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# COMMON LEAST-RANK SOLUTION OF MATRIX EQUATIONS $A_1X_1B_1 = C_1$ AND $A_2X_2B_2 = C_2$ WITH APPLICATIONS

**Sihem Guerarra and Said Guedjiba**

**Abstract.** In this paper we provide the necessary and sufficient conditions for the pair of matrix equations $A_1X_1B_1 = C_1$ and $A_2X_2B_2 = C_2$ to have a common least-rank solution, as well as the expression of this solution. We also give the necessary and sufficient conditions for the matrix equation $AXB = C$ to have a Hermitian least-rank solution. Using the first results, we investigate the expression of the general Hermitian least-rank solution of the matrix equation $AXB = C$.
**Key words:** Matrix equation; Rank formulas; Moore-Penrose generalized inverse; Hermitian; Least-rank solution.

## 1. Introduction

Linear matrix equations play a very important role in matrix theory. For a given matrix equation one always wants to know the consistency conditions, the general solution or the least squares solution, the properties of least squares solution and so on.

We consider the pair of linear matrix equations

$$A_1X_1B_1 = C_1 \text{ and } A_2X_2B_2 = C_2 \tag{1.1}$$

where $A_1 \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{p\times q}$, $C_1 \in \mathbb{C}^{m\times q}$, $A_2 \in \mathbb{C}^{s\times n}$, $B_2 \in \mathbb{C}^{p\times t}$, $C_2 \in \mathbb{C}^{s\times t}$ are given, $X_1 \in \mathbb{C}^{n\times p}$, $X_2 \in \mathbb{C}^{n\times p}$ are unknown matrices.

There have been many results given in the literature. In [4] X. Fu Liu, Hu Yang gave an expression of the general common least squares solution to the pair of matrix equations (1.1). Furthermore, they determined the conditions for the existence of a Hermitian least squares solution of the matrix equation $AXB = C$ and the expression of the general Hermitian least squares solution is also given, [5]. In [14], the authors provided some necessary and sufficient conditions for the existence of the reflexive extremal rank solutions to the matrix equation $AX =$

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$B$, and several representations for reflexive solutions are also given. A. Liao, Y. Lei established the least squares solution with the minimum norm for the matrix equation $(AXB, GXH) = (C, D)$. In [6], [7] Y. Liu gave the conditions for the pair of matrix equations (1.1) to have a common least squares solution by using the external ranks (the general common least squares solution, however, was not given), and also gave the ranks of solutions of the linear matrix equation $AX + YB = C$. In [8, 9, 10] S. K. Mitra, A. Navarra, P. L. Odell, D. M. Yong gave a representation of the general common solution of (1.1). In [13] Y. Tian studied the relation between least squares and least-rank solutions of the matrix equation $AXB = C$ where he established necessary and sufficient conditions for the two types of solutions to coincide. In [15] F. Zhang established necessary and sufficient conditions for least squares solutions of $AXB = C$ to be Hermitian or local Hermitian.

The field of generalized inverses has grown much since its rebirth in the early 1950s. Numerous papers have developed both its theory and its applications see. e.g. [1, 2, 11, 3]. In [1] and [2] the authors represented the theory of the generalized inverses with applications in many areas, especially the Moore-Penrose generalized inverse. In [11], P. S. Stanimirović investigated computation of generalized inverse by means of the introduced general representation and the rational canonical form, or the Jordan canonical form. In [3], S Karanasios and D. Pappas computed the generalized inverse of a finite rank operator on complex Hilbert space, and they also gave necessary and sufficient conditions for reverse order law on the generalized inverse of two rank-1 operators.

Motivated by the work of [4], [7], [13] we use the matrix rank method to drive necessary and sufficient conditions for (1.1) to have a common least-rank solution, and we give the expression of these common least rank solutions. In the last part of this paper we derive necessary and sufficient conditions for the matrix equation $AXB = C$ to have a Hermitian least-rank solution, and we give the expression of the general Hermitian least-rank solution of equation $AXB = C$.

Throughout this paper $\mathbb{C}^{m\times n}$ stands for the set of all $m \times n$ complex matrices, the symbols, $A^*$, $r(A)$, $R(A)$, stand for the conjugate transpose, the rank, and the range of $A$, respectively. $I_m$ denotes the identity matrix of order $m$. The Moore-Penrose generalized inverse of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by $A^+$, is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four matrix equations:

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA$$

The Moore-Penrose generalized inverse has been the object of several papers, see [1], [2].

Further, $E_A$ and $F_A$ stand for the two orthogonal projectors $E_A = I - AA^+$, $F_A = I - A^+A$ induced by $A$. Their ranks are given by $r(E_A) = m - r(A)$, $r(F_A) = n - r(A)$.

Some rank formulas that are needed in the paper are given by the following lemmas:

**Lemma 1.1.** [12] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$. *Then,*
*i)* $r\begin{bmatrix} A, & B \end{bmatrix} = r(A) + r(E_A B) = r(B) + r(E_B A)$.
*ii)* $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C)$.

**Lemma 1.2.** [15] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$. *Then*

*i)* $\min_{X\in\mathbb{C}^{k\times n}, Y\in\mathbb{C}^{m\times l}} r(A - BX - YC) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C)$.

*ii) if* $A \in \mathbb{C}^{m\times m}$, $A^* = -A$. *Then*

$$\min_{X\in\mathbb{C}^{k\times m}} r(A - BX - X^*B^*) = r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r(B).$$

**Lemma 1.3.** [12] *Let* $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$, *and* $D$ *are matrices such that expression* $D - C_1A_1^+B_1 - C_2A_2^+B_2$ *is defined. Then*

$$r\left(D - C_1A_1^+B_1 - C_2A_2^+B_2\right) = r\begin{bmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 \\ 0 & A_2^*A_2A_2^* & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D \end{bmatrix} - r(A_1) - r(A_2).$$

We consider the matrix equation

$$AXB = C \tag{1.2}$$

Where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $C \in \mathbb{C}^{m\times q}$, are given and $X \in \mathbb{C}^{n\times p}$ is unknown matrix.

In [13] The least-rank solution of (1.2) is the matrix $X$ which minimizes the rank of $(C - AXB)$ or equivalently

$$X = \arg\min_{X\in\mathbb{C}^{n\times p}} r(C - AXB).$$

We know that the general least-rank solution of the matrix equation (1.2) can be written as

$$X = -TM^+S + T_1U + VS_1,$$

where

$$M = \begin{bmatrix} C & A \\ B & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & I_n \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad T_1 = TF_M, \quad S_1 = E_M S,$$

and $U$, $V$ are arbitrary.

## 2. Common least-rank solution of matrix equations $A_1X_1B_1 = C_1$ and $A_2X_2B_2 = C_2$

Following the work of [7], [13] in this section we use the matrix rank method to drive the conditions for the matrix equations $A_1X_1B_1 = C_1$, $A_2X_2B_2 = C_2$ to have a common least-rank solution. Also in this section the expression of the general common least-rank solution to system (1.1) is established.

**Lemma 2.1.** [4] *Let* $A_1 \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{p\times q}$, $A_2 \in \mathbb{C}^{m\times l}$, $B_2 \in \mathbb{C}^{s\times q}$, $C \in \mathbb{C}^{m\times q}$ *be known and* $X_1 \in \mathbb{C}^{n\times p}$, $X_2 \in \mathbb{C}^{l\times s}$ *are unknown.*
*Let* $M = E_{A_1}A_2$, $N = B_2F_{B_1}$, $S = A_2F_M$, *then the following statements are equivalent.*

*i) The system*

$$A_1X_1B_1 + A_2X_2B_2 = C \tag{2.1}$$

*is solvable.*

*ii) The following rank equalities are satisfied*

$$r\begin{bmatrix} A_1 & C \\ 0 & B_2 \end{bmatrix} = r\begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad r\begin{bmatrix} A_2 & C \\ 0 & B_1 \end{bmatrix} = r\begin{bmatrix} A_2 & 0 \\ 0 & B_1 \end{bmatrix},$$

$$r\begin{bmatrix} C & A_1 & A_2 \end{bmatrix} = r\begin{bmatrix} A_1 & A_2 \end{bmatrix}, \quad r\begin{bmatrix} B_1 \\ B_2 \\ C \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

*In this case the general solution of* (2.1) *can be expressed as*

$$\begin{aligned} X_1 &= A_1^+CB_1^+ - A_1^+A_2M^+E_{A_1}CB_1^+ - A_1^+SA_2^+CF_{B_1}N^+B_2B_1^+ \\ &\quad - A_1^+SVE_NB_2B_1^+ + F_{A_1}U + ZE_{B_1}, \\ X_2 &= M^+E_{A_1}CB_2^+ + F_MS^+SA_2^+CF_{B_1}N^+ + F_M(V - S^+SVNN^+) + WE_{B_2}, \end{aligned}$$

*where* $U$, $V$, $W$ *and* $Z$ *are arbitrary.*

**Theorem 2.1.** *Suppose that* $X_1$ *and* $X_2$ *are the least-rank solutions of* (1.1)*. Then the pair of matrix equations in* (1.1) *has a common least rank solution if and only if*

$$r\begin{bmatrix} M_1 & 0 & S_1 \\ 0 & -M_2 & S_2 \\ T_1 & T_2 & 0 \end{bmatrix} = r\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} + r\begin{bmatrix} T_1, & T_2 \end{bmatrix} + r(M_1) + r(M_2). \tag{2.2}$$

*Proof.* The general least-rank solution of the matrix equation $AXB = C$ can be written as

$$X = -TM^+S + T_1U + VS_1,$$

where

$$M=\begin{bmatrix} C & A \\ B & 0 \end{bmatrix},\ T=\begin{bmatrix} 0 & I_n \end{bmatrix},\ S=\begin{bmatrix} 0 \\ I_p \end{bmatrix},\ T_1=TF_M,\ S_1=E_MS$$

and $U$, $V$ are arbitrary.

Hence the general least rank solutions of matrix equations $A_1X_1B_1 = C_1$ and $A_2X_2B_2 = C_2$ are

$$X_1 = -T_1M_1^+S_1 + T_{11}U_1 + V_1S_{11},$$
$$X_2 = -T_2M_2^+S_2 + T_{22}U_2 + V_2S_{22},$$

where $T_{ii} = T_iF_{M_i}$ , $S_{ii} = E_{M_i}S_i$, with $i$ =1, 2, and $U_1$, $V_1$, $U_2$, $V_2$ are arbitrary. In this case,

$$X_1-X_2 = T_2M_2^+S_2 - T_1M_1^+S_1 + [T_{11}, T_{22}]\begin{bmatrix} U_1 \\ -U_2 \end{bmatrix} + [V_1,-V_2]\begin{bmatrix} S_{11} \\ S_{22} \end{bmatrix}.$$

Thus, from Lemma 1.2 we get

$$\min_{X_1,X_2} r(X_1-X_2) = r\begin{bmatrix} T_2M_2^+S_2 - T_1M_1^+S_1 & T_{11} & T_{22} \\ S_{11} & 0 & 0 \\ S_{22} & 0 & 0 \end{bmatrix} - r[T_{11}, T_{22}] - r\begin{bmatrix} S_{11} \\ S_{22} \end{bmatrix}. \tag{2.3}$$

We have that

$$r\begin{bmatrix} S_{11} \\ S_{22} \end{bmatrix} = r\begin{bmatrix} S_1 & M_1 & 0 \\ S_2 & 0 & M_2 \end{bmatrix} - r(M_1) - r(M_2)$$
$$= r\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} + r\begin{bmatrix} M_1, & M_2 \end{bmatrix} - r(M_1) - r(M_2),$$

and

$$r[T_{11}, T_{22}] = r\begin{bmatrix} T_1 & T_2 \\ M_1 & 0 \\ 0 & M_2 \end{bmatrix} - r(M_1) - r(M_2)$$
$$= r\begin{bmatrix} T_1, & T_2 \end{bmatrix} + r\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} - r(M_1) - r(M_2).$$

Applying Lemmas 1.2 and 1.3 and the block elementary operations we get

$$
\begin{aligned}
& r\begin{bmatrix} T_2M_2^+S_2 - T_1M_1^+S_1 & T_{11} & T_{22} \\ S_{11} & 0 & 0 \\ S_{22} & 0 & 0 \end{bmatrix} \\
& = r\begin{bmatrix} T_2M_2^+S_2 - T_1M_1^+S_1 & T_1 & T_2 & 0 & 0 & 0 \\ S_1 & 0 & 0 & M_1 & 0 & 0 \\ S_2 & 0 & 0 & 0 & M_2 & 0 \\ 0 & M_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - 2r(M_1) - 2r(M_2) \\
& = r\begin{bmatrix} M_1^*M_1M_1^* & 0 & M_1^*S_1 & 0 & 0 & 0 & 0 \\ 0 & -M_2^*M_2M_2^* & M_2^*S_2 & 0 & 0 & 0 & 0 \\ T_1M_1^* & T_2M_2^* & 0 & T_1 & T_2 & 0 & 0 \\ 0 & 0 & S_1 & 0 & 0 & M_1 & 0 \\ 0 & 0 & S_2 & 0 & 0 & 0 & M_2 \\ 0 & 0 & 0 & M_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_2 & 0 & 0 \end{bmatrix} - 3r(M_1) - 3r(M_2) \\
& = r\begin{bmatrix} M_1 & 0 & S_1 & 0 & 0 & 0 & 0 \\ 0 & -M_2 & S_2 & 0 & 0 & 0 & 0 \\ T_1 & T_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_2 \\ 0 & 0 & 0 & M_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_2 & 0 & 0 \end{bmatrix} - 3r(M_1) - 3r(M_2) \\
& = r\begin{bmatrix} M_1 & 0 & S_1 \\ 0 & -M_2 & S_2 \\ T_1 & T_2 & 0 \end{bmatrix} + r\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} + r\begin{bmatrix} M_1, & M_2 \end{bmatrix} - 3r(M_1) - 3r(M_2).
\end{aligned}
$$

Substituting into (2.3) yields (2.2). □

**Theorem 2.2.** *Let*

$$
\begin{aligned}
P_1 &= E_{T_{22}}T_{11}, \\
Q_1 &= S_{11}F_{S_{22}}, \\
K_1 &= E_{T_{22}}F_{P_1}, \\
P_2 &= E_{T_{11}}T_{22}, \\
Q_2 &= S_{22}F_{S_{11}}, \\
K_2 &= E_{T_{11}}F_{P_2}, \\
H_1 &= E_{T_{22}}T_1M_1^+S_1F_{S_{22}} - E_{T_{22}}T_2M_2^+S_2F_{S_{22}}, \\
H_2 &= E_{T_{11}}T_2M_2^+S_2F_{S_{11}} - E_{T_{11}}T_1M_1^+S_1F_{S_{11}}.
\end{aligned}
$$

*Assume that the system (1.1) has a common least-rank solution. Then, the general* *common least-rank solution can be expressed as*

(2.4)
$$\begin{aligned} X_1 = &-T_1M_1^+S_1 + P_1^+H_1F_{S_{22}} - P_1^+E_{T_{22}}K_1^+E_{P_1}H_1F_{S_{22}} \\ &+ K_1^+E_{P_1}H_1Q_1^+ - P_1^+E_{T_{22}}F_{K_1}V_1Q_1F_{S_{22}} \\ &+ T_{11}F_{P_1}U_1 + T_{11}Z_1E_{F_{S_{22}}} + F_{K_1}V_1S_{11} + W_1F_{Q_1}S_{11}, \end{aligned}$$

*or*

(2.5)
$$\begin{aligned} X_2 = &-T_2M_2^+S_2 + P_2^+H_2F_{S_{11}} - P_2^+E_{T_{11}}K_2^+E_{P_2}H_2F_{S_{11}} \\ &+ K_2^+E_{P_2}H_2Q_2^+ - P_2^+E_{T_{11}}F_{K_2}V_2Q_2F_{S_{11}} \\ &+ T_{22}F_{P_2}U_2 + T_{22}Z_2E_{F_{S_{11}}} + F_{K_2}V_2S_{22} + W_2F_{Q_2}S_{22}, \end{aligned}$$

*where $U_1$, $U_2$, $V_1$, $V_2$, $W_1$, $W_2$, $Z_1$, $Z_2$, are arbitrary matrices with appropriate sizes.*

*Proof.* In [13] the general least-rank solution of matrix equation $AXB = C$ is the solution of the consistent equation $E_{T_1}XF_{S_1} = -E_{T_1}TM^+SF_1$, such that $\widehat{P}(X) = E_{T_1}(X + TM^+S)F_{S_1}$ is called the rank reduced form of $P(X) = C - AXB$.

The general expression of the least-rank solution to $A_1X_1B_1 = C_1$ can be written as

(2.6)
$$X_1 = -T_1M_1^+S_1 + T_{11}\widetilde{U}_1 + \widetilde{V}_1S_{11}.$$

The pair of matrix equation (1.1) have a common least-rank solution if and only if there exist some $\widetilde{U}_1$and $\widetilde{V}_1$ such that $X_1$ is a least-rank solution to system $A_2X_2B_2 = C_2$ i.e.

$$E_{T_{22}}\left(-T_1M_1^+S_1 + T_{11}\widetilde{U}_1 + \widetilde{V}_1S_{11}\right)F_{S_{22}} = -E_{T_{22}}T_2M_2^+S_2F_{S_{22}}.$$

So

(2.7)
$$E_{T_{22}}T_{11}\widetilde{U}_1F_{S_{22}} + E_{T_{22}}\widetilde{V}_1S_{11}F_{S_{22}} = E_{T_{22}}T_1M_1^+S_1F_{S_{22}} - E_{T_{22}}T_2M_2^+S_2F_{S_{22}}.$$

So by applying Lemma 2.1 to (2.7), the general solution of system (2.7) is

(2.8)
$$\begin{aligned} \widetilde{U}_1 = &P_1^+H_1F_{S_{22}} - P_1^+E_{T_{22}}K_1^+E_{P_1}H_1F_{S_{22}} \\ &- P_1^+E_{T_{22}}F_{K_1}V_1Q_1F_{S_{22}} + F_{P_1}U_1 + Z_1E_{F_{S_{22}}}, \end{aligned}$$

(2.9)
$$\widetilde{V}_1 = K_1^+E_{P_1}H_1Q_1^+ + F_{K_1}V_1 + W_1E_{Q_1},$$

where $U_1$, $V_1$, $W_1$, $Z_1$, are arbitrary matrices.
Substituting (2.8) and (2.9) into (2.6) yields (2.4), similarly the expression (2.5) can be found. □

**Remark 2.1.** The expressions of $X_1$ and $X_2$ are different, under the condition (2.2), the sets $\{X_1\}$ and $\{X_2\}$ are equivalent.

## 3. Hermitian least-rank solution of matrix equation $AXB = C$

In this section we will obtain necessary and sufficient conditions for least-rank solution of matrix equation $AXB = C$ to be Hermitian, and give the expression of the general Hermitian least-rank solution of $AXB = C$.

Consider the matrix equation

$$AXB = C \tag{3.1}$$

where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $C \in \mathbb{C}^{m\times q}$ are given, and $X \in \mathbb{C}^{n\times p}$ is unknown matrix.

**Lemma 3.1.** [4] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $C \in \mathbb{C}^{m\times q}$, be known matrices. The matrix equation (3.1) has a Hermitian solution if and only if the pair of the matrix equation*

$$AXB = C \ \text{ and } B^*XA^* = C^* \tag{3.2}$$

*have a common solution, provided a Hermitian solution exists, a representation of the general Hermitian solution to (3.1) is of the form*

$$X_s = \frac{X + X^*}{2},$$

*where $X$ is the representation of the general common solution to equations (3.2).*

**Theorem 3.1.** *The matrix equation (3.1) has a Hermitian least-rank solution if and only if*

$$r\begin{bmatrix} M & 0 & S \\ 0 & -M^* & T^* \\ T & S^* & 0 \end{bmatrix} = 2r\begin{bmatrix} T, & S^* \end{bmatrix} + 2r(M). \tag{3.3}$$

*In this case the general Hermitian least-rank solution can be expressed as*

$$X_s = \frac{X + X^*}{2}, \tag{3.4}$$

*where*

$$\begin{aligned} X = & -TM^+S + P^+HF_{S_1} - P^+F_{S_1}K^+E_PHF_{S_1} + K^+E_PHP^+ \\ & - P^+E_{T_{22}}F_KV_1PF_{S_1} + T_1F_PU + T_1ZE_{F_{S1}} + F_KVS_1 + WF_PS_1. \end{aligned}$$

*Proof.* The general least-rank solution of (3.1) can be written as

$$X = -TM^+S + T_1U + VS_1,$$

where $U$, $V$ are arbitrary. So,

$$X^* = -(TM^+S)^* + U^*T_1^* + S_1^*V^*.$$

Therefore,

$$X - X^* = (TM^+S)^* - (TM^+S) + \begin{bmatrix} T_1 & S_1^* \end{bmatrix} \begin{bmatrix} U \\ -V^* \end{bmatrix} + \begin{bmatrix} V & -U^* \end{bmatrix} \begin{bmatrix} S_1 \\ T_1^* \end{bmatrix}.$$

Thus, from Lemma 1.2 we get

$$\min_X r(X - X^*) = r \begin{bmatrix} (TM^+S)^* - (TM^+S) & T_1 & S_1^* \\ S_1 & 0 & 0 \\ T_1^* & 0 & 0 \end{bmatrix} - 2r \begin{bmatrix} T_1 & S_1^* \end{bmatrix}. \tag{3.5}$$

We have that

$$\begin{aligned} r\begin{bmatrix} T_1 & S_1^* \end{bmatrix} &= r\begin{bmatrix} T & S^* \\ M & 0 \\ 0 & M^* \end{bmatrix} - 2r(M) \\ &= r\begin{bmatrix} T, & S^* \end{bmatrix} + r\begin{bmatrix} M \\ M^* \end{bmatrix} - 2r(M). \end{aligned} \tag{3.6}$$

And it is not difficult to find by Lemmas 1.2 and 1.3 and block elementary operations

$$\begin{aligned} & r\begin{bmatrix} (TM^+S)^* - (TM^+S) & T_1 & S_1^* \\ S_1 & 0 & 0 \\ T_1^* & 0 & 0 \end{bmatrix} \\ &= r\begin{bmatrix} (TM^+S)^* - (TM^+S) & T & S^* & 0 & 0 & 0 \\ S & 0 & 0 & M & 0 & 0 \\ T^* & 0 & 0 & 0 & M^* & 0 \\ 0 & M & 0 & 0 & 0 & 0 \\ 0 & 0 & M^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - 4r(M) \\ &= r\begin{bmatrix} M^*MM^* & 0 & M^*S & 0 & 0 & 0 & 0 \\ 0 & -MM^*M & MT^* & 0 & 0 & 0 & 0 \\ TM^* & S^*M & 0 & T & S^* & 0 & 0 \\ 0 & 0 & S & 0 & 0 & M & 0 \\ 0 & 0 & T^* & 0 & 0 & 0 & M^* \\ 0 & 0 & 0 & M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M^* & 0 & 0 \end{bmatrix} - 6r(M) \end{aligned}$$

$$= r\begin{bmatrix} M & 0 & S \\ 0 & -M^* & T^* \\ T & S^* & 0 \end{bmatrix} + r\begin{bmatrix} M \\ M^* \end{bmatrix} + r\begin{bmatrix} M, & M^* \end{bmatrix} - 6r(M). \tag{3.7}$$

A substitution of (3.6) and (3.7) into (3.5) gives (3.3). □

Now we want to find the expression of the general common Hermitian least-rank solution.

From Lemma 3.1 the matrix equation (3.1) has a Hermitian least-rank solution if and only if the pair of matrix equations

$$E_{T_1} X F_{S_1} = -E_T (TM^+ S) F_{S_1} \text{ and } F_{S_1} X E_{T_1} = -F_{S_1} (TM^+ S)^* E_{T_1},$$

has a common solution. So by the formula (2.4) we have

$$\begin{aligned} X = {} & -TM^+ S + P^+ H F_{S_1} - P^+ F_{S_1} K^+ E_P H F_{S_1} + K^+ E_P H P^+ \\ & - P^+ E_{T_{22}} F_K V_1 P F_{S_1} + T_1 F_P U + T_1 Z E_{F_{S1}} + F_K V S_1 + W F_P S_1, \end{aligned} \tag{3.8}$$

where

$$P = E_{T_1}, \quad H = -E_T (TM^+ S) F_{S_1} - F_{S_1} (TM^+ S)^* E_{T_1}.$$

Substituting (3.8) into (3.4) we get $X_s$.

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Sihem Guerarra
Department of Mathematics, Faculté of Sciences,
HADJ LAKHDAR University, Batna, Algeria.
&
Department of Mathematics,
Faculty of exact sciences and sciences of nature and life,
University of Oum El Bouaghi,
Algeria.
sihgue@yahoo.fr

Said Guedjiba
Department of Mathematics,
Faculty of Science,
HADJ LAKHDAR University,
Batna, Algeria.
saidguedjiba@yahoo.fr