacity C. Our analysis focuses on the values of C which give either stability or instability of the disease-free equilibrium (DFE). If the vectorial capacity C is less than some threshold, we obtain the stability of DFE, which means that the disease is eradicated for any period of culling dogs. Otherwise, for C greater than the threshold, the period of culling must be in a limited interval. For the particular case, when the period of culling is equal to the threshold, we observe bifurcation phenomena, which means that the disease is installed. In our study of the exponential stability of the DFE we use the fixed point method, and for the bifurcation we use the Lyapunov-Schmidt method.

Keywords: impulsive differential equations, mathematical models, periodic solutions, bifurcation, stability

1. Introduction

In this paper, we investigate a mathematical model of zoonotic visceral leishmaniasis (ZVL). The model studied here is inspired by [8], [11] and [15]. Zoonotic visceral leishmaniasis (ZVL), caused by leishmania infantum, is a disease of humans and domestic dogs (the reservoir) transmitted by phlebotomize sandflies. According to the World Health Organization, leishmaniasis is one of the diseases affecting the poorest in developing countries, with 556 million people estimated to be at risk of contracting leishmaniasis (see [22]).
Some mathematical models for ZVL are considered in [8], [11], [12] and [15], where the behavior of the infection, stability of the disease free equilibrium, and endemic equilibria are studied. Many other papers have considered leishmaniasis diseases (see, for instance, [9], [17] and [23]).

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The model considered here is from [8] and [11], where total dog population size D is constant and divided into two categories, ever-infectious dogs (that become infectious) and never-infected dogs. The ever-infectious dogs category is partitioned into three subclasses: susceptible (uninfected), latent (infected but not infectious) and infectious dogs, with sizes (numbers) denoted by S, L and I, respectively. The never-infected dogs category is partitioned into two subclasses, i.e. uninfected and infected dogs, with sizes denoted by R and Q, respectively.

The sum $S+L+I+R+Q$ is the total population D. The natural death rate δ is assumed to be identical in all subclasses. A proportion α of dogs born susceptible to ZVL with $0<\alpha<1$. Consequently, the birth flux into the susceptible class is $\alpha\beta D$ and into the resistant class is $(1-\alpha)\beta D$ where β is the natural birth rate of dogs as in [11] and [15]. We also consider that $\beta=\delta$, elsewhere D is non-constant and D will tend to zero or infinity for $\delta>\beta$ or $\delta<\beta$, respectively. Latent dogs become infectious and re-enter the infected class with the rate σ. The force of infection is CI/D, where C is the vectorial capacity of the sandfly population transmitting the infection among dogs. It is denoted by CIS/D (resp. CIR/D) for the contact between infectious and susceptible (resp. infectious and uninfected) dogs.

Following the above assumptions, the obtained epidemic model is governed by the following system of ordinary differential equations (see [8] and [11])

$$\left\{\begin{array}{lcl}\dot{S} & = & \alpha\beta D-\frac{CIS}{D}-\delta S,\\ \dot{L} & = & \frac{CIS}{D}-(\sigma+\delta)\,L,\\ \dot{I} & = & \sigma L-\delta I,\\ \dot{R} & = & (1-\alpha)\,\beta D-\frac{CIR}{D}-\delta R,\\ \dot{Q} & = & \frac{CIR}{D}-\delta Q\end{array}\right. \tag{1.1}$$

with initial conditions

$$S(0)\geq 0, L(0)\geq 0, I(0)\geq 0, R(0)\geq 0 \text{ and } Q(0)\geq 0. \tag{1.2}$$

In [8], the well-posedness of (1.1), (1.2) is proved and the local stability of equilibria is investigated. In fact, the disease free equilibrium
$E_f=\left(S^0,L^0,I^0,R^0,Q^0\right)=(\alpha D,0,0,(1-\alpha)D,0)$
is locally asymptotically stable for $\mathcal{R}_0(C)=\frac{C\alpha\sigma}{\delta(\sigma+\delta)}<1$, and unstable for $\mathcal{R}_0(C)>1$, where $\mathcal{R}_0(C)$ is the base reproduction number. For $\mathcal{R}_0(C)>1$, the endemic equilibrium

$$\begin{aligned}E^* &= (S^*,L^*,I^*,R^*,Q^*)\\ &=\left(\frac{\alpha D}{\mathcal{R}_0(C)},\frac{\delta^2 D(\mathcal{R}_0(C)-1)}{\sigma C},\frac{\delta D(\mathcal{R}_0(C)-1)}{C},\ \frac{(1-\alpha)D}{\mathcal{R}_0(C)},\frac{(1-\alpha)D(\mathcal{R}_0(C)-1)}{\mathcal{R}_0(C)}\right)\end{aligned}$$

exists and is locally asymptotically stable.

In this paper, we consider the case of controlling the disease ZL by killing infected dogs. As in [11], a fraction $\theta \in (0,1)$ of infected dogs is culled periodically in order to reduce or eradicate the disease from the dog population. We obtain the following model

$$\dot{x}_1(t) = F_1(x_1, x_2, x_3, x_4, x_5), \tag{1.3}$$
$$\dot{x}_2(t) = F_2(x_1, x_2, x_3, x_4, x_5), \tag{1.4}$$
$$\dot{x}_3(t) = F_3(x_1, x_2, x_3, x_4, x_5), \tag{1.5}$$
$$\dot{x}_4(t) = F_4(x_1, x_2, x_3, x_4, x_5), \tag{1.6}$$
$$\dot{x}_5(t) = F_5(x_1, x_2, x_3, x_4, x_5), \tag{1.7}$$

for $t > 0$ and $t \neq t_i$, where t_i is the time of the i^{th} control,

$$\begin{aligned}
F_1(x_1, x_2, x_3, x_4, x_5) &= \alpha\beta D - \frac{Cx_1x_3}{D} - \delta x_1, \\
F_2(x_1, x_2, x_3, x_4, x_5) &= \frac{Cx_1x_3}{D} - (\sigma + \delta)\, x_2, \\
F_3(x_1, x_2, x_3, x_4, x_5) &= \sigma x_2 - \delta x_3, \\
F_4(x_1, x_2, x_3, x_4, x_5) &= (1-\alpha)\,\beta D - \frac{Cx_3x_4}{D} - \delta x_4, \text{ and} \\
F_5(x_1, x_2, x_3, x_4, x_5) &= \frac{Cx_3x_4}{D} - \delta x_5.
\end{aligned}$$

For $t = t_i$ we have

$$x_1(t_i^+) = \Theta_1(x_1(t_i), x_2(t_i), x_3(t_i), x_4(t_i), x_5(t_i)), \tag{1.8}$$
$$x_2(t_i^+) = \Theta_2(x_1(t_i), x_2(t_i), x_3(t_i), x_4(t_i), x_5(t_i)), \tag{1.9}$$
$$x_3(t_i^+) = \Theta_3(x_1(t_i), x_2(t_i), x_3(t_i), x_4(t_i), x_5(t_i)), \tag{1.10}$$
$$x_4(t_i^+) = \Theta_4(x_1(t_i), x_2(t_i), x_3(t_i), x_4(t_i), x_5(t_i)), \tag{1.11}$$
$$x_5(t_i^+) = \Theta_5(x_1(t_i), x_2(t_i), x_3(t_i), x_4(t_i), x_5(t_i)), \tag{1.12}$$

where $x_j(t_i^+) = \lim\limits_{\substack{> \\ t \to t_i}} x_j(t)$, $(j = 1, \ldots, 5)$ is the size of x_j just after the i^{th} culling. In our case we have

$$\begin{aligned}
\Theta_1(x_1(t_i), x_2(t_i), x_3(t_i), x_4(t_i), x_5(t_i)) &= x_1(t_i), \\
\Theta_2(x_1(t_i), x_2(t_i), x_3(t_i), x_4(t_i), x_5(t_i)) &= x_2(t_i), \\
\Theta_3(x_1(t_i), x_2(t_i), x_3(t_i), x_4(t_i), x_5(t_i)) &= (1-\theta)x_3(t_i), \\
\Theta_4(x_1(t_i), x_2(t_i), x_3(t_i), x_4(t_i), x_5(t_i)) &= x_4(t_i), \\
\Theta_5(x_1(t_i), x_2(t_i), x_3(t_i), x_4(t_i), x_5(t_i)) &= x_5(t_i).
\end{aligned}$$

The variables are
$x_1 = S$: ever-infectious dogs,
$x_2 = L$: latent (infected but not infectious) dogs,

$x_3 = I$: infectious dogs,
$x_4 = R$: never-infectious dogs are either uninfected dogs,
$x_5 = Q$: never-infectious dogs are either infected dogs, and
τ: term of the first culling, it is the period between two controls, that is $t_i = i\tau$, $i \in \mathbb{N}$.
We obtain a special kind of differential equations called impulsive differential equations. To have more details on this kind of differential equations see [1]-[6] and [13]. Impulsive models in population dynamics were investigated widely in the last thirty years. We can cite the following papers [7]-[8], [18]- [21].
Our aim is to study (1.3)-(1.12), more specifically, we study the existence of periodic positive solutions and their stability in order to obtain conditions for eradication of the disease.
After the use of the impulse control E_f remains as the equilibrium of (1.3)-(1.12) but the endemic equilibrium E^* disappears, so eventually we will have a periodic solution to (1.3)-(1.12) as a new equilibrium of the model with impulse control. The aim of this paper is to study the stability of the disease free equilibrium of (1.3)-(1.12) which corresponds to the eradication of the disease by culling dogs. If we find periodic solutions using the bifurcation analysis as in [7] and [18]-[20], then the stability of the disease free equilibrium is lost and the disease is installed. So, the aim of this paper is to determine the values of the vectorial capacity and the period of culling dogs to eradicate the disease.
This paper is organized as follows. In the following section we study the existence and stability of the equilibrium of similar situations, and in Section 3 we analyze the bifurcation of nontrivial periodic solutions. Conclusions are given in Section 4, while in the final section we provide Appendix.

2. Stability of the disease free equilibrium E_f

We can show that $\zeta(t) := \zeta_0 = (\alpha D, 0, 0, (1-\alpha)D, 0) = E_f$ is a constant equilibrium of (1.3)-(1.12), it is called a trivial solution.
To study the stability of ζ we use the same approach of the fixed point process as in [7] and [18]-[20].
Since the solution of (1.3)-(1.7) exists globally in $\mathbb{R}_+$ and is nonnegative (see [8]) we have

$$X(t) = \Phi(t, X_0), t \geq 0 \tag{2.1}$$

where $X(t) = (x_1, x_2, x_3, x_4, x_5)(t)$, $X(0) = X_0 = (\alpha D, 0, 0, (1-\alpha)D), 0)$ and Φ is the flow associated to (1.3)-(1.12). Since the culling of dogs is used periodically with period $\tau > 0$ then the solutions of (1.3)-(1.12) for $t \in (0, \tau)$ are given by (2.1). The term $X(\tau^+)$ denotes the state of the population after the culling, $X(\tau^+) = \Theta(X(\tau)) = \Theta(\Phi(\tau, X_0))$.
To have periodic solution we must have $X(\tau^+) = X_0$ that is $X_0 = \Theta(\Phi(\tau, X_0))$.
Let Ψ be the operator defined by

$$\Psi(\tau, X_0) = \Theta(\Phi(\tau, X_0)) \tag{2.2}$$

and denote by $D_X\Psi$ the derivative of Ψ with respect to X. Then $X = \Phi(., X_0)$ is a τ-periodic solution of (1.3)-(1.12) if and only if

$$\Psi(\tau, X_0) = X_0, \tag{2.3}$$

i.e. X_0 is a fixed point of $\Psi(\tau, .)$, and it is exponentially stable if and only if the spectral radius $\rho(D_X\Psi(\tau, .))$ is strictly less than 1 (see [16]).

We have $D_X\Psi(\tau, X_0) = D_X\Theta(\Phi(\tau, X_0))\dfrac{\partial\Phi}{\partial X}(\tau, X_0)$.

Then for $X_0 = \zeta_0$ and $\tau = \tau_0$ we have

$$D_X\Psi(\tau_0, \zeta_0) = D_X\Theta(\Phi(\tau_0, \zeta_0))\frac{\partial\Phi}{\partial X}(\tau_0, \zeta_0)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial\Phi_1(\tau_0,\zeta_0)}{\partial x_1} & \frac{\partial\Phi_1(\tau_0,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_1(\tau_0,\zeta_0)}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial\Phi_2(\tau_0,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_2(\tau_0,\zeta_0)}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial\Phi_4(\tau_0,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_4(\tau_0,\zeta_0)}{\partial x_3} & \frac{\partial\Phi_4(\tau_0,\zeta_0)}{\partial x_4} & 0 \\ 0 & \frac{\partial\Phi_5(\tau_0,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_5(\tau_0,\zeta_0)}{\partial x_3} & 0 & \frac{\partial\Phi_5(\tau_0,\zeta_0)}{\partial x_5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial\Phi_1(\tau_0,\zeta_0)}{\partial x_1} & \frac{\partial\Phi_1(\tau_0,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_1(\tau_0,\zeta_0)}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial\Phi_2(\tau_0,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_2(\tau_0,\zeta_0)}{\partial x_3} & 0 & 0 \\ 0 & (1-\theta)\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_2} & (1-\theta)\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial\Phi_4(\tau_0,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_4(\tau_0,\zeta_0)}{\partial x_3} & \frac{\partial\Phi_4(\tau_0,\zeta_0)}{\partial x_4} & 0 \\ 0 & \frac{\partial\Phi_5(\tau_0,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_5(\tau_0,\zeta_0)}{\partial x_3} & 0 & \frac{\partial\Phi_5(\tau_0,\zeta_0)}{\partial x_5} \end{pmatrix}.$$

The equilibrium ζ is exponentially stable if and only if the spectral radius is less than one.

We have

$$\begin{aligned} (2.4) \quad det(D_X\Psi(\tau_0, \zeta_0) - \mu I) \quad &= \quad \left(\frac{\partial\Phi_1(\tau_0, \zeta_0)}{\partial x_1} - \mu\right)\left(\frac{\partial\Phi_4(\tau_0, \zeta_0)}{\partial x_4} - \mu\right) \\ &\quad \times \left(\frac{\partial\Phi_5(\tau_0, \zeta_0)}{\partial x_5} - \mu\right)\chi(\mu) \end{aligned}$$

where

$$\begin{aligned} (2.5)\,\chi(\mu) \quad &= \quad \mu^2 - \left(\frac{\partial\Phi_2}{\partial x_2}(\tau_0, \zeta_0) + (1-\theta)\frac{\partial\Phi_3}{\partial x_3}(\tau_0, \zeta_0)\right)\mu \\ &\quad +(1-\theta)\left(\frac{\partial\Phi_2}{\partial x_2}(\tau_0, \zeta_0)\frac{\partial\Phi_3}{\partial x_3}(\tau_0, \zeta_0) - \frac{\partial\Phi_3}{\partial x_2}(\tau_0, \zeta_0)\frac{\partial\Phi_2}{\partial x_3}(\tau_0, \zeta_0)\right). \end{aligned}$$

From (2.4) and (2.5), the equilibrium $\zeta = E_f$ is exponentially stable if and only if $\left|\frac{\partial\Phi_j}{\partial x_j}(\tau_0, \zeta_0)\right| < 1$ for $j = 1,\ 4,\ 5$ and $|\mu_\pm| < 1$. Where

$\mu_\pm = \frac{\frac{\partial\Phi_2}{\partial x_2}(\tau_0,\zeta_0)+(1-\theta)\frac{\partial\Phi_3}{\partial x_3}(\tau_0,\zeta_0)\pm\sqrt{\Delta}}{2}$ and

$$(2.6)\ \ \Delta = \left(\frac{\partial\Phi_2}{\partial x_2}(\tau_0, \zeta_0) - (1-\theta)\frac{\partial\Phi_3}{\partial x_3}(\tau_0, \zeta_0)\right)^2 + 4(1-\theta)\frac{\partial\Phi_3}{\partial x_2}(\tau_0, \zeta_0)\frac{\partial\Phi_2}{\partial x_3}(\tau_0, \zeta_0).$$

From the variational equation $\frac{d}{dt}(D_X\Phi(t,\zeta_0)) = \frac{\partial F}{\partial X}(\zeta_0)\frac{\partial \Phi}{\partial X}(t,\zeta_0)$, we have for all $0 < t \le \tau_0$

$$\frac{\partial \Phi_1(t,\zeta_0)}{\partial x_1} = \frac{\partial \Phi_4(t,\zeta_0)}{\partial x_4} = \frac{\partial \Phi_5(t,\zeta_0)}{\partial x_5} = e^{-\delta t} \in (0,1),$$

$$\frac{\partial \Phi_2(t,\zeta_0)}{\partial x_2} = \frac{e^{-\delta t}\left((\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma}{2}t}+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma}{2}t}\right)}{2\sqrt{\sigma(\sigma+4\alpha C)}} > 0,$$

$$\frac{\partial \Phi_2(t,\zeta_0)}{\partial x_3} = \frac{-\sigma e^{-\delta t}\left(e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma}{2}t}-e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma}{2}t}\right)}{\sqrt{\sigma(\sigma+4\alpha C)}} > 0,$$

$$\frac{\partial \Phi_3(t,\zeta_0)}{\partial x_3} = \frac{e^{-\delta t}\left((\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma}{2}t}+(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma}{2}t}\right)}{2\sqrt{\sigma(\sigma+4\alpha C)}} > 0,$$

$$\frac{\partial \Phi_3(t,\zeta_0)}{\partial x_2} = \frac{\alpha C}{\sigma}\frac{\partial \Phi_2(t,\zeta_0)}{\partial x_3} > 0.$$

(see Appendix, Subsection 5.1.).

Proposition 2.1. *For all $\tau_0 > 0$ we have $0 < \mu_- < \mu_+$.*

Proof. Since $\frac{\partial \Phi_3}{\partial x_2}(\tau_0,\zeta_0)\frac{\partial \Phi_2}{\partial x_3}(\tau_0,\zeta_0) > 0$ then $\Delta > 0$. Moreover,

$$\chi(0) = \frac{\partial \Phi_2}{\partial x_2}(\tau_0,\zeta_0)\frac{\partial \Phi_3}{\partial x_3}(\tau_0,\zeta_0) - \frac{\partial \Phi_2}{\partial x_3}(\tau_0,\zeta_0)\frac{\partial \Phi_3}{\partial x_2}(\tau_0,\zeta_0) = e^{-(\sigma+2\delta)\tau_0} > 0,$$

then there exist two positive roots $\mu_\pm$ of χ such that $\mu_+ > \mu_- > 0$. □

We deduce the following result.

Theorem 2.1. *1. If $C \le \frac{\delta(\sigma+\delta)}{\alpha\sigma}$ (i.e. $\mathfrak{R}_0(C) \le 1$), then the trivial solution ζ is exponentially stable for all $\tau_0 > 0$.*

2. If $C > \frac{\delta(\sigma+\delta)}{\alpha\sigma}$ (i.e. $\mathfrak{R}_0(C) > 1$), then there exists $\tau_0^* > 0$ such that the trivial solution ζ is exponentially stable for all $\tau_0 < \tau_0^*$, and unstable for $\tau_0 > \tau_0^*$.

Proof. The inequality $\mu_+ < 1$ is equivalent to

$$\sqrt{\Delta} < H_1(\tau_0), \tag{2.7}$$

where

$$\begin{aligned} H_1(\tau_0) := \quad & 2 - \left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)(1-\theta)}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right)e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0} \\ & -\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)(1-\theta)}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right)e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0} \end{aligned}$$

and

$$\begin{aligned} \Delta = & \left(\frac{\left(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma-(1-\theta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)\right)^2+16\sigma\alpha C(1-\theta)}{4\sigma(\sigma+4\alpha C)}\right)e^{-(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)\tau_0} \\ & +\left(\frac{\left(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-(1-\theta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)\right)^2+16\sigma\alpha C(1-\theta)}{4\sigma(\sigma+4\alpha C)}\right)e^{(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)\tau_0} \\ & +\left(\frac{\left(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma-(1-\theta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)\right)\left(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-(1-\theta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)\right)-16\sigma\alpha C(1-\theta)}{2\sigma(\sigma+4\alpha C)}\right) \\ & \times e^{-(\sigma+2\delta)\tau_0}. \end{aligned}$$

The inequality (2.7) is equivalent to $H_1(\tau_0) > 0$ and $H_2(\tau_0) := (H_1(\tau_0))^2 - \Delta > 0$, where

$$\begin{aligned}
H_2(\tau_0) &= (H_1(\tau_0) - \sqrt{\Delta})(H_1(\tau_0) + \sqrt{\Delta}) \\
&= 4 + 4(1-\theta)e^{-(\sigma+2\delta)\tau_0} - \left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)(1-\theta)}{\sqrt{\sigma(\sigma+4\alpha C)}}\right) e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0} \\
&\quad -2\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)(1-\theta)}{\sqrt{\sigma(\sigma+4\alpha C)}}\right) e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}.
\end{aligned}$$

We have

$$\begin{aligned}
H_1'(\tau_0) &= \frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)(1-\theta)}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right) e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0} \\
&\quad - \frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)(1-\theta)}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right) e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}, \\
H_2'(\tau_0) &= -4(\sigma+2\delta)(1-\theta)e^{-(\sigma+2\delta)\tau_0} \\
&\quad +(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)(1-\theta)}{\sqrt{\sigma(\sigma+4\alpha C)}}\right) e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0} \\
&\quad -(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)(1-\theta)}{\sqrt{\sigma(\sigma+4\alpha C)}}\right) e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}
\end{aligned}$$

and

$$\begin{aligned}
H_3'(\tau_0) &= -4\sigma\alpha\left(C - \frac{\delta(\sigma+\delta)}{\sigma\alpha}\right)\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)(1-\theta)}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right) e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0} \\
&\quad -4\sigma\alpha\left(C - \frac{\delta(\sigma+\delta)}{\sigma\alpha}\right)\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)(1-\theta)}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right) e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0}
\end{aligned}$$

where $H_3(\tau_0) = e^{(\sigma+2\delta)\tau_0} H_2'(\tau_0)$.
We consider two cases $\mathfrak{R}_o \le 1$ or $\mathfrak{R}_o > 1$.

1. Case1: If $C \le \frac{\delta(\sigma+\delta)}{\alpha\sigma}$ (*i.e.* $\mathfrak{R}_o \le 1$), then $H_1'(\tau_0) > 0$ and $H_3'(\tau_0) > 0$ for all $\tau_0 > 0$. Since $H_1(0) = \theta$ and $H_3(0) = 4(\sigma+\delta)\theta > 0$ we have $H_1(\tau_0) > 0$ and $H_3(\tau_0) > 0$ for all $\tau_0 > 0$. That is, $H_2'(\tau_0) > 0$. Since $H_2(0) = 0$ we have $H_2(\tau_0) > 0$ for all $\tau_0 > 0$.

2. Case2: If $C > \frac{\delta(\sigma+\delta)}{\alpha\sigma}$ (*i.e.* $\mathfrak{R}_o > 1$), then $H_1'(\tau_0) = 0$ if and only if

$$\tau_0 = \tau_0^1 := \frac{1}{\sqrt{\sigma(\sigma+4\alpha C)}} \ln\left(\frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)\left(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)(1-\theta)\right)}{(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)\left(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)(1-\theta)\right)}\right).$$

   $H_1'(\tau_0) > 0$ for all $\tau_0 < \tau_0^1$ and $H_1'(\tau_0) < 0$ for all $\tau_0 > \tau_0^1$.
   Since $\lim_{\tau_0\to+\infty} H_1(\tau_0) = -\infty$, there exists $\tau_0^2 > \tau_0^1$ such that $H_1(\tau_0^2) = 0$ and $H_1(\tau_0) > 0$ for all $\tau_0 < \tau_0^2$.
   Moreover $H_3'(\tau_0) < 0$ for all $\tau_0 > 0$. Since $H_3(0) > 0$ and $\lim_{\tau_0\to+\infty} H_3(\tau_0) = -\infty$ then there exists $\tau_0^3 > 0$ such that $H_3(\tau_0^3) = 0$ (that is $H_2'(\tau_0^3) = 0$), $H_3(\tau_0) > 0$ (that is $H_2'(\tau_0) > 0$) for $\tau_0 < \tau_0^3$ and $H_3(\tau_0) < 0$ (that is $H_2'(\tau_0) < 0$) for $\tau_0 > \tau_0^3$, then H_2 has a maximum positive at τ_0^3. Since $H_2(0) = 0$ and $H_2(\tau_0^2) = -\Delta$, then $\tau_0^3 < \tau_0^2$ and there exists $\tau_0^* \in (\tau_0^3, \tau_0^2)$ such that $H_2(\tau_0) > 0$ for all $\tau_0 < \tau_0^*$ and $H_2(\tau_0) < 0$ for all $\tau_0 > \tau_0^*$.

□

Remark 2.1. If $C > \frac{\delta(\sigma+\delta)}{\alpha\sigma}$ (*i.e.* $\mathfrak{R}_o > 1$) we have $\mu_+ < 1$ for $\tau_0 < \tau_0^*$ and $\mu_+ = 1$ for $\tau_0 = \tau_0^*$. That is we have a critical case at $\tau_0 = \tau_0^*$.

Remark 2.2. From theorem 2.1, we show that in the case of low vectorial capacity $C(\leq \frac{\delta(\sigma+\delta)}{\alpha\sigma})$ we can choose any period τ_0 of culling dogs to have eradication of the disease. Otherwise, for high vectorial capacity $C(> \frac{\delta(\sigma+\delta)}{\alpha\sigma})$, the eradication of the disease is acquired only for period τ_0 less than some threshold τ_0^*. It is interesting to show what happens for $\tau = \tau_0^*$.

3. Bifurcation Analysis of nontrivial periodic solution

In this section, we analyze the bifurcation of nontrivial periodic solutions of the system (1.3) − (1.12) from ζ at $\tau_0 = \tau_0^*$. This case is possible for $C > \frac{\delta(\sigma+\delta)}{\alpha\sigma}$ (*i.e.* $\mathfrak{R}_o > 1$) (see theorem 2.1). The bifurcation of nontrivial periodic solutions means that the disease free equilibrium lost its stability and becomes unstable. The bifurcated solutions means that the disease is installed.
Let $\bar{\tau}$ and $\bar{X}$ such that $\tau = \tau_0^* + \bar{\tau}$ and $X = \zeta_0 + \bar{X}$. The equation (2.3) is equivalent to

$$M(\bar{\tau}, \bar{X}) = 0, \tag{3.1}$$

where $M(\bar{\tau}, \bar{X}) = \big(M_1(\bar{\tau}, \bar{X}), \ldots, M_5(\bar{\tau}, \bar{X})\big) := \zeta_0 + \bar{X} - \Psi(\tau_0^* + \bar{\tau}, \zeta_0 + \bar{X})$.
If $(\bar{\tau}, \bar{X})$ is a zero of M, then $(\zeta_0 + \bar{X})$ is a fixed point of $\Psi(\tau_0^* + \bar{\tau}, .)$. Let

$$D_X M(\bar{\tau}, \bar{X}) = \begin{pmatrix} a & b & c & * & * \\ * & d & e & * & * \\ * & f & g & * & * \\ * & h & i & j & * \\ * & k & l & * & m \end{pmatrix}. \tag{3.2}$$

For $(\bar{\tau}, \bar{X}) = (0, (0, 0, 0, 0, 0))$, we have

$$D_X M(0, (0, 0, 0, 0, 0)) = \begin{pmatrix} a_0 & b_0 & c_0 & * & * \\ * & d_0 & e_0 & * & * \\ * & f_0 & g_0 & * & * \\ * & h_0 & i_0 & j_0 & * \\ * & k_0 & l_0 & * & m_0 \end{pmatrix}$$

where $a_0 = 1 - \frac{\partial \Phi_1}{\partial x_1}(\tau_0^*, \zeta_0)$, $b_0 = -\frac{\partial \Phi_1}{\partial x_2}(\tau_0^*, \zeta_0)$, $c_0 = -\frac{\partial \Phi_1}{\partial x_3}(\tau_0^*, \zeta_0)$, $d_0 = 1 - \frac{\partial \Phi_2}{\partial x_2}$, $e_0 = -\frac{\partial \Phi_2}{\partial x_3}(\tau_0^*, \zeta_0)$, $f_0 = -(1-\theta)\frac{\partial \Phi_3}{\partial x_2}(\tau_0^*, \zeta_0)$, $g_0 = 1 - (1-\theta)\frac{\partial \Phi_3}{\partial x_3}(\tau_0^*, \zeta_0)$, $h_0 = -\frac{\partial \Phi_4}{\partial x_2}(\tau_0^*, \zeta_0)$, $i_0 = -\frac{\partial \Phi_4}{\partial x_3}(\tau_0^*, \zeta_0)$, $j_0 = 1 - \frac{\partial \Phi_4}{\partial x_4}(\tau_0^*, \zeta_0)$, $k_0 = -\frac{\partial \Phi_5}{\partial x_2}(\tau_0^*, \zeta_0)$, $l_0 = -\frac{\partial \Phi_5}{\partial x_3}(\tau_0^*, \zeta_0)$, and $m_0 = 1 - \frac{\partial \Phi_5}{\partial x_5}(\tau_0^*, \zeta_0)$.
We have $\mu_+ = 1$ if and only if $\frac{\partial \Phi_2}{\partial x_2}(\tau_0^*, \zeta_0) + (1-\theta)\frac{\partial \Phi_3}{\partial x_3}(\tau_0^*, \zeta_0) + \sqrt{\Delta} = 2$. From (2.6) we obtain

$$d_0 + g_0 = \sqrt{(d_0 - g_0)^2 + 4 f_0 e_0} = \sqrt{(d_0 + g_0)^2 - 4(g_0 d_0 - f_0 e_0)}, \tag{3.3}$$

that is

$$g_0 d_0 - f_0 e_0 = 0. \tag{3.4}$$

Put
$e_0 := \frac{\sigma G_1(\tau_0)}{\sqrt{\sigma(\sigma+4\alpha C)}}$, $f_0 := \frac{(1-\theta)\alpha C G_1(\tau_0)}{\sqrt{\sigma(\sigma+4\alpha C)}}$, $d_0 := \frac{G_2(\tau_0)}{2\sqrt{\sigma(\sigma+4\alpha C)}}$ and $g_0 := \frac{G_3(\tau_0)}{2\sqrt{\sigma(\sigma+4\alpha C)}}$,
where

$$G_1(\tau_0) = e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0} - e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0},$$

$$G_2(\tau_0) = 2\sqrt{\sigma(\sigma+4\alpha C)} - (\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0}$$
$$-(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}, \text{ and}$$

$$G_3(\tau_0) = 2\sqrt{\sigma(\sigma+4\alpha C)} - (1-\theta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0}$$
$$-(1-\theta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}.$$

Then we have

Proposition 3.1. *Let $C > \frac{\delta(\sigma+\delta)}{\alpha\sigma}$ (i.e.$\mathfrak{R}_o > 1$).*

1. *For all $\tau_0 > 0$ we have $a_0 > 0$, $b_0 > 0$, $c_0 > 0$, $e_0 < 0$ $f_0 < 0$, $h_0 > 0$, $i_0 > 0$, $j_0 > 0$, $k_0 < 0$, $l_0 < 0$ and $m_0 > 0$.*
2. *There exists $\hat{\tau}_0 > 0$ such that $d_0 = 0$ for $\tau_0 = \hat{\tau}_0$, $d_0 > 0$ for $\tau_0 < \hat{\tau}_0$ and $d_0 < 0$ for $\tau_0 > \hat{\tau}_0$.*
3. *There exists $\tilde{\tau}_0 > 0$ such that $g_0 = 0$ for $\tau_0 = \tilde{\tau}_0$, $g_0 > 0$ for $\tau_0 < \tilde{\tau}_0$ and $g_0 < 0$ for $\tau_0 > \tilde{\tau}_0$.*

Proof. Let $C > \frac{\delta(\delta+\sigma)}{\sigma\alpha}$.

1. We have $G_1(\tau_0) < 0$ for all $\tau_0 > 0$, then $f_0 < 0$ and $e_0 < 0$ for all $\tau_0 > 0$. The signs of a_0, b_0, c_0, h_0, i_0, j_0, k_0, l_0 and m_0 can be deduced from signs of $\frac{\partial \Phi_i}{\partial x_j}(\tau_0^*, \zeta_0)$.
2. We have
$$G_2'(\tau_0) = \frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)}{2}e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0}$$
$$-\frac{(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)}{2}e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}$$
then $G_2'(\tau_0) = 0$ if and only if
$$\tau_0 := \tau_0^4 = \frac{1}{\sqrt{\sigma(\sigma+4\alpha C)}} \ln\left(\frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)}{(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)}\right).$$
Since $G_2(0) = 0$ then there exists $\hat{\tau}_0 > 0$ such that $G_2(\hat{\tau}_0) = 0$, $G_2(\tau_0) > 0$ for $\tau_0 \in (0, \hat{\tau}_0)$ and $G_2(\tau_0) < 0$ for $\tau_0 \in (\hat{\tau}_0, +\infty)$.

3. We have
$G_3'(\tau_0) = (1-\theta)\frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)}{2}e^{-\frac{3\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}$
$-(1-\theta)\frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)}{2}(1-\theta)e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}$
then $G_3'(\tau_0) = 0$ if and only if

$$\tau_0 := \tau_0^5 = \frac{1}{\sqrt{\sigma(\sigma+4\alpha C)}}\ln\left(\frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)}{(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)}\right).$$

Since $G_3(0) = 0$ then there exists $\tilde{\tau}_0 > 0$ such that $G_2(\tilde{\tau}_0) = 0$, $G_3(\tau_0) > 0$ for $\tau_0 \in (0, \tilde{\tau}_0)$ and $G_3(\tau_0) < 0$ for $\tau_0 \in (\tilde{\tau}_0, +\infty)$.

□

Remark 3.1. From (3.3), (3.4) and Proposition 3.1 we have $d_0 + g_0 > 0$ and $d_0 g_0 = f_0 e_0 > 0$ then we must have $d_0 > 0$ and $g_0 > 0$, that is τ_0^* must be less than $\hat{\tau}_0$ and $\tilde{\tau}_0$.

Put $G_4(\tau_0) \;:=\; 2\sqrt{\sigma(\sigma+4\alpha C)}(g_0 d_0 - f_0 e_0)$.

Proposition 3.2. *For $C > \frac{\delta(\delta+\sigma)}{\sigma\alpha}$ we have $\tau_0^* < \min(\hat{\tau}_0, \tilde{\tau}_0)$.*

Proof. Let $C > \frac{\delta(\delta+\sigma)}{\sigma\alpha}$. We have

$$\frac{G_4(\tau_0)-G_2(\tau_0)}{(1-\theta)e^{-(\sigma+2\delta)\tau_0}} = 2\sqrt{\sigma(\sigma+4\alpha C)} - \left(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma\right)e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0} - \left(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma\right)e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0}$$

and

$$\frac{G_4(\tau_0)-G_3(\tau_0)}{e^{-(\sigma+2\delta)\tau_0}} = 2\sqrt{\sigma(\sigma+4\alpha C)}(1-\theta) - \left(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma\right)e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0} - \left(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma\right)e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0}.$$

Then

$$\left(\frac{G_4(\tau_0)-G_2(\tau_0)}{(1-\theta)e^{-(\sigma+2\delta)\tau_0}}\right)' e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0} = \frac{(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)\left(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma\right)}{2} - \frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)}{2}e^{\sqrt{\sigma(\sigma+4\alpha C)}\tau_0}$$

and

$$\left(\frac{G_4(\tau_0)-G_3(\tau_0)}{e^{-(\sigma+2\delta)\tau_0}}\right)' e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0} = \frac{(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)}{2} - \frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)}{2}e^{\sqrt{\sigma(\sigma+4\alpha C)}\tau_0}.$$

Since
$\frac{(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)}{2} < \frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)}{2}$
and
$\frac{(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)}{2} < \frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)}{2}$
we obtain

$\left(\frac{G_4(\tau_0)-G_2(\tau_0)}{(1-\theta)e^{-(\sigma+2\delta)\tau_0}}\right)' < 0$ and $\left(\frac{G_4(\tau_0)-G_3(\tau_0)}{e^{-(\sigma+2\delta)\tau_0}}\right)' < 0$. Moreover, we have

$\frac{G_4(0)-G_2(0)}{(1-\theta)} = G_4(0) - G_3(0) = 0$, that is $G_4(\tau_0) < G_2(\tau_0)$ and $G_4(\tau_0) < G_3(\tau_0)$. This implies that $\tau_0^* < \min(\hat{\tau}_0, \tilde{\tau}_0)$. □

We have $M(0,(0,0,0)) = 0$. Let $D_X M(0,(0,0,0)) = E$, then $\dim\ker(E) = \operatorname{co}\dim \mathcal{R}(\mathrm{E}) = 1$. Denote by P_1 and P_2 the projectors onto $\ker(E)$ and $\mathcal{R}(E)$ respectively, such that $P_1 + P_2 = Id_{\mathbb{R}^5}$, $P_1\mathbb{R}^5 = \operatorname{span}\{\mathrm{Y}_0\} = \ker(\mathrm{E})$, with $Y_0 = (q_1, q_2, 1, q_4, q_5)$, $q_1 = \frac{e_0 b_0}{d_0 a_0} - \frac{c_0}{a_0}$, $q_2 = -\frac{e_0}{d_0}$, $q_4 = \frac{e_0 h_0}{d_0 j_0} - \frac{i_0}{j_0}$, $q_5 = \frac{e_0 k_0}{d_0 m_0} - \frac{l_0}{m_0}$ and $P_2\mathbb{R}^5 = \operatorname{span}\{(1,0,0,0,0),(0,1,0,0,0),(0,0,0,1,0),(0,0,0,0,1)\} = \mathcal{R}(E)$.
Then $(I - P_1)\mathbb{R}^5 = \operatorname{span}\{(1,0,0,0,0),(0,1,0,0,0),(0,0,0,1,0),(0,0,0,0,1)\}$ and $(I - P_2)\mathbb{R}^5 = \operatorname{span}\{(0,0,1,0,0)\}$.
Equation (3.1) is equivalent to

$$\left\{\begin{array}{lcl} M_1(\bar{\tau}, \gamma Y_0 + Z) &=& 0, \\ M_2(\bar{\tau}, \gamma Y_0 + Z) &=& 0, \\ M_3(\bar{\tau}, \gamma Y_0 + Z) &=& 0, \\ M_4(\bar{\tau}, \gamma Y_0 + Z) &=& 0, \\ M_5(\bar{\tau}, \gamma Y_0 + Z) &=& 0, \end{array}\right. \tag{3.5}$$

where $Z = (z_1, z_2, 0, z_4, z_5)$, $(\bar{\tau}, \bar{X}) = (\bar{\tau}, \gamma Y_0 + Z)$ and $(\gamma, z_1, z_2, z_4, z_5) \in \mathbb{R}^5$.
From the two first and two last equations of (3.5), we have

$$\det\begin{pmatrix} \frac{\partial M_1(0,(0,0,0,0,0))}{\partial z_1} & \frac{\partial M_1(0,(0,0,0,0,0))}{\partial z_2} & \frac{\partial M_1(0,(0,0,0,0,0))}{\partial z_4} & \frac{\partial M_1(0,(0,0,0,0,0))}{\partial z_5} \\ \frac{\partial M_2(0,(0,0,0,0,0))}{\partial z_1} & \frac{\partial M_2(0,(0,0,0,0,0))}{\partial z_2} & \frac{\partial M_2(0,(0,0,0,0,0))}{\partial z_4} & \frac{\partial M_2(0,(0,0,0,0,0))}{\partial z_5} \\ \frac{\partial M_4(0,(0,0,0,0,0))}{\partial z_1} & \frac{\partial M_4(0,(0,0,0,0,0))}{\partial z_2} & \frac{\partial M_4(0,(0,0,0,0,0))}{\partial z_4} & \frac{\partial M_4(0,(0,0,0,0,0))}{\partial z_5} \\ \frac{\partial M_5(0,(0,0,0,0,0))}{\partial z_1} & \frac{\partial M_5(0,(0,0,0,0,0))}{\partial z_2} & \frac{\partial M_5(0,(0,0,0,0,0))}{\partial z_4} & \frac{\partial M_5(0,(0,0,0,0,0))}{\partial z_5} \end{pmatrix}$$
$$= \det\begin{pmatrix} a_0 & b_0 & 0 & 0 \\ 0 & d_0 & 0 & 0 \\ 0 & h_0 & j_0 & 0 \\ 0 & k_0 & 0 & m_0 \end{pmatrix} = a_0 d_0 j_0 m_0 \neq 0.$$

From the implicit function theorem, there exists a unique continuous function Z^* such that
$Z^*(\bar{\tau}, \gamma) = (z_1^*(\bar{\tau}, \gamma), z_2^*(\bar{\tau}, \gamma), 0, z_4^*(\bar{\tau}, \gamma), z_4^*(\bar{\tau}, \gamma))$, $Z^*(0,0) = (0,0,0,0,0)$
and

$$M_i\left(\bar{\tau}, (q_1\gamma + z_1^*(\bar{\tau}, \gamma), q_2\gamma + z_2^*(\bar{\tau}, \gamma), \gamma, q_4\gamma + z_4^*(\bar{\tau}, \gamma), q_5\gamma + z_5^*(\bar{\tau}, \gamma))\right) = 0, \tag{3.6}$$

for $i = 1,\ 2,\ 4,\ 5$, with γ and $\bar{\tau}$ small enough.
We have $q_1 < 0$, $q_2 > 0$, $q_4 < 0$ and $q_5 > 0$ for $\tau_0 < \min(\hat{\tau}_0, \tilde{\tau}_0)$.
Moreover, we find $\dfrac{\partial Z^*}{\partial \bar{\tau}}(0,0) = \dfrac{\partial Z^*}{\partial \gamma}(0,0)$ (see Appendix, subsection 5.3.).
Then $M(\bar{\tau}, \bar{X}) = 0$ if and only if

$$\omega(\bar{\tau}, \gamma) = 0 \tag{3.7}$$

where

$$\omega(\bar{\tau}, \gamma) = M_3\left(\bar{\tau}, (q_1\gamma + z_1^*(\bar{\tau}, \gamma), q_2\gamma + z_2^*(\bar{\tau}, \gamma), \gamma, q_4\gamma + z_4^*(\bar{\tau}, \gamma), q_5\gamma + z_5^*(\bar{\tau}, \gamma))\right).$$

Equation (3.7) is called determining equation, it determines the number of periodic solutions of (1.3)-(1.12) (see [10]).
To solve (3.7) we need Taylor development of ω near $(\bar{\tau},\gamma) = (0,0)$. We find $\omega(0,0) = 0$ and $\frac{\partial\omega(0,0)}{\partial\bar{\tau}} = \frac{\partial\omega(0,0)}{\partial\gamma} = 0$ (see Appendix, subsection 5.5.).
Let $\mathcal{A} = \frac{\partial^2\omega(0,0)}{\partial\bar{\tau}^2}$, $\mathcal{B} = \frac{\partial^2\omega(0,0)}{\partial\bar{\tau}\partial\gamma}$ and $\mathcal{C} = \frac{\partial^2\omega(0,0)}{\partial\gamma^2}$. It's shown that $\mathcal{A} = 0$ (see Appendix, subsection 5.6.). Hence

$$\omega(\bar{\tau},\gamma) = \mathcal{B}\bar{\tau}\gamma + \mathcal{C}\frac{\gamma^2}{2} + o\left(|\gamma|^2 + |\bar{\tau}|^2\right),$$

where

$$\begin{aligned}\mathcal{C} &= -\frac{\partial\Theta_3}{\partial x_3}\Big\{2q_1q_2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_2\partial x_1} + 2q_1\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3\partial x_1} + q_2^2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_2^2} \\ &\quad +2q_2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3\partial x_2} + \frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3^2} + \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial^2 z_2^*(0,0)}{\partial\gamma^2}\Big\}\end{aligned}$$

and

$$\mathcal{B} = -\frac{\partial\Theta_3}{\partial x_3}\left\{q_2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar{\tau}\partial x_2} + \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial^2 z_2^*(0,0)}{\partial\bar{\tau}\partial\gamma} + \frac{\partial^2\Phi_3(\tau_0,\zeta_0))}{\partial\bar{\tau}\partial x_3}\right\}.$$

Proposition 3.3. *For $\tau_0 < \tau_0^m := \min(\hat{\tau}_0,\tilde{\tau}_0)$ (see Appendix, subsection 5.2. and 5.4.) we have*

$$\begin{aligned}
\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_1} &= \frac{C}{D}\int_0^t e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}(s-t)}e^{-\delta s} \\
&\quad\times\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right)\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}ds > 0,\\
\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_1} &= \frac{\sigma C}{D}\int_0^t e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}(s-t)}e^{-\delta s}\left(\frac{e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}-1}{\sqrt{\sigma(\sigma+4\alpha C)}}\right)\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}ds > 0,\\
\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_2} &= \frac{C}{D}\int_0^t e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}(s-t)}\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right)\\
&\quad\times\left(\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_2} + \frac{\partial\Phi_1(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}\right)ds < 0,\\
\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_2} &= \frac{\sigma C}{D}\int_0^t e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}(s-t)}\left(\frac{e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}-1}{\sqrt{\sigma(\sigma+4\alpha C)}}\right)\\
&\quad\times\left(\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_2} + \frac{\partial\Phi_1(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}\right)ds < 0,\\
\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3^2} &= \frac{2C}{D}\int_0^t e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}(s-t)}\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right)\\
&\quad\times\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_3}ds < 0,\\
\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3^2} &= \frac{2\sigma C}{D}\int_0^t e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}(s-t)}\left(\frac{e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}-1}{\sqrt{\sigma(\sigma+4\alpha C)}}\right)\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_3}ds < 0,\\
\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_2} &= \frac{C}{D}\int_0^t e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}(s-t)}\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right)\\
&\quad\times\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_1}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}ds > 0,\\
\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_2} &= \frac{\sigma C}{D}\int_0^t e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}(s-t)}\left(\frac{e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}-1}{\sqrt{\sigma(\sigma+4\alpha C)}}\right)\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_1}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}ds > 0,
\end{aligned}$$

$$\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_2^2} = \frac{2C}{D}\int_0^t e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}(s-t)}\left(\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right)$$
$$\times\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_2}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}ds<0,$$
$$\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_2^2} = \frac{2\sigma C}{D}\int_0^t e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+}{2}(s-t)}\left(\frac{e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}-1}{\sqrt{\sigma(\sigma+4\alpha C)}}\right)\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_2}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}ds<0.$$
$$\frac{\partial^2 z_2^*(0,0)}{\partial\gamma^2} = \frac{\partial\Theta_2(\Phi(\tau_0,\zeta_0))}{\partial x_2}\frac{1}{d_0}\left(2q_1q_2\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial x_2\partial x_1}+2q_1\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial x_3\partial x_1}+q_2^2\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial x_2^2}\right.$$
$$\left.+2q_2\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial x_3\partial x_2}+\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial x_3^2}\right)<0,$$
$$q_2\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_2}+\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_3} = -\frac{2\sigma\sqrt{\sigma(\sigma+4\alpha C)}}{2d_0\sqrt{\sigma(\sigma+4\alpha C)}}e^{-(\sigma+2\delta)\tau_0}$$
$$+\frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)\sigma}{2d_0\sqrt{\sigma(\sigma+4\alpha C)}}e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0}$$
$$+\frac{(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)\sigma}{2d_0\sqrt{\sigma(\sigma+4\alpha C)}}e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}>0,$$
$$\frac{\partial^2 z_2^*(0,0)}{\partial\gamma\partial\bar\tau} = \frac{\partial\Theta_2(\Phi(\tau_0,\zeta_0))}{\partial x_2}\frac{1}{d_0}\left(q_2\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_2}+\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_3}\right)>0,\quad and$$
$$q_2\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial\bar\tau\partial x_2}+\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial\bar\tau\partial x_3} = \frac{4\delta\sqrt{\sigma(\sigma+4\alpha C)}}{4d_0\sqrt{\sigma(\sigma+4\alpha C)}}e^{-(\sigma+2\delta)\tau_0}$$
$$-\frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)}{4d_0\sqrt{\sigma(\sigma+4\alpha C)}}e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0}$$
$$+\frac{(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)}{4d_0\sqrt{\sigma(\sigma+4\alpha C)}}e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}>0.$$

Proof. The signs of $\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_i\partial x_j}$ and $\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_i\partial x_j}$ can be deduced from those of $\frac{\partial\Phi_i}{\partial x_j}(t,\zeta_0)$.

Let

$$q_2\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_2}+\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_3} := \frac{\sigma e^{-(\sigma+2\delta)\tau_0}}{2d_0\sqrt{\sigma(\sigma+4\alpha C)}}G_5(\tau_0),$$

where

$$G_5(\tau_0) = (\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}$$
$$+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0}-2\sqrt{\sigma(\sigma+4\alpha C)}.$$

We have

$$G_5'(\tau_0) = -\frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)}{2}e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}$$
$$+\frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)}{2}e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0}>0$$

and $G_5(0)=0$ then $G_5(\tau_0)>0$. That is $q_2\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_2}+\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_3}>0$, then $\frac{\partial^2 z_2^*(0,0)}{\partial\gamma\partial\bar\tau}>0$.

Let

$$q_2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_2}+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_3} := \frac{e^{-(\sigma+2\delta)\tau_0}}{4d_0\sqrt{\sigma(\sigma+4\alpha C)}}G_6(\tau_0),$$

where

$$G_6(\tau_0) = -(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}$$
$$+(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0}$$
$$+4\delta\sqrt{\sigma(\sigma+4\alpha C)}.$$

We have

$$G_6'(\tau_0)=\frac{(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma)}{2}e^{-\frac{\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta}{2}\tau_0}$$
$$+\frac{(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}-\sigma-2\delta)(\sqrt{\sigma(\sigma+4\alpha C)}+\sigma)}{2}e^{\frac{\sqrt{\sigma(\sigma+4\alpha C)}+\sigma+2\delta}{2}\tau_0}>0$$

and $G_6(0)=0$. Then $G_6(\tau_0)>0$, that is $q_2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar{\tau}\partial x_2}+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar{\tau}\partial x_3}>0$.
□

From the last proposition we have $\mathcal{C}>0$ and $\mathcal{B}<0$ for $\tau_0<\tau_0^m$. Put $\bar{\tau}:=\xi\gamma$ then $\omega(\xi\gamma,\gamma)=\frac{\gamma^2}{2}g(\xi,\gamma)$ where $g(\xi,\gamma)=2\mathcal{B}\xi+\mathcal{C}+o_\gamma\left(1+\xi^2\right)$. Then $\frac{\partial g}{\partial\xi}(\xi,0)=2\mathcal{B}$ and $g(\xi,0)=2\mathcal{B}\xi+\mathcal{C}$. The implicit function theorem is applicable for g if $\mathcal{B}\neq0$ and $\xi_0=-\frac{\mathcal{C}}{2\mathcal{B}}$. We find a function $\xi(\gamma)$ such that for γ small enough $g(\xi(\gamma),\gamma)=0$ and $\xi(0)=\xi_0=-\frac{\mathcal{C}}{2\mathcal{B}}$.
Then, $\omega(\xi(\gamma)\gamma,\gamma)=0$ for γ small enough.

In conclusion we have the following theorem.

Theorem 3.1. *Let $C>\frac{\delta(\sigma+\delta)}{\alpha\sigma}$ ($i.e.\Re_o>1$). For $\tau_0=\tau_0^*$ we have a supercritical bifurcation of nontrivial periodic solutions of (1.3)-(1.12) with period $\tau(\gamma)=\tau_0^*+\bar{\tau}(\gamma)$ starting from $\zeta_0+\gamma Y_0+Z^*(\bar{\tau}(\gamma),\gamma)$ for all $\gamma(>0)$ small enough where $\bar{\tau}(\gamma)=-\frac{\mathcal{C}}{2\mathcal{B}}\gamma+\circ(\gamma)$.*

Remark 3.2. From theorem 3.1, we deduce that for a high vectorial capacity $C(>\frac{\delta(\sigma+\delta)}{\alpha\sigma})$ and a period of culling dogs $\tau_0=\tau_0^*$ there is loss in the stability of the disease free equilibrium and we note the presence of endemic periodic solutions, which means that the disease is installed for the period $\tau_0(\gamma)$ close to τ_0^*.

4. Conclusions

In this paper we have developed an impulsive model for zoonotic visceral Leishmaniasis diseases under control by culling infectious dogs. Our model is inspired by those of [11]. In the absence of control the model is studied in [8] where they study the existence of a trivial equilibrium $E_f$and an endemic equilibrium E^* which exists only when the basic reproduction number $\Re_0>1$. The trivial equilibrium E_f is stable for $\Re_0<1$, but for $\Re_0>1$ E_f is unstable and E^* exists and is stable. Here, we obtain conditions for eradication of the diseases corresponding to the exponential stability of the trivial solution, that is, we have eradication of the disease. When stability is lost we obtain conditions for persistence of the diseases. This is translated by the bifurcation phenomena.
The reproduction number $\Re_0$ can be expressed by some parameters of the model, which we take depending on the vectorial capacity of transmitting infection C between dogs. In the presence of impulse control the equilibrium E^* disappears, so we have only E_f which is exponentially stable for low vectorial capacity, that is, $C\leq\frac{\delta(\sigma+\delta)}{\alpha\sigma}$ (*i.e.* $\Re_0\leq1$) and for all $\tau_0>0$. For a high vectorial capacity $C>\frac{\delta(\sigma+\delta)}{\alpha\sigma}$ (*i.e.* $\Re_0>1$) the exponential stability of E_f is possible only for $\tau_0<\tau_0^*$ where $\tau_0^*(>0)$ is the critical period for which bifurcation of nontrivial periodic positive solutions appears. Note that τ_0^* depends on the amplitude of the control θ. We

conclude that eradication of the disease is possible for the control period τ_0 less than some critical value τ_0^*. For $\tau_0 = \tau_0^*$, however, we have a bifurcation of nontrivial periodic positive solutions which correspond to the persistence of the infectious dog population, that is, the disease is not eradicated. It is therefore recommended to be careful with the period of the culling of dogs in the case of high vectorial capacity. Following this study, we can develop new models with delays in the time between the determination of infectious dogs and culling. In addition, if latent dogs are removed from the model, we could have a more realistic model with delays. Another concern is the case where the death rate δ is different from the birth rate β. In that case, we will have a nonconstant population D which goes to zero for $\delta > \beta$ or to infinity if $\delta < \beta$. In the latter case the model must be modified to be more realistic and a logistic model could be a good suggestion.

5. Appendix

5.1. First derivatives of Φ

For all $t \in (0, \tau]$, we have $\frac{d}{dt} D_X(\Phi(t,\zeta_0)) = \frac{\partial F}{\partial X}(\zeta_0)\frac{\partial \Phi}{\partial X}(t,\zeta_0)$ with the initial condition $D_X(\Phi(0,\zeta_0)) = I_{\mathbb{R}^5}$, where

$$\frac{d}{dt} D_X(\Phi(t,\zeta_0)) = \frac{d}{dt}\begin{pmatrix} \frac{\partial \Phi_1(t,\zeta_0)}{\partial x_1} & \frac{\partial \Phi_1(t,\zeta_0)}{\partial x_2} & \frac{\partial \Phi_1(t,\zeta_0)}{\partial x_3} & \frac{\partial \Phi_1(t,\zeta_0)}{\partial x_4} & \frac{\partial \Phi_1(t,\zeta_0)}{\partial x_5} \\ \frac{\partial \Phi_2(t,\zeta_0)}{\partial x_1} & \frac{\partial \Phi_2(t,\zeta_0)}{\partial x_2} & \frac{\partial \Phi_2(t,\zeta_0)}{\partial x_3} & \frac{\partial \Phi_2(t,\zeta_0)}{\partial x_4} & \frac{\partial \Phi_2(t,\zeta_0)}{\partial x_5} \\ \frac{\partial \Phi_3(t,\zeta_0)}{\partial x_1} & \frac{\partial \Phi_3(t,\zeta_0)}{\partial x_2} & \frac{\partial \Phi_3(t,\zeta_0)}{\partial x_3} & \frac{\partial \Phi_3(t,\zeta_0)}{\partial x_4} & \frac{\partial \Phi_3(t,\zeta_0)}{\partial x_5} \\ \frac{\partial \Phi_4(t,\zeta_0)}{\partial x_1} & \frac{\partial \Phi_4(t,\zeta_0)}{\partial x_2} & \frac{\partial \Phi_4(t,\zeta_0)}{\partial x_3} & \frac{\partial \Phi_4(t,\zeta_0)}{\partial x_4} & \frac{\partial \Phi_4(t,\zeta_0)}{\partial x_5} \\ \frac{\partial \Phi_5(t,\zeta_0)}{\partial x_1} & \frac{\partial \Phi_5(t,\zeta_0)}{\partial x_2} & \frac{\partial \Phi_5(t,\zeta_0)}{\partial x_3} & \frac{\partial \Phi_5(t,\zeta_0)}{\partial x_4} & \frac{\partial \Phi_5(t,\zeta_0)}{\partial x_5} \end{pmatrix},$$

$$\begin{aligned}\frac{\partial F}{\partial X}(\zeta_0) &= \begin{pmatrix} \frac{\partial F_1(\zeta(t))}{\partial x_1} & 0 & \frac{\partial F_1(\zeta(t))}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial F_2(\zeta(t))}{\partial x_2} & \frac{\partial F_2(\zeta(t))}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial F_3(\zeta(t))}{\partial x_2} & \frac{\partial F_3(\zeta(t))}{\partial x_3} & 0 & 0 \\ 0 & 0 & \frac{\partial F_4(\zeta(t))}{\partial x_3} & \frac{\partial F_4(\zeta(t))}{\partial x_4} & 0 \\ 0 & 0 & \frac{\partial F_5(\zeta(t))}{\partial x_3} & 0 & \frac{\partial F_1(\zeta(t))}{\partial x_3} \end{pmatrix} \\ &= \begin{pmatrix} -\delta & 0 & -\alpha C & 0 & 0 \\ 0 & -(\delta+\sigma) & \alpha C & 0 & 0 \\ 0 & \sigma & -\delta & 0 & 0 \\ 0 & 0 & -(1-\alpha)C & -\delta & 0 \\ 0 & 0 & (1-\alpha)C & 0 & -\delta \end{pmatrix},\end{aligned}$$

and

$$\frac{\partial\Phi}{\partial X}(t,\zeta_0)=\begin{pmatrix}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_1} & \frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_1(t,\zeta_0)}{\partial x_3} & \frac{\partial\Phi_1(t,\zeta_0)}{\partial x_4} & \frac{\partial\Phi_1(t,\zeta_0)}{\partial x_5}\\ \frac{\partial\Phi_2(t,\zeta_0)}{\partial x_1} & \frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_2(t,\zeta_0)}{\partial x_3} & \frac{\partial\Phi_2(t,\zeta_0)}{\partial x_4} & \frac{\partial\Phi_2(t,\zeta_0)}{\partial x_5}\\ \frac{\partial\Phi_3(t,\zeta_0)}{\partial x_1} & \frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3} & \frac{\partial\Phi_3(t,\zeta_0)}{\partial x_4} & \frac{\partial\Phi_3(t,\zeta_0)}{\partial x_5}\\ \frac{\partial\Phi_4(t,\zeta_0)}{\partial x_1} & \frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_4(t,\zeta_0)}{\partial x_3} & \frac{\partial\Phi_4(t,\zeta_0)}{\partial x_4} & \frac{\partial\Phi_4(t,\zeta_0)}{\partial x_5}\\ \frac{\partial\Phi_5(t,\zeta_0)}{\partial x_1} & \frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2} & \frac{\partial\Phi_5(t,\zeta_0)}{\partial x_3} & \frac{\partial\Phi_5(t,\zeta_0)}{\partial x_4} & \frac{\partial\Phi_5(t,\zeta_0)}{\partial x_5}\end{pmatrix}.$$

From Cauchy Lipschitz theorem (uniqueness of solution) we obtain that $\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_i}=0,\ i\in\{4,\ 5\}$,
$\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_i}=0,\ i\in\{1,\ 4,\ 5\}$, $\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_i}=0,\ i\in\{1,\ 4,\ 5\}$, $\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_i}=0,\ i\in\{1,\ 5\}$ and $\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_i}=0,\ i\in\{1,\ 4\}$. Moreover, we have

$$\frac{d}{dt}\left(\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_1}\right)=\frac{\partial F_1(\zeta(t))}{\partial x_1}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_1}, \tag{5.1}$$

$$\frac{d}{dt}\left(\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}\right)=\frac{\partial F_1(\zeta(t))}{\partial x_1}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}+\frac{\partial F_1(\zeta(t))}{\partial x_3}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}, \tag{5.2}$$

$$\frac{d}{dt}\left(\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_3}\right)=\frac{\partial F_1(\zeta(t))}{\partial x_1}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_3}+\frac{\partial F_1(\zeta(t))}{\partial x_3}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3}, \tag{5.3}$$

$$\frac{d}{dt}\left(\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2}\right)=\frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}, \tag{5.4}$$

$$\frac{d}{dt}\left(\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_3}\right)=\frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_3}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3}, \tag{5.5}$$

$$\frac{d}{dt}\left(\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}\right)=\frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}, \tag{5.6}$$

$$\frac{d}{dt}\left(\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3}\right)=\frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_3}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3}, \tag{5.7}$$

$$\frac{d}{dt}\left(\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2}\right)=\frac{\partial F_4(\zeta(t))}{\partial x_3}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}+\frac{\partial F_4(\zeta(t))}{\partial x_4}\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2}, \tag{5.8}$$

$$\frac{d}{dt}\left(\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_3}\right)=\frac{\partial F_4(\zeta(t))}{\partial x_3}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3}+\frac{\partial F_4(\zeta(t))}{\partial x_4}\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_3}, \tag{5.9}$$

$$\frac{d}{dt}\left(\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_4}\right)=\frac{\partial F_4(\zeta(t))}{\partial x_4}\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_4}, \tag{5.10}$$

$$\frac{d}{dt}\left(\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2}\right)=\frac{\partial F_5(\zeta(t))}{\partial x_3}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}+\frac{\partial F_5(\zeta(t))}{\partial x_5}\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2}, \tag{5.11}$$

$$\frac{d}{dt}\left(\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_3}\right)=\frac{\partial F_5(\zeta(t))}{\partial x_3}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3}+\frac{\partial F_5(\zeta(t))}{\partial x_5}\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_3}, \tag{5.12}$$

$$\frac{d}{dt}\left(\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_5}\right)=\frac{\partial F_5(\zeta(t))}{\partial x_5}\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_5}. \tag{5.13}$$

From (5.1), (5.10) and (5.13) we obtain $\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_1}=\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_4}=\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_5}=e^{-\delta t}$.

From (5.4) and (5.6) we have $\begin{pmatrix} \frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2} \\ \frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2} \end{pmatrix} = e^{tA} \begin{pmatrix} \frac{\partial\Phi_2(0,\zeta_0)}{\partial x_2} \\ \frac{\partial\Phi_3(0,\zeta_0)}{\partial x_2} \end{pmatrix} = e^{tA} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where

$$A = \begin{pmatrix} \frac{\partial F_2(\zeta(t))}{\partial x_2} & \frac{\partial F_2(\zeta(t))}{\partial x_3} \\ \frac{\partial F_3(\zeta(t))}{\partial x_2} & \frac{\partial F_3(\zeta(t))}{\partial x_3} \end{pmatrix} = \begin{pmatrix} -\sigma-\delta & \alpha C \\ \sigma & -\delta \end{pmatrix} = PVP^{-1},$$

$$V = \begin{pmatrix} -\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2} & 0 \\ 0 & -\frac{\sigma+2\delta-\sqrt{\sigma(\sigma+4\alpha C)}}{2} \end{pmatrix},$$

$$P = \begin{pmatrix} 1 & 1 \\ \frac{\sigma-\sqrt{\sigma(\sigma+4\alpha C)}}{2\alpha C} & \frac{\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2\alpha C} \end{pmatrix},\ P^{-1} = \begin{pmatrix} \frac{\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2\sqrt{\sigma(\sigma+4\alpha C)}} & \frac{-\alpha C}{\sqrt{\sigma(\sigma+4\alpha C)}} \\ \frac{-\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2\sqrt{\sigma(\sigma+4\alpha C)}} & \frac{\alpha C}{\sqrt{\sigma(\sigma+4\alpha C)}} \end{pmatrix},$$

and

$$\begin{aligned} e^{tA} &= Pe^{tV}P^{-1} = P \begin{pmatrix} e^{-\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t} & 0 \\ 0 & e^{-\frac{\sigma+2\delta-\sqrt{\sigma(\sigma+4\alpha C)}}{2}t} \end{pmatrix} P^{-1} \\ &= e^{-\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t} \begin{pmatrix} k_1(t) & k_2(t) \\ k_3(t) & k_4(t) \end{pmatrix}, \end{aligned}$$

$k_1(t) = \frac{\sigma+\sqrt{\sigma(\sigma+4\alpha C)}+(-\sigma+\sqrt{\sigma(\sigma+4\alpha C)})e^{\sqrt{\sigma(\sigma+4\alpha C)}t}}{2\sqrt{\sigma(\sigma+4\alpha C)}}$, $k_2(t) = \frac{-\alpha C+\alpha Ce^{\sqrt{\sigma(\sigma+4\alpha C)}t}}{\sqrt{\sigma(\sigma+4\alpha C)}}$,
$k_3(t) = \frac{-\sigma+\sigma e^{\sqrt{\sigma(\sigma+4\alpha C)}t}}{\sqrt{\sigma(\sigma+4\alpha C)}}$ and $k_4(t) = \frac{-\sigma+\sqrt{\sigma(\sigma+4\alpha C)}+(\sigma+\sqrt{\sigma(\sigma+4\alpha C)})e^{\sqrt{\sigma(\sigma+4\alpha C)}}}{2\sqrt{\sigma(\sigma+4\alpha C)}}$.
We obtain

$$\begin{cases} \frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2} = e^{-\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t} \left(\frac{\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2\sqrt{\sigma(\sigma+4\alpha C)}} + \frac{(-\sigma+\sqrt{\sigma(\sigma+4\alpha C)})e^{\sqrt{\sigma(\sigma+4\alpha C)}t}}{2\sqrt{\sigma(\sigma+4\alpha C)}} \right), \\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_2} = e^{-\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t} \left(\frac{-\sigma}{\sqrt{\sigma(\sigma+4\alpha C)}} + \frac{\sigma e^{\sqrt{\sigma(\sigma+4\alpha C)}t}}{\sqrt{\sigma(\sigma+4\alpha C)}} \right). \end{cases}$$

From (5.5) and (5.7) we have $\begin{pmatrix} \frac{\partial\Phi_2(t,\zeta_0)}{\partial x_3} \\ \frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3} \end{pmatrix} = e^{tA} \begin{pmatrix} \frac{\partial\Phi_2(0,\zeta_0)}{\partial x_3} \\ \frac{\partial\Phi_3(0,\zeta_0)}{\partial x_3} \end{pmatrix} = e^{tA} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We obtain

$$\begin{cases} \frac{\partial\Phi_2(t,\zeta_0)}{\partial x_3} = e^{-\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t} \left(\frac{-\alpha C}{\sqrt{\sigma(\sigma+4\alpha C)}} + \frac{\alpha Ce^{\sqrt{\sigma(\sigma+4\alpha C)}t}}{\sqrt{\sigma(\sigma+4\alpha C)}} \right), \\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3} = e^{-\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t} \left(\frac{-\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2\sqrt{\sigma(\sigma+4\alpha C)}} + \frac{(\sigma+\sqrt{\sigma(\sigma+4\alpha C)})e^{\sqrt{\sigma(\sigma+4\alpha C)}}}{2\sqrt{\sigma(\sigma+4\alpha C)}} \right). \end{cases}$$

From (5.2) we have

$$\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2} = e^{\frac{\partial F_1(\zeta(t))}{\partial x_1}t} \frac{\partial\Phi_1(0,\zeta_0)}{\partial x_2} + \int_0^t e^{\frac{\partial F_1(\zeta(t))}{\partial x_1}(t-s)} \frac{\partial F_1(\zeta(s))}{\partial x_3} \frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2} ds,$$

then

$$\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2} = e^{-\delta t}\left(\frac{-2\alpha^2C^2(e^{\frac{-\sigma-\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}-1)}{\sqrt{\sigma(\sigma+4\alpha C)}(\sigma+\sqrt{\sigma(\sigma+4\alpha C)})} + \frac{2\alpha^2C^2(e^{\frac{-\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}-1)}{\sqrt{\sigma(\sigma+4\alpha C)}(\sigma-\sqrt{\sigma(\sigma+4\alpha C)})}\right).$$

From (5.3) we have

$$\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_3} = e^{\frac{\partial F_1(\zeta(t))}{\partial x_1}t}\frac{\partial\Phi_1(0,\zeta_0)}{\partial x_3} + \int_0^t e^{\frac{\partial F_1(\zeta(t))}{\partial x_1}(t-s)}\frac{\partial F_1(\zeta(s))}{\partial x_3}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}ds,$$

then

$$\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_3} = \alpha Ce^{-\delta t}\left(\frac{-(\sigma-\sqrt{\sigma(\sigma+4\alpha C)})(e^{\frac{-\sigma-\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}-1)}{\sqrt{\sigma(\sigma+4\alpha C)}(\sigma+\sqrt{\sigma(\sigma+4\alpha C)})}\right.$$
$$\left. + \frac{(\sigma+\sqrt{\sigma(\sigma+4\alpha C)})(e^{\frac{-\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}-1)}{\sqrt{\sigma(\sigma+4\alpha C)}(\sigma-\sqrt{\sigma(\sigma+4\alpha C)})}\right).$$

From (5.8) we have

$$\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2} = e^{\frac{\partial F_4(\zeta(t))}{\partial x_4}t}\frac{\partial\Phi_4(0,\zeta_0)}{\partial x_2} + \int_0^t e^{\frac{\partial F_4(\zeta(t))}{\partial x_4}(t-s)}\frac{\partial F_4(\zeta(s))}{\partial x_3}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}ds,$$

then

$$\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2} = e^{-\delta t}\left(\frac{-\alpha(1-\alpha)C^2(e^{\frac{-\sigma-\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}-1)}{\sqrt{\sigma(\sigma+4\alpha C)}(\sigma+\sqrt{\sigma(\sigma+4\alpha C)})} + \frac{\alpha(1-\alpha)C^2(e^{\frac{-\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}-1)}{\sqrt{\sigma(\sigma+4\alpha C)}(\sigma-\sqrt{\sigma(\sigma+4\alpha C)})}\right).$$

From (5.9) we have

$$\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_3} = e^{\frac{\partial F_4(\zeta(t))}{\partial x_4}t}\frac{\partial\Phi_4(0,\zeta_0)}{\partial x_3} + \int_0^t e^{\frac{\partial F_4(\zeta(t))}{\partial x_4}(t-s)}\frac{\partial F_4(\zeta(s))}{\partial x_3}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}ds,$$

then

$$\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_3} = -(1-\alpha)Ce^{-\delta t}\left(\frac{(\sigma-\sqrt{\sigma(\sigma+4\alpha C)})(e^{\frac{-\sigma-\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}-1)}{2\sqrt{\sigma(\sigma+4\alpha C)}(\sigma+\sqrt{\sigma(\sigma+4\alpha C)})}\right.$$
$$\left. - \frac{(\sigma+\sqrt{\sigma(\sigma+4\alpha C)})(e^{\frac{-\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}-1)}{2\sqrt{\sigma(\sigma+4\alpha C)}(\sigma-\sqrt{\sigma(\sigma+4\alpha C)})}\right).$$

From (5.11) we have

$$\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2} = e^{\frac{\partial F_5(\zeta(t))}{\partial x_5}t}\frac{\partial\Phi_5(0,\zeta_0)}{\partial x_2} + \int_0^t e^{\frac{\partial F_5(\zeta(t))}{\partial x_5}(t-s)}\frac{\partial F_5(\zeta(s))}{\partial x_3}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}ds,$$

then

$$\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2} = e^{-\delta t}\left(\frac{\alpha(1-\alpha)C^2(e^{\frac{-\sigma-\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}-1)}{2\sqrt{\sigma(\sigma+4\alpha C)}(\sigma+\sqrt{\sigma(\sigma+4\alpha C)})} - \frac{\alpha(1-\alpha)C^2(e^{\frac{-\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}-1)}{2\sqrt{\sigma(\sigma+4\alpha C)}(\sigma-\sqrt{\sigma(\sigma+4\alpha C)})}\right).$$

From (5.12) we have

$$\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_3} = e^{\frac{\partial F_5(\zeta(t))}{\partial x_5}t}\frac{\partial\Phi_5(0,\zeta_0)}{\partial x_3} + \int_0^t e^{\frac{\partial F_5(\zeta(t))}{\partial x_5}(t-s)}\frac{\partial F_5(\zeta(s))}{\partial x_3}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}ds,$$

then

$$\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_3} = (1-\alpha)Ce^{-\delta t}\left(\frac{(\sigma-\sqrt{\sigma(\sigma+4\alpha C)})(e^{\frac{-\sigma-\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}-1)}{2\sqrt{\sigma(\sigma+4\alpha C)}(\sigma+\sqrt{\sigma(\sigma+4\alpha C)})} - \frac{(\sigma+\sqrt{\sigma(\sigma+4\alpha C)})(e^{\frac{-\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}-1)}{2\sqrt{\sigma(\sigma+4\alpha C)}(\sigma-\sqrt{\sigma(\sigma+4\alpha C)})}\right).$$

5.2. Second derivatives of Φ_2 and Φ_3

The second partial derivatives of Φ_2 and Φ_3 can be obtained from the following differential equations

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_1}\right) &= \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_3\partial x_1} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_1}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_1} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_1}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_1} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_1}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_1} = 0$, then

$$\begin{aligned}\text{(5.14)}\quad \frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_1}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_1} + \frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_1}\\ &+\frac{\partial^2 F_2(\zeta(t))}{\partial x_3\partial x_1}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_1}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_1} = 0$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_1}\right) &= \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_3\partial x_1} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_1}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_1} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_1}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_1} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_1}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_1} = 0$, then

$$\text{(5.15)}\qquad \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_1}\right) = \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_1} + \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_1}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_1}=0$. From (5.14)-(5.15) we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_1}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_1}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_1}\\
&+\frac{\partial^2 F_2(\zeta(t))}{\partial x_3\partial x_1}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_1}\\
\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_1}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_1}+\frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_1}
\end{aligned}$$

with the initial condition
$\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_1}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_1}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.
Then

$$\begin{pmatrix}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_1}\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_1}\end{pmatrix}=e^{tA}\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_1}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_1}\end{pmatrix}+\int_0^t e^{(t-s)A}\begin{pmatrix}\frac{\partial^2 F_2(\zeta(s))}{\partial x_3\partial x_1}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_1}\\ 0\end{pmatrix}ds.$$

where

$$e^{tA}=e^{-\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}t}\begin{pmatrix}l_1(t) & l_2(t)\\ l_3(t) & l_4(t)\end{pmatrix},$$

$l_1(t)=\frac{\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}+\frac{(-\sigma+\sqrt{\sigma(\sigma+4\alpha C)})e^{\sqrt{\sigma(\sigma+4\alpha C)}t}}{2\sqrt{\sigma(\sigma+4\alpha C)}}$,
$l_2(t)=\frac{-\alpha C}{\sqrt{\sigma(\sigma+4\alpha C)}}+\frac{\alpha C e^{\sqrt{\sigma(\sigma+4\alpha C)}t}}{\sqrt{\sigma(\sigma+4\alpha C)}}$,
$l_3(t)=\frac{-\sigma}{\sqrt{\sigma(\sigma+4\alpha C)}}+\frac{\sigma e^{\sqrt{\sigma(\sigma+4\alpha C)}t}}{\sqrt{\sigma(\sigma+4\alpha C)}}$,
$l_4(t)=\frac{-\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}+\frac{(\sigma+\sqrt{\sigma(\sigma+4\alpha C)})e^{\sqrt{\sigma(\sigma+4\alpha C)}}}{2\sqrt{\sigma(\sigma+4\alpha C)}}$.
We obtain

$$\begin{cases}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_1} = \frac{C}{D}\int_0^t e^{\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}(s-t)}e^{-\delta s}l_1(t-s)\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}ds\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_1} = \frac{C}{D}\int_0^t e^{\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}(s-t)}e^{-\delta s}l_3(t-s)\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}ds.\end{cases}$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_2}\right) &= \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_3\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}\\
&+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}\\
&+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2}
\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_2}=0$, then

$$\begin{aligned}(5.16)\quad \frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_2}\\ &+\frac{\partial^2 F_2(\zeta(t))}{\partial x_3\partial x_1}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}\\ &+\frac{\partial^2 F_2(\zeta(t))}{\partial x_3\partial x_1}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_3}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_2}=0$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_2}\right) &= \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_3\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_2}=0$, then

$$(5.17)\qquad \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_2}\right)=\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_2}+\frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_2}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_2}=0$. From (5.16)-(5.17) we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_2}\\ &+\frac{\partial^2 F_2(\zeta(t))}{\partial x_3\partial x_1}\left(\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}+\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_3}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}\right)\\ \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_2}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_2}+\frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_2}\end{aligned}$$

with the initial condition
$\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_2}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.
Then

$$\begin{aligned}\begin{pmatrix}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_2}\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_2}\end{pmatrix} &= e^{tA}\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_2}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_2}\end{pmatrix}\\ &+\int_0^t e^{(t-s)A}\begin{pmatrix}\frac{\partial^2 F_2(\zeta(s))}{\partial x_3\partial x_1}\left(\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_2}+\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}\right)\\ 0\end{pmatrix}ds.\end{aligned}$$

We obtain

$$\begin{cases} \frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_2} = \frac{C}{D}\int_0^t e^{\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}(s-t)}\left(\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_2}+\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}\right) \\ \qquad\qquad \times\left(\frac{\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}+\frac{(-\sigma+\sqrt{\sigma(\sigma+4\alpha C)})e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right)ds \\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_2} = \frac{C}{D}\int_0^t e^{\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}(s-t)}\left(\frac{-\sigma}{\sqrt{\sigma(\sigma+4\alpha C)}}+\frac{\sigma e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{\sqrt{\sigma(\sigma+4\alpha C)}}\right) \\ \qquad\qquad \times\left(\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_2}+\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}\right)ds. \end{cases}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3^2}\right) &= \sum_{i=1}^5\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_3^2}+\left(\sum_{i=1}^5\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_3} \\ &+\left(\sum_{i=1}^5\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_3}+\left(\sum_{i=1}^5\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3} \\ &+\left(\sum_{i=1}^5\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_3}+\left(\sum_{i=1}^5\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_3} \end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3^2}=0$, then

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3^2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3^2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3^2} \\ &\quad +2\frac{\partial^2F_2(\zeta(t))}{\partial x_3\partial x_1}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_3} \end{aligned} \tag{5.18}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3^2}=0$.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3^2}\right) &= \sum_{i=1}^5\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_3^2}+\left(\sum_{i=1}^5\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_3} \\ &+\left(\sum_{i=1}^5\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_3}+\left(\sum_{i=1}^5\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3} \\ &+\left(\sum_{i=1}^5\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_3}+\left(\sum_{i=1}^5\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_3} \end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3^2}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3^2}\right)=\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3^2}+\frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3^2} \tag{5.19}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3^2}=0$. From (5.18)-(5.19) we have

$$\begin{aligned}
(5.20)\quad \frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3^2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3^2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3^2}\\
&\quad +2\frac{\partial^2 F_2(\zeta(t))}{\partial x_3\partial x_1}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_3}\\
\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3^2}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3^2}+\frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3^2}
\end{aligned}$$

with the initial condition
$\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3^2}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3^2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.
Then

$$\begin{aligned}
\begin{pmatrix}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3^2}\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3^2}\end{pmatrix} &= e^{tA}\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3^2}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3^2}\end{pmatrix}\\
&\quad +\int_0^t e^{(t-s)A}\begin{pmatrix}2\frac{\partial^2 F_2(\zeta(s))}{\partial x_3\partial x_1}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_3}\\ 0\end{pmatrix}ds.
\end{aligned}$$

We obtain

$$\left\{\begin{aligned}
\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3^2} &= \frac{C}{D}\int_0^t e^{\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}(s-t)}\left(2\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_3}\right)\\
&\quad\times\left(\frac{\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}+\frac{(-\sigma+\sqrt{\sigma(\sigma+4\alpha C)})e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right)ds\\
\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3^2} &= \frac{C}{D}\int_0^t e^{\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}(s-t)}\left(2\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_3}\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_3}\right)\\
&\quad\times\left(\frac{-\sigma}{\sqrt{\sigma(\sigma+4\alpha C)}}+\frac{\sigma e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{\sqrt{\sigma(\sigma+4\alpha C)}}\right)ds.
\end{aligned}\right.$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_4}\right) &= \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_3\partial x_4}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_4}\\
&\quad+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_4}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_4}\\
&\quad+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_4}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_4}
\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_4}=0$, then

$$(5.21)\qquad \frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_4}\right)=\frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_4}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_4}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_4} = 0$.

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_4}\right) = \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_3\partial x_4} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_4}$$
$$+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_4} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_4}$$
$$+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_4} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_4}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_4} = 0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_4}\right) = \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_4} + \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_4} \tag{5.22}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_4} = 0$. From (5.21)-(5.22) we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_4}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_4} + \frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_4}\\
\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_4}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_4} + \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_4}
\end{aligned}$$

with the initial condition
$$\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_4}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_4}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$
Then
$$\begin{pmatrix}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_4}\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_4}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_5}\right) = \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_3\partial x_5} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_5}$$
$$+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_5} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_5}$$
$$+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_5} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_5}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_5} = 0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_5}\right) = \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_5} + \frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_5} \tag{5.23}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_5}=0$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_5}\right) &= \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_3\partial x_5}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_5}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_5}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_5}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_5}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_3}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_5}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_5}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_5}\right)=\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_5}+\frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_5}\tag{5.24}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_5}=0$. From (5.23)-(5.24) we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_5}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_5}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_5}\\ \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_5}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_5}+\frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_5}\end{aligned}$$

with the initial condition
$\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_3\partial x_5}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_3\partial x_5}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.
Then
$\begin{pmatrix}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_3\partial x_5}\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_3\partial x_5}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1^2}\right) &= \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_1^2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_1}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_1}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_1}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_1}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_1}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1^2}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1^2}\right)=\frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1^2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1^2}\tag{5.25}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1^2}=0$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1^2}\right) &= \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_1^2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_1}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_1}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_1}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_1}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_1}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1^2}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1^2}\right) = \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1^2}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1^2} \tag{5.26}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1^2}=0$. From (5.25)-(5.26) we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1^2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1^2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1^2}\\ \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1^2}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1^2}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1^2}\end{aligned}$$

with the initial condition

$$\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1^2}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1^2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Then

$$\begin{pmatrix}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1^2}\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1^2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_2}\right) &= \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_1\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1\partial x_2}=0$, then

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_2}\\ &+\frac{\partial^2 F_2(\zeta(t))}{\partial x_1\partial x_3}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_1}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}\end{aligned} \tag{5.27}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1\partial x_2}=0$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_2}\right) &= \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_1\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1\partial x_2}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_2}\right) = \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_2}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_2} \tag{5.28}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1\partial x_2}=0$. From (5.27)-(5.28) we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_2}\\ &\quad+\frac{\partial^2F_2(\zeta(t))}{\partial x_1\partial x_3}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_1}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}\\ \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_2}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_2}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_2}\end{aligned}$$

with the initial condition
$\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1\partial x_2}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1\partial x_2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.
Then

$$\begin{aligned}\begin{pmatrix}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_2}\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_2}\end{pmatrix} &= e^{tA}\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1\partial x_2}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1\partial x_2}\end{pmatrix}\\ &\quad+\int_0^t e^{(t-s)A}\begin{pmatrix}\frac{\partial^2F_2(\zeta(s))}{\partial x_3\partial x_1}\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_1}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}\\ 0\end{pmatrix}ds.\end{aligned}$$

We obtain

$$\begin{cases}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_2} = \frac{C}{D}\int_0^t e^{\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}(s-t)}\left(\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_1}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}\right)\\ \qquad\times\left(\frac{\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}+\frac{(-\sigma+\sqrt{\sigma(\sigma+4\alpha C)})e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right)ds\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_2} = \frac{C}{D}\int_0^t e^{\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}(s-t)}\left(\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_1}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}\right)\\ \qquad\times\left(\frac{-\sigma}{\sqrt{\sigma(\sigma+4\alpha C)}}+\frac{\sigma e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{\sqrt{\sigma(\sigma+4\alpha C)}}\right)ds.\end{cases}$$

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_4}\right) = \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_1\partial x_4} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_4}$$
$$+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_4} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_4}$$
$$+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_4} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_4}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1\partial x_4} = 0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_4}\right) = \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_4} + \frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_4} \tag{5.29}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1\partial x_4} = 0$.

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_4}\right) = \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_1\partial x_4} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_4}$$
$$+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_4} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_4}$$
$$+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_4} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_4}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1\partial x_4} = 0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_4}\right) = \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_4} + \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_4} \tag{5.30}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1\partial x_4} = 0$. From (5.29)-(5.30) we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_4}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_4} + \frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_4}\\ \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_4}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_4} + \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_4}\end{aligned}$$

with the initial condition
$\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1\partial x_4}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1\partial x_4}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.
Then

$$\begin{pmatrix} \frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_4} \\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_5}\right) &= \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_1\partial x_5} + \left(\sum_{i=1}^{5} + \frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_5} \\ &+ \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_5} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_5} \\ &+ \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_5} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_5} \end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1\partial x_5} = 0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_5}\right) = \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_5} + \frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_5} \tag{5.31}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1\partial x_5} = 0$.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_5}\right) &= \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_1\partial x_5} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_5} \\ &+ \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_5} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_5} \\ &+ \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_5} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_1}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_5} \end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1\partial x_5} = 0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_5}\right) = \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_5} + \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_5} \tag{5.32}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1\partial x_5} = 0$. From (5.31)-(5.32) we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_5}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_5} + \frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_5} \\ \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_5}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_5} + \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_5} \end{aligned}$$

with the initial condition

$$\begin{pmatrix} \frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_1\partial x_5} \\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_1\partial x_5} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} \frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_1\partial x_5} \\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_1\partial x_5} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_2^2}\right) &= \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_2^2} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_2}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2} \\ &+ \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_2}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_2}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2} \\ &+ \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_2}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_2}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2} \end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_2^2} = 0$, then

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_2^2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_2^2} + \frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_2^2} \\ &+2\frac{\partial^2 F_2(\zeta(t))}{\partial x_1\partial x_3}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2} \end{aligned} \tag{5.33}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_2^2} = 0$.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_2^2}\right) &= \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_2^2} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_2}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2} \\ &+\sum_{i=1}^{5}\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_2}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_2}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2} \\ &+ \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_2}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2} + \left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_2}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2} \end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_2^2} = 0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_2^2}\right) = \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_2^2} + \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_2^2} \tag{5.34}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_2^2} = 0$. From (5.33)-(5.34) we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_2^2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_2^2} + \frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_2^2} \\ &+2\frac{\partial^2 F_2(\zeta(t))}{\partial x_1\partial x_3}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2} \\ \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_2^2}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_2^2} + \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_2^2} \end{aligned} \tag{5.35}$$

with the initial condition
$\begin{pmatrix} \frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_2^2} \\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$
Then

$$\begin{pmatrix} \frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_2^2} \\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_2^2} \end{pmatrix} = e^{tA}\begin{pmatrix} \frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_2^2} \\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_2^2} \end{pmatrix} + \int_0^t e^{(t-s)A}\begin{pmatrix} 2\frac{\partial^2 F_2(\zeta(t))}{\partial x_1\partial x_3}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2} \\ 0 \end{pmatrix} ds.$$

We obtain

$$\begin{cases} \frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_2^2} = \frac{C}{D}\int_0^t e^{\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}(s-t)}\left(2\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_2}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}\right) \\ \qquad\times\left(\frac{\sigma+\sqrt{\sigma(\sigma+4\alpha C)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}+\frac{(-\sigma+\sqrt{\sigma(\sigma+4\alpha C)})e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{2\sqrt{\sigma(\sigma+4\alpha C)}}\right)ds \\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_2^2} = \frac{C}{D}\int_0^t e^{\frac{\sigma+2\delta+\sqrt{\sigma(\sigma+4\alpha C)}}{2}(s-t)}\left(2\frac{\partial\Phi_1(s,\zeta_0)}{\partial x_2}\frac{\partial\Phi_3(s,\zeta_0)}{\partial x_2}\right) \\ \qquad\times\left(\frac{-\sigma}{\sqrt{\sigma(\sigma+4\alpha C)}}+\frac{\sigma e^{-\sqrt{\sigma(\sigma+4\alpha C)}(s-t)}}{\sqrt{\sigma(\sigma+4\alpha C)}}\right)ds. \end{cases}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4\partial x_2}\right) &= \sum_{i=1}^5\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_4\partial x_2}+\left(\sum_{i=1}^5\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2} \\ &+\left(\sum_{i=1}^5\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^5\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2} \\ &+\left(\sum_{i=1}^5\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^5\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2} \end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_4\partial x_2} = 0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4\partial x_2}\right) = \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4\partial x_2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4\partial x_2} \tag{5.36}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_4\partial x_2} = 0$.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4\partial x_2}\right) &= \sum_{i=1}^5\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_4\partial x_2}+\left(\sum_{i=1}^5\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_2} \\ &+\left(\sum_{i=1}^5\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^5\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2} \\ &+\left(\sum_{i=1}^5\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^5\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2} \end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_4\partial x_2}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4\partial x_2}\right) = \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4\partial x_2}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4\partial x_2} \tag{5.37}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_4\partial x_2}=0$. From (5.36)-(5.37) we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4\partial x_2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4\partial x_2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4\partial x_2}\\ \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4\partial x_2}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4\partial x_2}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4\partial x_2}\end{aligned}$$

with the initial condition
$\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_4\partial x_2}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_4\partial x_2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.
Then
$\begin{pmatrix}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4\partial x_2}\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4\partial x_2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_2}\right) &= \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_5\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_5\partial x_2}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_2}\right)=\frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_2} \tag{5.38}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_5\partial x_2}=0$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_2}\right) &= \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_5\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_2}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_2}+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_2}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_2}++\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_1\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_2}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_5\partial x_2}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_2}\right) = \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_2}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_2} \tag{5.39}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_5\partial x_2}=0$. From (5.38)-(5.39) we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_2}\\ \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_2}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_2}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_2}\end{aligned}$$

with the initial condition
$\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_5\partial x_2}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_5\partial x_2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.
Then
$\begin{pmatrix}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_2}\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4^2}\right) &= \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_4^2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_4}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_4}++\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_4}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_4}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_4}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_4^2}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4^2}\right)=\frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4^2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4^2}\tag{5.40}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_4^2}=0$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4^2}\right) &= \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_4^2}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_4}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_1\partial x_2}\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_4}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_4}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_4}++\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_4}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_4}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_4^2}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4^2}\right)=\frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4^2}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4^2}\tag{5.41}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_4^2}=0$. From (5.42)-(5.43) we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4^2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4^2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4^2}\\
\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4^2}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4^2}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4^2}
\end{aligned}$$

with the initial condition
$\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_4^2}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_4^2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.
Then
$\begin{pmatrix}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_4^2}\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_4^2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_4}\right) &= \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_5\partial x_4}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_4}\\
&+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_4}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_4}\\
&+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_4}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_4}
\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_5\partial x_4}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_4}\right)=\frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_4}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_4}\tag{5.42}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_5\partial x_4}=0$.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_4}\right) &= \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_5\partial x_4}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_4}\\
&+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_4}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_4}\\
&+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_4}+\left(\sum_{i=1}^{5}\frac{\partial^2 F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_4}
\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_5\partial x_4}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_4}\right)=\frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_4}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_4}\tag{5.43}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_5\partial x_4}=0$. From (5.42)-(5.43) we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_4}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_4}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_4}\\ \frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_4}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_4}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_4}\end{aligned}$$

with the initial condition
$\begin{pmatrix}\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_5\partial x_4}\\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_5\partial x_4}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.
Then
$\begin{pmatrix}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5\partial x_4}\\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5\partial x_4}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5^2}\right) &= \sum_{i=1}^{5}\frac{\partial F_2(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_5^2}+\left(\sum_{i=1}^{5}\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_5}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_5}+\left(\sum_{i=1}^{5}\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_5}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_5}+\left(\sum_{i=1}^{5}\frac{\partial^2F_2(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_5}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_5^2}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5^2}\right)=\frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5^2}+\frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5^2}\tag{5.44}$$

with the initial condition $\frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_5^2}=0$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5^2}\right) &= \sum_{i=1}^{5}\frac{\partial F_3(\zeta(t))}{\partial x_i}\frac{\partial^2\Phi_i(t,\zeta_0)}{\partial x_5^2}+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_1}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_1(t,\zeta_0)}{\partial x_5}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_2}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_2(t,\zeta_0)}{\partial x_5}+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_3}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_3(t,\zeta_0)}{\partial x_5}\\ &+\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_4}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_4(t,\zeta_0)}{\partial x_5}++\left(\sum_{i=1}^{5}\frac{\partial^2F_3(\zeta(t))}{\partial x_i\partial x_5}\frac{\partial\Phi_i(t,\zeta_0)}{\partial x_5}\right)\frac{\partial\Phi_5(t,\zeta_0)}{\partial x_5}\end{aligned}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_5^2}=0$, then

$$\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5^2}\right)=\frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5^2}+\frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5^2}\tag{5.45}$$

with the initial condition $\frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_5^2} = 0$. From (5.44)-(5.45) we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5^2}\right) &= \frac{\partial F_2(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5^2} + \frac{\partial F_2(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5^2} \\
\frac{d}{dt}\left(\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5^2}\right) &= \frac{\partial F_3(\zeta(t))}{\partial x_2}\frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5^2} + \frac{\partial F_3(\zeta(t))}{\partial x_3}\frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5^2}
\end{aligned}$$

with the initial condition $\begin{pmatrix} \frac{\partial^2\Phi_2(0,\zeta_0)}{\partial x_5^2} \\ \frac{\partial^2\Phi_3(0,\zeta_0)}{\partial x_5^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then $\begin{pmatrix} \frac{\partial^2\Phi_2(t,\zeta_0)}{\partial x_5^2} \\ \frac{\partial^2\Phi_3(t,\zeta_0)}{\partial x_5^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

5.3. First derivatives of Z^*

Let $\eta(\bar\tau) = \tau_0 + \bar\tau$, $\eta_1(\bar\tau,\gamma) = \alpha D + q_1\gamma + z_1^*(\bar\tau,\gamma)$, $\eta_2(\bar\tau,\gamma) = q_2\gamma + z_2^*(\bar\tau,\gamma)$, $\eta_3(\bar\tau,\gamma) = \gamma$, $\eta_4(\bar\tau,\gamma) = (1-\alpha)D + q_3\gamma + z_4^*(\bar\tau,\gamma)$ and $\eta_5(\bar\tau,\gamma) = q_4\gamma + z_5^*(\bar\tau,\gamma)$. From (3.6) we have

$$\left\{\begin{array}{l}
\frac{\partial}{\partial\bar\tau}(\eta_1 - \Theta_1\circ\Phi(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))(0,0) = 0, \\
\frac{\partial}{\partial\bar\tau}(\eta_2 - \Theta_2\circ\Phi(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))(0,0) = 0, \\
\frac{\partial}{\partial\bar\tau}(\eta_4 - \Theta_4\circ\Phi(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))(0,0) = 0, \\
\frac{\partial}{\partial\bar\tau}(\eta_5 - \Theta_5\circ\Phi(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))(0,0) = 0.
\end{array}\right.$$

Therefore

$$\left\{\begin{array}{l}
\frac{\partial z_1^*(0,0)}{\partial\bar\tau} - \sum_{i=1}^{5}\frac{\partial\Theta_1(\Phi(\tau_0,\zeta_0))}{\partial x_i}\left(\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial\bar\tau} + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau} + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\right. \\
\qquad\left. + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_4}\frac{\partial z_4^*(0,0)}{\partial\bar\tau} + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_5}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\right) = 0, \\
\frac{\partial z_2^*(0,0)}{\partial\bar\tau} - \sum_{i=1}^{5}\frac{\partial\Theta_2(\Phi(\tau_0,\zeta_0))}{\partial x_i}\left(\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial\bar\tau} + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau} + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\right. \\
\qquad\left. + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_4}\frac{\partial z_4^*(0,0)}{\partial\bar\tau} + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_5}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\right) = 0, \\
\frac{\partial z_4^*(0,0)}{\partial\bar\tau} - \sum_{i=1}^{5}\frac{\partial\Theta_4(\Phi(\tau_0,\zeta_0))}{\partial x_i}\left(\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial\bar\tau} + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau} + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\right. \\
\qquad\left. + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_4}\frac{\partial z_4^*(0,0)}{\partial\bar\tau} + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_5}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\right) = 0, \\
\frac{\partial z_5^*(0,0)}{\partial\bar\tau} - \sum_{i=1}^{5}\frac{\partial\Theta_5(\Phi(\tau_0,\zeta_0))}{\partial x_i}\left(\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial\bar\tau} + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau} + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\right. \\
\qquad\left. + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_4}\frac{\partial z_4^*(0,0)}{\partial\bar\tau} + \frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_5}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\right) = 0.
\end{array}\right.$$

We obtain

$$\left\{\begin{array}{l}
a_0\frac{\partial z_1^*(0,0)}{\partial\bar\tau} + b_0\frac{\partial z_2^*(0,0)}{\partial\bar\tau} = 0, \\
d_0\frac{\partial z_2^*(0,0)}{\partial\bar\tau} = 0, \\
j_0\frac{\partial z_4^*(0,0)}{\partial\bar\tau} + h_0\frac{\partial z_2^*(0,0)}{\partial\bar\tau} = 0, \\
m_0\frac{\partial z_5^*(0,0)}{\partial\bar\tau} + k_0\frac{\partial z_2^*(0,0)}{\partial\bar\tau} = 0.
\end{array}\right.$$

That is

$$(5.46)\qquad \begin{cases} \frac{\partial z_1^*(0,0)}{\partial \bar{\tau}} = 0, \\ \frac{\partial z_2^*(0,0)}{\partial \bar{\tau}} = 0, \\ \frac{\partial z_4^*(0,0)}{\partial \bar{\tau}} = 0, \\ \frac{\partial z_5^*(0,0)}{\partial \bar{\tau}} = 0. \end{cases}$$

In the same way as above, we obtain

$$\begin{cases} \frac{\partial}{\partial \gamma}(\eta_1 - \Theta_1 \circ \Phi(\eta, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5))(0,0) = 0, \\ \frac{\partial}{\partial \gamma}(\eta_2 - \Theta_2 \circ \Phi(\eta, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5))(0,0) = 0, \\ \frac{\partial}{\partial \gamma}(\eta_4 - \Theta_4 \circ \Phi(\eta, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5))(0,0) = 0, \\ \frac{\partial}{\partial \gamma}(\eta_5 - \Theta_5 \circ \Phi(\eta, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5))(0,0) = 0. \end{cases}$$

Therefore

$$\begin{cases} \left(\frac{d_0'j_0'}{c_0'a_0'} - \frac{b_0'}{a_0'} + \frac{\partial z_1^*(0,0)}{\partial \gamma}\right) - \sum_{i=1}^{5} \frac{\partial \Theta_1(\Phi(\tau_0,\zeta_0))}{\partial x_i} \left\{ \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_1} \left(\frac{d_0'j_0'}{c_0'a_0'} - \frac{b_0'}{a_0'} + \frac{\partial z_1^*(0,0)}{\partial \gamma}\right) \right. \\ + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_2} \left(-\frac{d_0'}{c_0'} + \frac{\partial z_2^*(0,0)}{\partial \gamma}\right) + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_3} + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_4} \left(\frac{d_0'l_0'}{c_0'g_0'} - \frac{f_0'}{g_0'} + \frac{\partial z_4^*(0,0)}{\partial \gamma}\right) \\ \left. + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_5} \left(\frac{d_0'm_0'}{c_0'i_0'} - \frac{h_0'}{i_0'} + \frac{\partial z_5^*(0,0)}{\partial \gamma}\right) \right\} = 0, \\ \left(-\frac{d_0'}{c_0'} + \frac{\partial z_2^*(0,0)}{\partial \gamma}\right) - \sum_{i=1}^{5} \frac{\partial \Theta_2(\Phi(\tau_0,\zeta_0))}{\partial x_i} \left\{ \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_1} \left(\frac{d_0'j_0'}{c_0'a_0'} - \frac{b_0'}{a_0'} + \frac{\partial z_1^*(0,0)}{\partial \gamma}\right) \right. \\ + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_2} \left(-\frac{d_0'}{c_0'} + \frac{\partial z_2^*(0,0)}{\partial \gamma}\right) + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_3} + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_4} \left(\frac{d_0'l_0'}{c_0'g_0'} - \frac{f_0'}{g_0'} + \frac{\partial z_4^*(0,0)}{\partial \gamma}\right) \\ \left. + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_5} \left(\frac{d_0'm_0'}{c_0'i_0'} - \frac{h_0'}{i_0'} + \frac{\partial z_5^*(0,0)}{\partial \gamma}\right) \right\} = 0, \\ \left(\frac{d_0'l_0'}{c_0'g_0'} - \frac{f_0'}{g_0'} + \frac{\partial z_4^*(0,0)}{\partial \gamma}\right) - \sum_{i=1}^{5} \frac{\partial \Theta_4(\Phi(\tau_0,\zeta_0))}{\partial x_i} \left\{ \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_1} \left(\frac{d_0'j_0'}{c_0'a_0'} - \frac{b_0'}{a_0'} + \frac{\partial z_1^*(0,0)}{\partial \gamma}\right) \right. \\ + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_2} \left(-\frac{d_0'}{c_0'} + \frac{\partial z_2^*(0,0)}{\partial \gamma}\right) + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_3} + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_4} \left(\frac{d_0'l_0'}{c_0'g_0'} - \frac{f_0'}{g_0'} + \frac{\partial z_4^*(0,0)}{\partial \gamma}\right) \\ \left. + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_5} \left(\frac{d_0'm_0'}{c_0'i_0'} - \frac{h_0'}{i_0'} + \frac{\partial z_5^*(0,0)}{\partial \gamma}\right) \right\} = 0, \\ \left(\frac{d_0'm_0'}{c_0'i_0'} - \frac{h_0'}{i_0'} + \frac{\partial z_5^*(0,0)}{\partial \gamma}\right) - \sum_{i=1}^{5} \frac{\partial \Theta_5(\Phi(\tau_0,\zeta_0))}{\partial x_i} \left\{ \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_1} \left(\frac{d_0'j_0'}{c_0'a_0'} - \frac{b_0'}{a_0'} + \frac{\partial z_1^*(0,0)}{\partial \gamma}\right) \right. \\ + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_2} \left(-\frac{d_0'}{c_0'} + \frac{\partial z_2^*(0,0)}{\partial \gamma}\right) + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_3} + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_4} \left(\frac{d_0'l_0'}{c_0'g_0'} - \frac{f_0'}{g_0'} + \frac{\partial z_4^*(0,0)}{\partial \gamma}\right) \\ \left. + \frac{\partial \Phi_i(\tau_0,\zeta_0)}{\partial x_5} \left(\frac{d_0'm_0'}{c_0'i_0'} - \frac{h_0'}{i_0'} + \frac{\partial z_5^*(0,0)}{\partial \gamma}\right) \right\} = 0. \end{cases}$$

We obtain

$$\begin{cases} \left(\frac{\partial z_1^*(0,0)}{\partial \gamma}\right) a_0 = 0, \\ \left(\frac{\partial z_2^*(0,0)}{\partial \gamma}\right) d_0 = 0, \\ \left(\frac{\partial z_4^*(0,0)}{\partial \gamma}\right) j_0 = 0, \\ \left(\frac{\partial z_5^*(0,0)}{\partial \gamma}\right) m_0 = 0. \end{cases}$$

That is

$$\left\{\begin{array}{l} \frac{\partial z_1^*(0,0)}{\partial\gamma}=0,\\ \frac{\partial z_2^*(0,0)}{\partial\gamma}=0,\\ \frac{\partial z_4^*(0,0)}{\partial\gamma}=0,\\ \frac{\partial z_5^*(0,0)}{\partial\gamma}=0. \end{array}\right. \tag{5.47}$$

5.4. Second derivatives of z_2^*

From (3.6) we have

$$\frac{\partial^2}{\partial\bar{\tau}^2}(\eta_2-\Theta_2\circ\Phi(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))(0,0)=0.$$

Then

$$\frac{\partial^2 z_2^*(0,0)}{\partial\bar{\tau}^2}-\frac{\partial}{\partial\bar{\tau}}\sum_{i=1}^{5}\frac{\partial\Theta_2(\Phi(\tau_0,\zeta_0))}{\partial x_i}\left(\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial\bar{\tau}}+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar{\tau}}+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar{\tau}}\right.$$
$$\left.+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_4}\frac{\partial z_4^*(0,0)}{\partial\bar{\tau}}+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_5}\frac{\partial z_5^*(0,0)}{\partial\bar{\tau}}\right)=0.$$

Therefore

$$\frac{\partial^2 z_2^*(0,0)}{\partial\bar{\tau}^2}-\sum_{i=1}^{5}\frac{\partial\Theta_2}{\partial x_i}\left\{\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial\bar{\tau}^2}+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial\bar{\tau}\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar{\tau}}+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial\bar{\tau}\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar{\tau}}\right.$$
$$+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial\bar{\tau}\partial x_4}\frac{\partial z_4^*(0,0)}{\partial\bar{\tau}}+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial\bar{\tau}\partial x_5}\frac{\partial z_5^*(0,0)}{\partial\bar{\tau}}+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_1^2}\left(\frac{\partial z_1^*(0,0)}{\partial\bar{\tau}}\right)^2$$
$$+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_1\partial x_2}\frac{\partial z_1^*(0,0)}{\partial\bar{\tau}}\frac{\partial z_2^*(0,0)}{\partial\bar{\tau}}+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_1\partial x_4}\frac{\partial z_1^*(0,0)}{\partial\bar{\tau}}\frac{\partial z_4^*(0,0)}{\partial\bar{\tau}}$$
$$+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_1\partial x_5}\frac{\partial z_1^*(0,0)}{\partial\bar{\tau}}\frac{\partial z_5^*(0,0)}{\partial\bar{\tau}}+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_1}\frac{\partial^2 z_1^*(0,0)}{\partial\bar{\tau}^2}+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_2^2}\left(\frac{\partial z_2^*(0,0)}{\partial\bar{\tau}}\right)^2$$
$$+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_2\partial x_4}\frac{\partial z_2^*(0,0)}{\partial\bar{\tau}}\frac{\partial z_4^*(0,0)}{\partial\bar{\tau}}+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_2\partial x_5}\frac{\partial z_2^*(0,0)}{\partial\bar{\tau}}\frac{\partial z_5^*(0,0)}{\partial\bar{\tau}}+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial^2 z_2^*(0,0)}{\partial\bar{\tau}^2}$$
$$+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_4^2}\left(\frac{\partial z_4^*(0,0)}{\partial\bar{\tau}}\right)^2+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_4\partial x_5}\frac{\partial z_4^*(0,0)}{\partial\bar{\tau}}\frac{\partial z_5^*(0,0)}{\partial\bar{\tau}}+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_4}\frac{\partial^2 z_4^*(0,0)}{\partial\bar{\tau}^2}$$
$$\left.+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_5^2}\left(\frac{\partial z_5^*(0,0)}{\partial\bar{\tau}}\right)^2+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_5}\frac{\partial^2 z_5^*(0,0)}{\partial\bar{\tau}^2}\right\}=0.$$

We obtain $d_0\frac{\partial^2 z_2^*(0,0)}{\partial\bar{\tau}^2}=0$, that is $\frac{\partial^2 z_2^*(0,0)}{\partial\bar{\tau}^2}=0$.
In the same way as above, it follows that

$$\frac{\partial^2}{\partial\gamma^2}(\eta_2-\Theta_2\circ\Phi(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))(0,0)=0.$$

Then

$$\frac{\partial^2 z_2^*(0,0)}{\partial\gamma^2}-\frac{\partial}{\partial\gamma}\sum_{i=1}^{5}\frac{\partial\Theta_2(\Phi(\tau_0,\zeta_0))}{\partial x_i}\left\{\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_1}\left(\frac{d_0'j_0'}{c_0'a_0'}-\frac{b_0'}{a_0'}+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)\right.$$
$$+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_2}\left(-\frac{d_0'}{c_0'}+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_3}+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_4}\left(\frac{d_0'l_0'}{c_0'g_0'}-\frac{f_0'}{g_0'}+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)$$
$$\left.+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_5}\left(\frac{d_0'm_0'}{c_0'i_0'}-\frac{h_0'}{i_0'}+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)\right\}=0.$$

Therefore

$$
\begin{aligned}
&\frac{\partial^2 z_2^*(0,0)}{\partial\gamma^2} - \sum_{i=1}^{5}\frac{\partial\Theta_2(\Phi(\tau_0,\zeta_0))}{\partial x_i}\left\{\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_1^2}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)^2\right.\\
&+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_2\partial x_1}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_3\partial x_1}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)\\
&+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_4\partial x_1}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_1}\left(\frac{\partial^2 z_1^*(0,0)}{\partial\gamma^2}\right)\\
&+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_5\partial x_1}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_2^2}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)^2\\
&+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_3\partial x_2}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_4\partial x_2}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)\\
&+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_5\partial x_2}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_2}\left(\frac{\partial^2 z_2^*(0,0)}{\partial\gamma^2}\right)\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_3^2}+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_3\partial x_4}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_3\partial x_5}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_4^2}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)^2+2\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_4\partial x_5}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)\\
&\left.+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_4}\left(\frac{\partial^2 z_4^*(0,0)}{\partial\gamma^2}\right)+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_5^2}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)^2+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_5}\left(\frac{\partial^2 z_5^*(0,0)}{\partial\gamma^2}\right)\right\}=0,
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\frac{\partial^2 z_2^*(0,0)}{\partial\gamma^2} = \frac{1}{d_0}\frac{\partial\Theta_2(\Phi(\tau_0,\zeta_0))}{\partial x_2}&\left\{2q_1q_2\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial x_2\partial x_1}+2q_1\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial x_3\partial x_1}+q_2^2\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial x_2^2}\right.\\
&\left.+2q_2\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial x_3\partial x_2}+\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial x_3^2}\right\}.
\end{aligned}
$$

In the same way as above, we obtain

$$\frac{\partial^2}{\partial\gamma\partial\bar\tau}(\eta_2-\Theta_2\circ\Phi(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))(0,0)=0.$$

Then

$$
\begin{aligned}
&\frac{\partial^2 z_2^*(0,0)}{\partial\gamma\partial\bar\tau}-\frac{\partial}{\partial\bar\tau}\sum_{i=1}^{5}\frac{\partial\Theta_2(\Phi(\tau_0,\zeta_0))}{\partial x_i}\left\{\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_1}\left(\frac{d_0'j_0'}{c_0'a_0'}-\frac{b_0'}{a_0'}+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)\right.\\
&+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_2}\left(-\frac{d_0'}{c_0'}+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_3}+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_4}\left(\frac{d_0'l_0'}{c_0'g_0'}-\frac{f_0'}{g_0'}+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)\\
&\left.+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_5}\left(\frac{d_0'm_0'}{c_0'i_0'}-\frac{h_0'}{i_0'}+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)\right\}=0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{\partial^2 z_2^*(0,0)}{\partial\gamma\partial\bar\tau} - \sum_{i=1}^{5}\frac{\partial\Theta_2}{\partial x_i}\left\{\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_1}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_1^2}\frac{\partial z_1^*(0,0)}{\partial\bar\tau}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)\right.\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_2\partial x_1}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_4\partial x_1}\frac{\partial z_4^*(0,0)}{\partial\bar\tau}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_5\partial x_1}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_1}\frac{\partial^2 z_1^*(0,0)}{\partial\bar\tau\partial\gamma}\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_2}\left(-\frac{d_0'}{c_0'}+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_2^2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_2\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_4\partial x_2}\frac{\partial z_4^*(0,0)}{\partial\bar\tau}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_5\partial x_2}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial^2 z_2^*(0,0)}{\partial\bar\tau\partial\gamma}+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_3}\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_3\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_3\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau}+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_4\partial x_3}\frac{\partial z_4^*(0,0)}{\partial\bar\tau}+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_5\partial x_3}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_4}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_4^2}\frac{\partial z_4^*(0,0)}{\partial\bar\tau}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_4\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_4\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_5\partial x_4}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_4}\frac{\partial^2 z_4^*(0,0)}{\partial\bar\tau\partial\gamma}\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_5}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_5^2}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_5\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_5\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)\\
&\left.+\frac{\partial^2\Phi_i(\tau_0,\zeta_0)}{\partial x_5\partial x_4}\frac{\partial z_4^*(0,0)}{\partial\bar\tau}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_i(\tau_0,\zeta_0)}{\partial x_5}\frac{\partial^2 z_5^*(0,0)}{\partial\bar\tau\partial\gamma}\right\}=0.
\end{aligned}
$$

We obtain

$$
\frac{\partial^2 z_2^*(0,0)}{\partial\gamma\partial\bar\tau}=\frac{1}{d_0}\frac{\partial\Theta_2}{\partial x_2}\left\{q_2\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_2}+\frac{\partial^2\Phi_2(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_3}\right\}.
$$

5.5. First derivatives of ω

We have

$$
\begin{aligned}
\frac{\partial\omega}{\partial\bar\tau} &= \frac{\partial}{\partial\bar\tau}(\eta_3-\Theta_3\circ\Phi(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))\\
&= -\sum_{i=1}^{5}\frac{\partial\Theta_3}{\partial x_i}\left\{\frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial\bar\tau}+\frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_4)}{\partial x_1}\frac{\partial z_1^*}{\partial\bar\tau}\right.\\
&\left.\quad+\frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2}\frac{\partial z_2^*}{\partial\bar\tau}+\frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_4)}{\partial x_4}\frac{\partial z_4^*}{\partial\bar\tau}+\frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5}\frac{\partial z_5^*}{\partial\bar\tau}\right\}.
\end{aligned}
$$

At $(\bar\tau,\gamma)=(0,0)$ we obtain

$$
\begin{aligned}
\frac{\partial\omega}{\partial\bar\tau}(0,0) &= -\frac{\partial\Theta_3}{\partial x_3}\left\{\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial\bar\tau}+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau}+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\right.\\
&\quad\left.+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_4}\frac{\partial z_4^*(0,0)}{\partial\bar\tau}+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_5}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\right\}=0.
\end{aligned}
$$

$$\frac{\partial\omega}{\partial\gamma} = \frac{\partial}{\partial\gamma}(\eta_3 - \Theta_3 \circ \Phi(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))$$
$$= 1 - \sum_{i=1}^{5} \frac{\partial\Theta_3}{\partial x_i} \left\{ \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_1} \left(q_1 + \frac{\partial z_1^*}{\partial\gamma}\right) + \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2} \left(q_2 + \frac{\partial z_2^*}{\partial\gamma}\right) \right.$$
$$+ \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_3} + \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4} \left(q_4 + \frac{\partial z_4^*}{\partial\gamma}\right)$$
$$\left. + \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5} \left(q_5 + \frac{\partial z_5^*}{\partial\gamma}\right) \right\}.$$

At $(\bar{\tau},\gamma) = (0,0)$ we obtain

$$\begin{aligned} \frac{\partial\omega}{\partial\gamma}(0,0) &= 1 - \frac{\partial\Theta_3}{\partial x_3} \left\{ \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_1} \left(q_1 + \frac{\partial z_1^*(0,0)}{\partial\gamma}\right) + \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_2} \left(q_2 + \frac{\partial z_2^*(0,0)}{\partial\gamma}\right) \right. \\ &\quad \left. + \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_3} + \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_4} \left(q_4 + \frac{\partial z_4^*(0,0)}{\partial\gamma}\right) + \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_5} \left(q_5 + \frac{\partial z_5^*(0,0)}{\partial\gamma}\right) \right\} \\ &= 1 - \frac{\partial\Theta_3}{\partial x_3} \left\{ q_2 \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_2} + \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_3} \right\} \\ &= f_0 q_2 + g_0 = 0 \end{aligned}$$

Therefore $D_{(\bar{\tau},\alpha)}\omega(0,0) = (0,0)$.

5.6. Second derivatives of ω

Let $\mathcal{A} = \frac{\partial^2\omega(0,0)}{\partial\bar{\tau}^2}$, $\mathcal{B} = \frac{\partial^2\omega(0,0)}{\partial\bar{\tau}\partial\gamma}$ and $\mathcal{C} = \frac{\partial^2\omega(0,0)}{\partial\gamma^2}$.

5.6.1. Calculation of $\mathcal{A}$

We have $\frac{\partial^2\omega}{\partial\bar{\tau}^2} = \frac{\partial^2}{\partial\bar{\tau}^2}(\eta_3 - \Theta_3 \circ \Phi(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))$, then

$$\frac{\partial^2\omega}{\partial\bar{\tau}^2} = -\sum_{i=1}^{5} \frac{\partial\Theta_3}{\partial x_i} \left\{ \frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial\bar{\tau}^2} + 2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial\bar{\tau}\partial x_1} \frac{\partial z_1^*}{\partial\bar{\tau}} \right.$$
$$+2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial\bar{\tau}\partial x_2} \frac{\partial z_2^*}{\partial\bar{\tau}} + 2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial\bar{\tau}\partial x_4} \frac{\partial z_4^*}{\partial\bar{\tau}} + 2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial\bar{\tau}\partial x_5} \frac{\partial z_5^*}{\partial\bar{\tau}}$$
$$+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_1^2} \left(\frac{\partial z_1^*}{\partial\bar{\tau}}\right)^2 + 2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_1\partial x_2} \frac{\partial z_1^*}{\partial\bar{\tau}} \frac{\partial z_2^*}{\partial\bar{\tau}}$$
$$+2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_1\partial x_4} \frac{\partial z_1^*}{\partial\bar{\tau}} \frac{\partial z_4^*}{\partial\bar{\tau}} + 2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_1\partial x_5} \frac{\partial z_1^*}{\partial\bar{\tau}} \frac{\partial z_5^*}{\partial\bar{\tau}} + \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_1} \frac{\partial^2 z_1^*}{\partial\bar{\tau}^2}$$
$$+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2^2} \left(\frac{\partial z_2^*}{\partial\bar{\tau}}\right)^2 + 2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2\partial x_4} \frac{\partial z_2^*}{\partial\bar{\tau}} \frac{\partial z_4^*}{\partial\bar{\tau}}$$
$$+2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2\partial x_5} \frac{\partial z_2^*}{\partial\bar{\tau}} \frac{\partial z_5^*}{\partial\bar{\tau}} + \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2} \frac{\partial^2 z_2^*}{\partial\bar{\tau}^2} + \frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4^2} \left(\frac{\partial z_4^*}{\partial\bar{\tau}}\right)^2$$
$$+2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4\partial x_5} \frac{\partial z_4^*}{\partial\bar{\tau}} \frac{\partial z_5^*}{\partial\bar{\tau}} + \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4} \frac{\partial^2 z_4^*}{\partial\bar{\tau}^2} + \frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5^2} \left(\frac{\partial z_5^*}{\partial\bar{\tau}}\right)^2$$
$$\left. + \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5} \frac{\partial^2 z_5^*}{\partial\bar{\tau}^2} \right\}.$$

At $(\bar{\tau},\gamma)=(0,0)$ we obtain

$$\begin{aligned}
\frac{\partial^2\omega}{\partial\bar{\tau}^2}(0,0) = -\frac{\partial\Theta_3}{\partial x_3}\Bigg\{ & \frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar{\tau}^2} + 2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar{\tau}\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar{\tau}} + 2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar{\tau}\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar{\tau}} \\
&+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar{\tau}\partial x_4}\frac{\partial z_4^*(0,0)}{\partial\bar{\tau}} + 2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar{\tau}\partial x_5}\frac{\partial z_5^*(0,0)}{\partial\bar{\tau}} + \frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_1^2}\left(\frac{\partial z_1^*(0,0)}{\partial\bar{\tau}}\right)^2 \\
&+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_1\partial x_2}\frac{\partial z_1^*(0,0)}{\partial\bar{\tau}}\frac{\partial z_2^*(0,0)}{\partial\bar{\tau}} + 2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_1\partial x_4}\frac{\partial z_1^*(0,0)}{\partial\bar{\tau}}\frac{\partial z_4^*(0,0)}{\partial\bar{\tau}} \\
&+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_1\partial x_5}\frac{\partial z_1^*(0,0)}{\partial\bar{\tau}}\frac{\partial z_5^*(0,0)}{\partial\bar{\tau}} + \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_1}\frac{\partial^2 z_1^*(0,0)}{\partial\bar{\tau}^2} + \frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_2^2}\left(\frac{\partial z_2^*(0,0)}{\partial\bar{\tau}}\right)^2 \\
&+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_2\partial x_4}\frac{\partial z_2^*(0,0)}{\partial\bar{\tau}}\frac{\partial z_4^*(0,0)}{\partial\bar{\tau}} + 2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_2\partial x_5}\frac{\partial z_2^*(0,0)}{\partial\bar{\tau}}\frac{\partial z_5^*(0,0)}{\partial\bar{\tau}} + \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial^2 z_2^*(0,0)}{\partial\bar{\tau}^2} \\
&+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_4^2}\left(\frac{\partial z_4^*(0,0)}{\partial\bar{\tau}}\right)^2 + 2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_4\partial x_5}\frac{\partial z_4^*(0,0)}{\partial\bar{\tau}}\frac{\partial z_5^*(0,0)}{\partial\bar{\tau}} + \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_4}\frac{\partial^2 z_4^*(0,0)}{\partial\bar{\tau}^2} \\
&+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_5^2}\left(\frac{\partial z_5^*(0,0)}{\partial\bar{\tau}}\right)^2 + \frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_5}\frac{\partial^2 z_5^*(0,0)}{\partial\bar{\tau}^2}\Bigg\}.
\end{aligned}$$

Then $\mathcal{A}=0$.

5.6.2. Calculation of $\mathcal{C}$

We have $\frac{\partial^2\omega}{\partial\gamma^2}=\frac{\partial^2}{\partial\gamma^2}(\eta_3-\Theta_3\circ\Phi(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))$, then

$$\begin{aligned}
\frac{\partial^2\omega}{\partial\gamma^2} = -\sum_{i=1}^{5}\frac{\partial\Theta_3}{\partial x_i}\Bigg\{ & \frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_1^2}\left(q_1+\frac{\partial z_1^*}{\partial\gamma}\right)^2 \\
&+2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2\partial x_1}\left(q_1+\frac{\partial z_1^*}{\partial\gamma}\right)\left(q_2+\frac{\partial z_2^*}{\partial\gamma}\right) + 2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_3\partial x_1}\left(q_1+\frac{\partial z_1^*}{\partial\gamma}\right) \\
&+2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4\partial x_1}\left(q_1+\frac{\partial z_1^*}{\partial\gamma}\right)\left(q_4+\frac{\partial z_4^*}{\partial\gamma}\right) + \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_1}\left(\frac{\partial^2 z_1^*}{\partial\gamma^2}\right) \\
&+2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5\partial x_1}\left(q_1+\frac{\partial z_1^*}{\partial\gamma}\right)\left(q_5+\frac{\partial z_5^*}{\partial\gamma}\right) + \frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2^2}\left(q_2+\frac{\partial z_2^*}{\partial\gamma}\right)^2 \\
&+2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_3\partial x_2}\left(q_2+\frac{\partial z_2^*}{\partial\gamma}\right) + 2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4\partial x_2}\left(q_2+\frac{\partial z_2^*}{\partial\gamma}\right)\left(q_4+\frac{\partial z_4^*}{\partial\gamma}\right) \\
&+2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5\partial x_2}\left(q_2+\frac{\partial z_2^*}{\partial\gamma}\right)\left(q_5+\frac{\partial z_5^*}{\partial\gamma}\right) + \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2}\left(\frac{\partial^2 z_2^*}{\partial\gamma^2}\right) \\
&+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_3^2} + 2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_3\partial x_4}\left(q_4+\frac{\partial z_4^*}{\partial\gamma}\right) \\
&+2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_3\partial x_5}\left(q_5+\frac{\partial z_5^*}{\partial\gamma}\right) + \frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4^2}\left(q_4+\frac{\partial z_4^*}{\partial\gamma}\right)^2 \\
&+2\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4\partial x_5}\left(q_4+\frac{\partial z_4^*}{\partial\gamma}\right)\left(q_5+\frac{\partial z_5^*}{\partial\gamma}\right) + \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4}\left(\frac{\partial^2 z_4^*}{\partial\gamma^2}\right) \\
&+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5^2}\left(q_5+\frac{\partial z_5^*}{\partial\gamma}\right)^2 + \frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5}\left(\frac{\partial^2 z_5^*}{\partial\gamma^2}\right)\Bigg\}.
\end{aligned}$$

At $(\bar{\tau},\gamma)=(0,0)$ we obtain

$$\begin{aligned}
\frac{\partial^2\omega}{\partial\gamma^2}(0,0) = -\frac{\partial\Theta_3}{\partial x_3}\Bigg\{ &\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_1^2}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)^2 + 2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3\partial x_1}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)\\
&+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_2\partial x_1}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_1}\left(\frac{\partial^2 z_1^*(0,0)}{\partial\gamma^2}\right)\\
&+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_4\partial x_1}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_2^2}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)^2\\
&+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_5\partial x_1}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)\left(q_4+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3\partial x_2}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)\\
&+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_4\partial x_2}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_2}\left(\frac{\partial^2 z_2^*(0,0)}{\partial\gamma^2}\right)\\
&+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_5\partial x_2}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_4}\left(\frac{\partial^2 z_4^*(0,0)}{\partial\gamma^2}\right)\\
&+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3^2}+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3\partial x_4}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3\partial x_5}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_4^2}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)^2+2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_4\partial x_5}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_5^2}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)^2+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_5}\left(\frac{\partial^2 z_5^*(0,0)}{\partial\gamma^2}\right)\Bigg\}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{C} = -\frac{\partial\Theta_3}{\partial x_3}\Bigg\{ &2q_1q_2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_2\partial x_1}+2q_1\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3\partial x_1}+q_2^2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_2^2}\\
&+2q_2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3\partial x_2}+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3^2}+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial^2 z_2^*(0,0)}{\partial\gamma^2}\Bigg\}.
\end{aligned}$$

5.6.3. Calculation of $\mathcal{B}$

We have $\frac{\partial^2\omega}{\partial\bar{\tau}\partial\gamma}=\frac{\partial}{\partial\bar{\tau}}\left(\frac{\partial}{\partial\gamma}(\eta_3-\Theta_3\circ\Phi(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))\right)$, then

$$\begin{aligned}
\frac{\partial^2\omega}{\partial\bar{\tau}\partial\gamma} = -\sum_{i=1}^{5}\frac{\partial\Theta_3}{\partial x_i}\Bigg\{ &\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial\bar{\tau}\partial x_1}\left(q_1+\frac{\partial z_1^*}{\partial\gamma}\right)+\frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_1}\frac{\partial^2 z_1^*}{\partial\bar{\tau}\partial\gamma}\\
&+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_1^2}\frac{\partial z_1^*}{\partial\bar{\tau}}\left(q_1+\frac{\partial z_1^*}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2\partial x_1}\frac{\partial z_2^*}{\partial\bar{\tau}}\left(q_1+\frac{\partial z_1^*}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4\partial x_1}\frac{\partial z_4^*}{\partial\bar{\tau}}\left(q_1+\frac{\partial z_1^*}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5\partial x_1}\frac{\partial z_5^*}{\partial\bar{\tau}}\left(q_1+\frac{\partial z_1^*}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial\bar{\tau}\partial x_2}\left(q_1+\frac{\partial z_2^*}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2^2}\frac{\partial z_2^*}{\partial\bar{\tau}}\left(q_2+\frac{\partial z_2^*}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2\partial x_1}\frac{\partial z_1^*}{\partial\bar{\tau}}\left(q_2+\frac{\partial z_2^*}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4\partial x_2}\frac{\partial z_4^*}{\partial\bar{\tau}}\left(q_2+\frac{\partial z_2^*}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5\partial x_2}\frac{\partial z_5^*}{\partial\bar{\tau}}\left(q_2+\frac{\partial z_2^*}{\partial\gamma}\right)+\frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_2}\frac{\partial^2 z_2^*}{\partial\bar{\tau}\partial\gamma}+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial\bar{\tau}\partial x_3}\\
&+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_3\partial x_2}\frac{\partial z_2^*}{\partial\bar{\tau}}+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_3\partial x_1}\frac{\partial z_1^*}{\partial\bar{\tau}}+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4\partial x_3}\frac{\partial z_4^*}{\partial\bar{\tau}}\\
&+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5\partial x_3}\frac{\partial z_5^*}{\partial\bar{\tau}}+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial\bar{\tau}\partial x_4}\left(q_4+\frac{\partial z_4^*}{\partial\gamma}\right)+\frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4}\frac{\partial^2 z_4^*}{\partial\bar{\tau}\partial\gamma}\\
&+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4^2}\frac{\partial z_4^*}{\partial\bar{\tau}}\left(q_4+\frac{\partial z_4^*}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4\partial x_1}\frac{\partial z_1^*}{\partial\bar{\tau}}\left(q_4+\frac{\partial z_4^*}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_4\partial x_2}\frac{\partial z_2^*}{\partial\bar{\tau}}\left(q_4+\frac{\partial z_4^*}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5\partial x_4}\frac{\partial z_5^*}{\partial\bar{\tau}}\left(q_4+\frac{\partial z_4^*}{\partial\gamma}\right)\\
&+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial\bar{\tau}\partial x_5}\left(q_5+\frac{\partial z_5^*}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))}{\partial x_5^2}\frac{\partial z_5^*}{\partial\bar{\tau}}\left(q_5+\frac{\partial z_5^*}{\partial\gamma}\right)+
\end{aligned}$$

$$+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5\partial x_1}\frac{\partial z_1^*}{\partial\bar\tau}\left(q_5+\frac{\partial z_5^*}{\partial\gamma}\right)+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5\partial x_2}\frac{\partial z_2^*}{\partial\bar\tau}\left(q_5+\frac{\partial z_5^*}{\partial\gamma}\right)$$
$$+\frac{\partial^2\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5\partial x_4}\frac{\partial z_4^*}{\partial\bar\tau}\left(q_5+\frac{\partial z_5^*}{\partial\gamma}\right)+\frac{\partial\Phi_i(\eta,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)}{\partial x_5}\frac{\partial^2 z_5^*}{\partial\bar\tau\partial\gamma}\bigg\}.$$

At $(\bar\tau,\gamma)=(0,0)$ we obtain

$$\frac{\partial^2\omega}{\partial\bar\tau\partial\gamma}(0,0)=-\frac{\partial\Theta_3}{\partial x_3}\bigg\{\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_1}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_1^2}\frac{\partial z_1^*(0,0)}{\partial\bar\tau}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)$$
$$+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_2\partial x_1}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_4\partial x_1}\frac{\partial z_4^*(0,0)}{\partial\bar\tau}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)$$
$$+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_5\partial x_1}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\left(q_1+\frac{\partial z_1^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_1}\frac{\partial^2 z_1^*(0,0)}{\partial\bar\tau\partial\gamma}+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_2}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)$$
$$+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_2^2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_2\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)$$
$$+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_4\partial x_2}\frac{\partial z_4^*(0,0)}{\partial\bar\tau}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_5\partial x_2}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\left(q_2+\frac{\partial z_2^*(0,0)}{\partial\gamma}\right)$$
$$+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial^2 z_2^*(0,0)}{\partial\bar\tau\partial\gamma}+\frac{\partial^2\Phi_3(\tau_0,\zeta_0))}{\partial\bar\tau\partial x_3}+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_3\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau}$$
$$+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_4\partial x_3}\frac{\partial z_4^*(0,0)}{\partial\bar\tau}+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_5\partial x_3}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_4}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)$$
$$+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_4^2}\frac{\partial z_4^*(0,0)}{\partial\bar\tau}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_4\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)$$
$$+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_4\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_5\partial x_4}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\left(q_4+\frac{\partial z_4^*(0,0)}{\partial\gamma}\right)$$
$$+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_4}\frac{\partial^2 z_4^*(0,0)}{\partial\bar\tau\partial\gamma}+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_5}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_5^2}\frac{\partial z_5^*(0,0)}{\partial\bar\tau}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)$$
$$+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_5\partial x_1}\frac{\partial z_1^*(0,0)}{\partial\bar\tau}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_5\partial x_2}\frac{\partial z_2^*(0,0)}{\partial\bar\tau}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)$$
$$+\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial x_5\partial x_4}\frac{\partial z_4^*(0,0)}{\partial\bar\tau}\left(q_5+\frac{\partial z_5^*(0,0)}{\partial\gamma}\right)+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_5}\frac{\partial^2 z_5^*(0,0)}{\partial\bar\tau\partial\gamma}\bigg\}.$$

Then

$$\mathcal{B}=-\frac{\partial\Theta_3}{\partial x_3}\left\{q_2\frac{\partial^2\Phi_3(\tau_0,\zeta_0)}{\partial\bar\tau\partial x_2}+\frac{\partial\Phi_3(\tau_0,\zeta_0)}{\partial x_2}\frac{\partial^2 z_2^*(0,0)}{\partial\bar\tau\partial\gamma}+\frac{\partial^2\Phi_3(\tau_0,\zeta_0))}{\partial\bar\tau\partial x_3}\right\}.$$

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