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BEST PROXIMITY POINTS AND COUPLED BEST PROXIMITY POINTS OF $(\psi - \varphi - \theta)$-ALMOST WEAKLY CONTRACTIVE MAPS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper, we obtain some best proximity point results using almost contractive condition with three control functions (in which two of them need not be continuous) in partially ordered metric spaces. As an application, we prove coupled best proximity theorems. The results presented in this paper generalize the results of Choudhury, Metiya, Postolache and Konar [8]. We draw several corollaries and give illustrative examples to demonstrate the validity of our results.

Keywords: best proximity point, coupled best proximity point, almost weakly contractive map, partially ordered metric space

1. Introduction and Preliminaries

The well known Banach contraction principle asserts that every contraction self-map on a complete metric space has a unique fixed point. In recent research works, several authors deal with non-self maps to determine best proximity points. The purpose of best proximity point theory is to address a problem of finding the distance between two closed sets by using non-self mappings from one set to the other. This problem is known as the proximity point problem, which is considered here in the context of partially ordered metric spaces. Best proximity point theory analyze the existence of an approximate solution that is optimal. Let A be a non-empty subset of a metric space (X, d) and $f : A \to X$ has a fixed point in A if the fixed point equation $fx = x$ has at least one solution. If the fixed point equation $fx = x$ does not possess a solution, then $d(x, fx) > 0$ for all $x \in A$. In such a situation, it is our aim to find an element $x \in A$ such that $d(x, fx)$ is minimum as much as possible.

Let A and B be two non-empty subsets of a metric space (X, d) and $T : A \to B$ is a non-self mapping, then $d(x, Tx) \geq d(A, B)$ for all $x \in A$. In general for a non-self mapping $T : A \to B$, the fixed point equation $Tx = x$ may not have a solution.

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In this case, it is focused on the possibility of finding an element x that is in closed proximity to Tx in some sense, i.e., to find an approximate solution $x \in A$ such that error $d(x, Tx)$ is minimum, possibly $d(x, Tx) = d(A, B)$.

A point $x \in A$ is called best proximity point of $T : A \to B$ if $d(x, Tx) = d(A, B)$, where $d(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}$. A best proximity point becomes a fixed point if the underlying mapping is a self-mapping. Therefore, it can be concluded that best proximity point theorems generalize fixed point theorems in a natural way. In recent years, the existence and convergence of best proximity points is an interesting aspect of optimization theory which attracted the attention of many authors [1, 2, 3, 5, 6, 11, 12]. The best proximity point evolves as a generalization of the concept of the best approximation. The authors [7, 9, 10, 14, 17] and some references therein obtained best proximity point theorems under certain contraction conditions for non-self maps. Our purpose here is to establish best proximity point theorems in the context of partially ordered metric spaces.

We recall the following notations and definitions. Let $(X, d, \preceq)$ be a partially ordered metric space and let A and B be nonempty subsets of X.

$$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

Definition 1.1. [18] Let A and B be two non-empty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property, if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2).$$

Example 1.1. Let $X = [0, 1] \times [0, 1]$ with the Euclidean metric d. Let $A = \{0\} \times [0, 1]$ and $B = \{1\} \times [0, \frac{1}{2}]$. Clearly $d(A, B) = 1$, $A_0 = \{0\} \times [0, \frac{1}{2}] \neq \emptyset$ and $B_0 = B$. Let $(0, x_1)$, $(0, x_2) \in A_0$ and $(1, y_1)$, $(1, y_2) \in B_0$ with

$$d((0, x_1), (1, y_1)) = d((0, x_2), (1, y_2)) = d(A, B) = 1. \tag{1.1}$$

Clearly from (1.1), we get $x_1 = y_1$ and $x_2 = y_2$. i.e, $d((0, x_1), (0, x_2)) = d((1, y_1), (1, y_2))$ so that the pair (A, B) has the P-property.

Example 1.2. [1] Let A, B be two nonempty, bounded, closed, and convex subsets of a uniformly convex Banach space X. Then (A, B) has the P-property.

Definition 1.2. [6] A mapping $T : A \to B$ is said to be proximally increasing if for all $u_1, u_2, x_1, x_2 \in A$,

$$\left.\begin{array}{l} x_1 \preceq x_2 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \Rightarrow u_1 \preceq u_2.$$

Example 1.3. Let $X = \mathbb{R}^2$, with $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$, where (x_1, x_2), $(y_1, y_2) \in X$. We define a partial order $\preceq$ on X by $(x_1, x_2) \preceq (y_1, y_2)$ if and only if $x_1 \geq y_1$ and $x_2 \geq y_2$. Let $A = \{(0, x) : 0 \leq x \leq 2\}$, $B = \{(1, x) : 0 \leq x \leq 2\}$. Clearly $d(A, B) = 1$. We define $T : A \to B$ by

$$T(0, x) = \begin{cases} (1, \frac{x}{2}) & \text{if } x \in [0, 1] \\ (1, \frac{3}{2}x - 1) & \text{if } x \in [1, 2]. \end{cases}$$

It can easily be seen that T is proximally increasing on A.

Example 1.4. Let $X = \mathbb{R}^2$, with $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$, where (x_1, x_2), $(y_1, y_2) \in X$. We define a partial order $\preceq$ on X by $(x_1, x_2) \preceq (y_1, y_2)$ if and only if $x_1 \geq y_1$ and $x_2 \geq y_2$. Let $A = \{(0, x) : 0 \leq x \leq 3\}$, $B = \{(-2, x) : 0 \leq x \leq 3\}$. Clearly $d(A, B) = 2$. We define $T : A \to B$ by

$$T(0, x) = \begin{cases} (-2, 1 - x) & \text{if } x \in [0, 1] \\ (-2, x - 1) & \text{if } x \in [1, 3]. \end{cases}$$

It can easily be seen that T is not proximally increasing on A by choosing $(0, x), (0, y), (0, u)$ and $(0, v) \in A$, where $x, y, u, v \in [0, 1]$.

Definition 1.3. [13] A mapping $F : A \times A \to B$ is said to have the mixed monotone property if F is monotone nondecreasing in its first argument and is monotone non-increasing in its second argument. That is, if

$$x_1,\ x_2 \in A,\ x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y) \ \text{ for all } y \in A \text{ and}$$
$$y_1,\ y_2 \in A,\ y_1 \preceq y_2 \Rightarrow F(x, y_2) \preceq F(x, y_1) \ \text{ for all } x \in A.$$

Definition 1.4. [16] A mapping $F : A \times A \to B$ is said to have the proximal mixed monotone property if $F(x, y)$ is proximally nondecreasing in x and is proximally non-increasing in y. i.e.,

$$\left.\begin{array}{l} x_1 \preceq x_2 \\ d(u_1, F(x_1, y)) = d(A, B) \\ d(u_2, F(x_2, y)) = d(A, B) \end{array}\right\} \Rightarrow u_1 \preceq u_2 \text{ and}$$

$$\left.\begin{array}{l} y_1 \preceq y_2 \\ d(v_1, F(x, y_1)) = d(A, B) \\ d(v_2, F(x, y_2)) = d(A, B) \end{array}\right\} \Rightarrow v_2 \preceq v_1, \text{ where } x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2 \in A.$$

Definition 1.5. [19] An element $(x^*, y^*) \in A \times A$ is called a coupled best proximity point of the mapping $F : A \times A \to B$ if $d(x^*, F(x^*, y^*)) = d(A, B)$ and $d(y^*, F(y^*, x^*)) = d(A, B)$.

In order to obtain new coupled best proximity point results we use some notations, definitions and lemmas from Choudhury et. al [8].

Let $(X, d, \preceq)$ be a partially ordered metric space. We define a partial order $\preceq_1$ on $X \times X$ by $(u, v) \preceq_1 (x, y)$ if and only if $u \preceq x$ and $y \preceq v$ for $(u, v), (x, y) \in X \times X$. Further, we define a metric d_1 on $X \times X$ by

$d_1\big((x,y),(u,v)\big) = d(x,u) + d(y,v)$ for $(u,v),(x,y) \in X \times X$. Then d_1 is a metric on $X \times X$ and we call $(X \times X, d_1, \preceq_1)$ is a partially ordered product space. Here we observe that if d is complete, then d_1 is so.

We denote by $A^* = A \times A$, $B^* = B \times B$.
$A_0^* = \{x = (x_1, y_1) \in A^* : d_1(x,y) = d_1(A^*, B^*)$ for some $y = (x_2, y_2) \in B^*\}$ and
$B_0^* = \{y = (x_2, y_2) \in B^* : d_1(x,y) = d_1(A^*, B^*)$ for some $x = (x_1, y_1) \in A^*\}$.

Lemma 1.1. [8] *If a pair (A,B) has P-property, then the pair (A^*, B^*) has also the P-property.*

Lemma 1.2. [8] *Let $F : A \times A \to B$ be a mapping with $F(A_0 \times A_0) \subseteq B_0$. If F has the proximal mixed monotone property on $A_0 \times A_0$, then the mapping $T : A^* \to B^*$ defined by $T(x_1, y_1) = \big(F(x_1, y_1), F(y_1, x_1)\big)$ for $(x_1, y_1) \in A^*$ is proximally increasing on A_0^*.*

We denote by Ψ the set of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying

(i) ψ is continuous and

(ii) $\psi(t) = 0$ if and only if $t = 0$.

We denote by Θ the set of all functions $\alpha : [0,\infty) \to [0,\infty)$ satisfying

(i) α is bounded on any bounded interval in $[0,\infty)$ and

(ii) α is continuous at 0 and $\alpha(0) = 0$.

In 2015, Choudhury, Metiya, Postolache, Konar [8] proved the existence and uniqueness of best proximity points for non-self mapping in partially ordered metric space.

Theorem 1.1. [8] *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let (A,B) be a pair of non-empty closed subsets of X such that A_0 is non-empty closed and the pair (A,B) satisfies the P-property. Let $T : A \to B$ be a mapping such that $T(A_0) \subseteq B_0$ and T is proximally increasing on A_0. Suppose that there exist $\psi \in \Psi$ and $\varphi, \theta \in \Theta$ such that*

$$\psi(x) \le \varphi(y) \Rightarrow x \le y, \tag{1.2}$$

for any sequence $\{x_n\}$ in $[0,\infty)$ with $x_n \to t > 0$,

$$\psi(t) - \overline{\lim}\varphi(x_n) + \underline{\lim}\theta(x_n) > 0 \tag{1.3}$$

and for all $x, y \in A_0$ with $x \succeq y$

$$\psi(d(Tx,Ty)) \le \varphi(d(x,y)) - \theta(d(x,y)). \tag{1.4}$$

Suppose either

(a) *T is continuous or*

(b) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \geq 0$.*

Also, suppose that there exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \preceq x_1$.
Then T has a best proximity point in A_0, that is, there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A, B)$.

Definition 1.6. We call a map $T : A \to B$ that satisfies the inequality (1.4) is a $(\psi - \varphi - \theta)$-weakly contractive map.

Theorem 1.2. [8] *In addition to the hypotheses of Theorem 1.1, suppose that for every $x, y \in A_0$ there exists $u \in A_0$ such that u is comparable to x and y. Then T has a unique best proximity point in A_0.*

Lemma 1.3. [4] *Suppose that (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geqslant \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and*

(i) $\lim\limits_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$;

(ii) $\lim\limits_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$;

(iii) $\lim\limits_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$.

Remark 1.1. By using the hypotheses of Lemma 1.3 and triangular inequality we can show that $\lim\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \epsilon$.

In the following we define the notion of almost weakly contractive map.

Definition 1.7. Let $(X, d, \preceq)$ be a partially ordered metric space. Let A and B be nonempty subsets of X. Let $T : A \to B$ be non-self map. If there exist $\psi \in \Psi$, $\varphi, \theta \in \Theta$ and $L \geq 0$ such that for all $x, y \in A_0$ with $x \succeq y$

$$\psi(d(Tx, Ty)) \leq \varphi(d(x, y)) - \theta(d(x, y)) + L\, n(x, y), \text{ where} \tag{1.5}$$

$$n(x,y) = \min\{d(x,Tx)-d(A,B),\ d(y,Ty)-d(A,B),\ d(x,Ty)-d(A,B),\ d(y,Tx)-d(A,B)\},$$

then we call T is a $(\psi - \varphi - \theta)$-almost weakly contractive map.

If $L = 0$ in (1.5), then T is a $(\psi - \varphi - \theta)$-weakly contractive map.

Example 1.5. Let $X = \mathbb{R}^2$, with $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We define a partial order $\preceq$ on X by

$$\preceq := \big\{(x,y) \in X \times X, \text{ where } x = (x_1,x_2) \text{ and } y = (y_1,y_2) \text{ with } x_1 = y_1 \text{ and } x_2 = y_2\big\}$$

$$\cup \big\{\big((0,1),(0,\frac{1}{2})\big), \big((0,\frac{1}{2}),(0,\frac{1}{4})\big), \big((0,1),(0,\frac{1}{4})\big), \big((0,\frac{9}{4}),(0,2)\big)\big\}, \text{ where } (x_1,x_2) \preceq (y_1,y_2)$$

if and only if $x_1 \geq y_1$ and $x_2 \geq y_2$ for all (x_1,x_2), $(y_1,y_2) \in X$.

Let $A = \{(0,x) : 0 \leq x \leq 3\}$, $B = \{(1,x) : 0 \leq x \leq 3\}$. We define $T : A \to B$ by

$$T(0,x) = \begin{cases} (1, \frac{x}{2}) & \text{if } x \in [0,2] \\ (1, 2x-3) & \text{if } x \in [2,3]. \end{cases}$$

We define functions ψ, $\varphi, \theta : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \begin{cases} t & \text{if } t \in [0,1] \\ t^2 & \text{if } t \geq 1 \end{cases} \quad \varphi(t) = \begin{cases} t & \text{if } t \in [0,1] \\ \frac{t^2}{2} & \text{if } t \geq 1 \end{cases} \quad \text{and} \quad \theta(t) = \begin{cases} \frac{t}{2} & \text{if } t \in [0,1] \\ \frac{t^2}{4} & \text{if } t \geq 1. \end{cases}$$

The inequality (1.5) holds with $L = 1$. Hence T is $(\psi - \varphi - \theta)$-almost weakly contractive map. The inequality fails to hold if $L = 0$, $x = (0, \frac{9}{4})$ and $y = (0,2)$.

In Section 2 of this paper, we prove our main results and in Section 3 we draw some corollaries and provide examples in support of our results.

2. Main Results

Theorem 2.1. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let (A, B) be a pair of non-empty closed subsets of X such that A_0 is non-empty closed and (A, B) satisfies the P-property. Let $T : A \to B$ be a mapping which satisfies the $(\psi - \varphi - \theta)$-almost weakly contractive condition such that $T(A_0) \subseteq B_0$ and T is proximally increasing on A_0. Suppose that there exist $L \geq 0$, $\psi \in \Psi$ and $\varphi, \theta \in \Theta$ such that*

$$\psi(x) \leq \varphi(y) \Rightarrow x \preceq y, \tag{2.1}$$

for any sequence $\{x_n\}$ in $[0,\infty)$ with $x_n \to t > 0$,

$$\psi(t) - \overline{\lim}\varphi(x_n) + \underline{\lim}\theta(x_n) > 0. \tag{2.2}$$

Furthermore, assume that either

(a) T is continuous or

(b) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \geq 0$.

Also, suppose that there exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A,B)$ and $x_0 \preceq x_1$.
Then T has a best proximity point in A_0, that is, there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A,B)$.

Proof. By assumption, there exist $x_0,\ x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B) \text{ and } x_0 \preceq x_1. \tag{2.3}$$

Since $T(A_0) \subseteq B_0$, there exists an element $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B). \tag{2.4}$$

Since T is proximally increasing on A_0, from (2.3) and (2.4), we have $x_1 \preceq x_2$. As (A, B) satisfies the P-property, from (2.3) and (2.4), we have

$$d(x_1, x_2) = d(A, B).$$

On continuing this process, we get a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ for all } n \geq 0, \tag{2.5}$$

$$x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ... \quad \text{and} \tag{2.6}$$

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \text{ for all } \ n \geq 1. \tag{2.7}$$

Since $x_{n+1} \succeq x_n$, by the inequality (1.5), we have

$$\psi(d(x_{n+2}, x_{n+1})) = \psi(d(Tx_{n+1}, Tx_n)) \leq \varphi(d(x_{n+1}, x_n)) - \theta(d(x_{n+1}, x_n)) + L\, n(x_n, x_{n+1}),$$

where $n(x_n, x_{n+1}) = \min\{d(x_{n+1}, Tx_{n+1}) - d(A, B), d(x_n, Tx_n) - d(A, B),$
$d(x_{n+1}, Tx_n) - d(A, B), d(x_n, Tx_{n+1}) - d(A, B)\} = 0.$

Therefore

$$\psi(d(x_{n+2}, x_{n+1})) \leq \varphi(d(x_{n+1}, x_n)) - \theta(d(x_{n+1}, x_n)) \leq \varphi(d(x_{n+1}, x_n)), \tag{2.8}$$

because $\theta(d(x_{n+1}, x_n)) \geq 0$. Hence $\psi(d(x_{n+2}, x_{n+1})) \leq \varphi(d(x_{n+1}, x_n))$, which implies by (2.1) that $d(x_{n+2}, x_{n+1}) \leq d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$.
Hence $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence of nonnegative real numbers. Then there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r. \tag{2.9}$$

If possible suppose that $r > 0$. On taking the upper limit on both sides of (2.8) and using (2.9), we obtain

$$\psi(r) \leqslant \overline{\lim}\varphi(d(x_{n+1}, x_n)) + \overline{\lim}(-\theta(d(x_{n+1}, x_n))). \tag{2.10}$$

Since $\overline{\lim}(-\theta(d(x_{n+1}, x_n))) = -\underline{\lim}(\theta(d(x_{n+1}, x_n)))$, from (2.10), it follows that

$$\psi(r) - \overline{\lim}\varphi(d(x_{n+1}, x_n)) + \underline{\lim}\theta(d(x_{n+1}, x_n)) \leq 0,$$

a contradiction. Hence $r = 0$.

We now show that the sequence $\{x_n\}$ is Cauchy. Suppose that the sequence $\{x_n\}$ is not Cauchy. Then by Lemma 1.3, there exists an $\epsilon > 0$ for which we can find sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and the identities (i)-(iii) of Lemma 1.3 and Remark 1.1 are satisfied. Since $m_k > n_k$, from (2.6) we have $x_{m_k} \succeq x_{n_k}$. Therefore by using the inequality (1.5), we obtain

$$\begin{aligned}(2.11)\qquad \psi(d(x_{m_k+1}, x_{n_k+1})) &= \psi(d(Tx_{m_k}, Tx_{n_k})) \leq \varphi(d(x_{m_k}, x_{n_k}))\\ &- \theta(d(x_{m_k}, x_{n_k})) + L \min\{d(x_{m_k}, Tx_{m_k}) - d(A,B), d(x_{n_k}, Tx_{n_k}) - d(A,B),\\ &\quad d(x_{n_k}, Tx_{m_k}) - d(A,B), d(x_{m_k}, Tx_{n_k}) - d(A,B)\}\\ &\leq \varphi(d(x_{m_k}, x_{n_k})) - \theta(d(x_{m_k}, x_{n_k})) + L \min\{d(x_{n_k}, x_{n_k+1})\\ &\quad + d(x_{n_k+1}, Tx_{n_k}) - d(A,B), d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, Tx_{m_k}) - d(A,B),\\ &\quad d(x_{n_k}, x_{m_k+1}) + d(x_{m_k+1}, Tx_{m_k}) - d(A,B),\\ &\quad d(x_{m_k}, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) - d(A,B)\}\\ &= \varphi(d(x_{m_k}, x_{n_k})) - \theta(d(x_{m_k}, x_{n_k})) + L \min\{d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1}),\\ &\quad d(x_{n_k}, x_{m_k+1}), d(x_{m_k}, x_{n_k+1})\}.\end{aligned}$$

On taking the upper limit on both sides of (2.11) and using Lemma 1.3, we get

$$(2.12)\qquad \psi(\epsilon) \leqslant \overline{\lim}\varphi(d(x_{m_k}, x_{n_k})) + \overline{\lim}(-\theta(d(x_{m_k}, x_{n_k}))).$$

Since $\overline{\lim}(-\theta(d(x_{m_k}, x_{n_k}))) = -\underline{\lim}\theta(d(x_{m_k}, x_{n_k}))$, from (2.12), it follows that

$$\psi(\epsilon) - \overline{\lim}\varphi(d(x_{m_k}, x_{n_k})) + \underline{\lim}\theta(d(x_{m_k}, x_{n_k})) \leq 0,$$

a contradiction. Hence $\{x_n\}$ is a Cauchy sequence.

Since X is complete and A_0 is a closed subset of X and hence complete. From the completeness of A_0, there exists $x^* \in A_0$ such that $\lim_{n\to\infty} x_n = x^*$, that is

$$(2.13)\qquad \lim_{n\to\infty} d(x_n, x^*) = 0.$$

First we assume condition (a). i.e., T is continuous. On taking limit as $n \to \infty$ in (2.5) and using the continuity of T, we obtain $d(x^*, Tx^*) = d(A,B)$. Therefore x^* is the best proximity point of T.

We now assume condition (b). By condition (b) of the theorem and from the fact that $x_n \to x^*$, we have $x_n \preceq x^*$ for all $n \geq 0$. Since $x^* \in A_0$, we have $Tx^* \in T(A_0) \subseteq B_0$ and therefore there exists a point $z \in A_0$ such that $d(z, Tx^*) = d(A,B)$. From (2.5), we have $d(x_{n+1}, Tx_n) = d(A,B)$ and applying the P-property of (A,B), we get $d(z, x_{n+1}) = d(Tx^*, Tx_n)$. Hence by applying the inequality (1.5), we have

$$(2.14)\qquad \psi(d(z, x_{n+1})) = \psi(d(Tx^*, Tx_n)) \leq \varphi(d(x^*, x_n)) - \theta(d(x^*, x_n))$$

$$
\begin{aligned}
&+ L \min \{d(x^*, Tx^*) - d(A,B), d(x_n, Tx_n) - d(A,B), \\
&\qquad d(x^*, Tx_n) - d(A,B), d(x_n, Tx^*) - d(A,B)\} \\
&\le \varphi(d(x^*, x_n)) - \theta(d(x^*, x_n)) + L \min \{d(x^*, Tx^*) - d(A,B), d(x_n, Tx_n) - d(A,B), \\
&\qquad d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n) - d(A,B), d(x_n, Tx^*) - d(A,B)\} \\
&= \varphi(d(x^*, x_n)) - \theta(d(x^*, x_n)) + L \min \{d(x^*, Tx^*) - d(A,B), \\
&\qquad d(x_n, Tx_n) - d(A,B), d(x^*, x_{n+1}), d(x_n, Tx^*) - d(A,B)\}.
\end{aligned}
$$

On taking limit as $n \to \infty$ in (2.14), using (2.13), the property (ii) of φ and θ and the property of ψ, we obtain

$$\lim_{n\to\infty} \psi(d(z, x_{n+1})) \le \lim_{n\to\infty} \varphi(d(x^*, x_n)) - \lim_{n\to\infty} \theta(d(x^*, x_n)) \le 0.$$

Therefore $d(z, x^*) = 0$ which implies that $z = x^*$. Hence, we have $d(x^*, Tx^*) = d(A,B)$. □

Theorem 2.2. *In addition to the hypotheses of Theorem 2.1, assume the following.*
Condition (H): for every $x, y \in A_0$ there exists $u \in A_0$ such that u is comparable to x and y.
Then T has a unique best proximity point in A_0.

Proof. In view of the proof of Theorem 2.1, the set of best proximity points of T is non-empty. Suppose that $x, y \in A_0$ are two distinct best proximity points of T. That is,

$$d(x, Tx) = d(A,B) \text{ and } d(y, Ty) = d(A,B). \tag{2.15}$$

Case (i): x is comparable to y. i.e., either $x \succeq y$ or $y \succeq x$.

We assume, without loss of generality, that $x \succeq y$. Since (A,B) has the P-property, from (2.15), it follows that

$$d(x, y) = d(Tx, Ty). \tag{2.16}$$

Since $x \succeq y$, by using the inequality (1.5), we get

$$
\begin{aligned}
\psi(d(x,y)) = \psi(d(Tx,Ty)) &\le \varphi(d(x,y)) - \theta(d(x,y)) + L \min \{d(x,Tx) - d(A,B), \\
&\qquad d(y,Ty) - d(A,B), d(x,Ty) - d(A,B), d(y,Ty) - d(A,B)\} \\
&= \varphi(d(x,y)) - \theta(d(x,y)).
\end{aligned}
$$

Hence on taking the upper limit in the above inequality, we obtain

$$\psi(d(x,y)) - \overline{\lim}\varphi(d(x,y)) + \underline{\lim}\theta(d(x,y)) \le 0,$$

a contradiction. Hence $x = y$.

Case (ii): x is not comparable to y.

By the hypothesis, there exists $u \in A_0$ such that u is comparable to x and y. Now, we set $u_0 = u$. Suppose that either

$$u_0 \succeq x \quad \text{or} \quad x \succeq u_0. \tag{2.17}$$

We assume, without loss of generality, that

$$u_0 \preceq x. \tag{2.18}$$

Since $T(A_0) \subseteq B_0$ and $u = u_0 \in A_0$, we have $Tu_0 \in B_0$. Hence there exists $u_1 \in A_0$ such that

$$d(u_1, Tu_0) = d(A, B). \tag{2.19}$$

Since T is proximally increasing on A_0, from (2.15), (2.18) and (2.19), we have $u_1 \preceq x$. By using the P-property of the pair (A, B), from (2.15) and (2.19), we have

$$d(x, u_1) = d(Tx, Tu_0).$$

On continuing this process we construct a sequence $\{u_n\}$ in A_0 such that

$$d(x, u_{n+1}) = d(Tx, Tu_n) \text{ and } u_n \preceq x \text{ for all } \ n \geq 0. \tag{2.20}$$

Since $x \succeq u_n$, by the inequality (1.5), we have

$$\begin{aligned}\psi(d(x, u_{n+1})) = \psi(d(Tx, Tu_n)) &\leq \varphi(d(x, u_n)) - \theta(d(x, u_n)) + L \min\{d(x, Tx) - d(A, B),\\ &\quad d(u_n, Tu_n) - d(A, B), d(x, Tu_n) - d(A, B), d(u_n, Tx) - d(A, B)\}\\ &= \varphi(d(x, u_n)) - \theta(d(x, u_n)) \leq \ \varphi(d(x, u_n)).\end{aligned} \tag{2.21}$$

From (2.21), it follows that $\psi(d(x, u_{n+1})) \leq \varphi(d(x, u_n))$. Hence by condition (2.1), we obtain $d(x, u_{n+1}) \leq d(x, u_n)$. Therefore $\{d(x, u_n)\}$ is a decreasing sequence of non-negative real numbers. Hence there exists $r \geq 0$ such that

$$\lim_{n \to \infty} d(x, u_n) = r. \tag{2.22}$$

Suppose that $r > 0$. On taking the upper limit on both sides of (2.21), we have

$$\psi(r) \leq \overline{\lim} \varphi(d(x, u_n)) + \overline{\lim}(-\theta(d(x, u_n))).$$

i.e.,

$$\psi(r) - \overline{\lim} \varphi(d(x, u_n)) + \underline{\lim} \theta(d(x, u_n)) \leq 0,$$

which is a contradiction. Hence $r = 0$.

Similarly, we can show that $\lim\limits_{n \to \infty} d(y, u_n) = 0$. Hence by uniqueness of limit, it follows that $x = y$. $\square$

In the following theorem we apply the above results to prove the existence of coupled best proximity points.

Theorem 2.3. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let (A, B) be a pair of non-empty closed subsets of X such that A_0 is non-empty closed and the pair (A, B) satisfies the P-property. Let $F : A \times A \to B$ be a mapping such that $F(A_0 \times A_0) \subseteq B_0$ and F has proximal mixed monotone property on $A_0 \times A_0$. Suppose that there exist $\psi \in \Psi, \varphi, \theta \in \Theta$ and $L \geq 0$ such that (2.1) and (2.2) are satisfied and for all $(x, y), (u, v) \in A_0 \times A_0$ with $(x, y) \succeq (u, v)$,*

$$\begin{aligned}
(2.23)\qquad \psi\big(d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))\big) &\leq \varphi\big(d(x,u)+d(y,v)\big) \\
&\quad - \theta\big(d(x,u)+d(y,v)\big) + L\ m(x,y,u,v),\ \text{where} \\
m(x,y,u,v) = \min\{&d(x,F(x,y)) + d(y,F(y,x)) - 2d(A,B), \\
&d(x,F(u,v)) + d(y,F(v,u)) - 2d(A,B), \\
&d(u,F(x,y)) + d(v,F(y,x)) - 2d(A,B), \\
&d(u,F(u,v)) + d(v,F(v,u)) - 2d(A,B)\}.
\end{aligned}$$

Suppose that either

(a) *F is continuous or*

(b) *X has the following properties*

(i) *if $\{x_n\}$ is a nondecreasing sequence such that $x_n \to x$, then $x_n \preceq x$ for all $n \geq 0$;*

(ii) *if $\{y_n\}$ is a non-increasing sequence such that $y_n \to y$, then $y \preceq y_n$ for all $n \geq 0$.*

Also, suppose that there exist $(x_0, y_0), (x_1, y_1) \in A_0 \times A_0$ such that $d(x_1, F(x_0, y_0)) = d(A, B)$ and $d(y_1, F(y_0, x_0)) = d(A, B)$ with $(x_0, y_0) \preceq (x_1, y_1)$. Then F has a coupled best proximity point in $A_0 \times A_0$, that is, there exists an element $(x^*, y^*) \in A_0 \times A_0$ such that $d(x^*, F(x^*, y^*)) = d(A, B)$ and $d(y^*, F(y^*, x^*)) = d(A, B)$.

Proof. Let $(X \times X, d_1, \preceq_1)$ be a partially ordered product space, where

$$d_1\big((x,y),(u,v)\big) = d(x,u) + d(y,v),$$

d is a metric on X and

$$(u,v) \preceq_1 (x,y) \iff u \preceq x \text{ and } y \preceq v \text{ for } (u,v),(x,y) \in X \times X.$$

Since d is complete, then d_1 is complete. We define a function $T : A^* \to B^*$ by

$$T(x,y) = (F(x,y), F(y,x)) \text{ for all } (x,y) \in A^*.$$

By hypothesis (A, B) is a pair of non-empty closed subsets of X such that A_0 is non-empty, closed and (A, B) has the P-property. Hence by definition of A^*, B^* and by Lemma 1.1, we have (A^*, B^*) is a pair of non-empty closed subsets of $X \times X$ such that A_0^* is non-empty, closed and the pair (A^*, B^*) has the P-property. Since $F(A_0 \times A_0) \subseteq B_0$, we have $T(A_0^*) \subseteq B_0^*$.

Since F is continuous, it follows that T is continuous so that condition (a) is satisfied.

Also, by Lemma 1.2, it follows that T is proximally increasing on A_0^*. By the hypothesis there exist $z_0 = (x_0, y_0),\ z_1 = (x_1, y_1) \in A_0 \times A_0$ such that

$$d(x_1, F(x_0, y_0)) = d(y_1, F(y_0, x_0)) = d(A, B) \tag{2.24}$$

with $(x_0, y_0) \preceq_1 (x_1, y_1)$. i.e., $z_0 \preceq_1 z_1$. From (2.24), we have $d_1((x_1, y_1), (F(x_0, y_0), F(y_0, x_0))) = d_1(A^*, B^*)$. i.e.,

$$d_1(z_1, Tz_0) = d_1(A^*, B^*). \tag{2.25}$$

Since $(x_1, y_1) \in A_0 \times A_0$, $(y_1, x_1) \in A_0 \times A_0$ and $F(A_0 \times A_0) \subseteq B_0$, it follows that $F(x_1, y_1) \in B_0$ and $F(y_1, x_1) \in B_0$. Hence there exists $z_2 = (x_2, y_2) \in A_0 \times A_0$ such that

$$d(x_2, F(x_1, y_1)) = d(A, B) \quad \text{and} \tag{2.26}$$

$$d(y_2, F(y_1, x_1)) = d(A, B). \tag{2.27}$$

From (2.26) and (2.27), we have $d_1((x_2, y_2), (F(x_1, y_1), F(y_1, x_1)) = d_1(A^*, B^*)$. i.e.,

$$d_1(z_2, Tz_1) = d_1(A^*, B^*). \tag{2.28}$$

From the fact that F has proximal mixed monotone property and by Lemma 1.2, it follows that T is proximally increasing. Therefore from $z_0 \preceq_1 z_1$, (2.25) and (2.28), we obtain $z_1 \preceq_1 z_2$. On continuing this process, we construct a nondecreasing sequence $\{z_n\}$ in $X \times X$. From completeness of $X \times X$, there exists $z \in X \times X$ such that $z_n \to z$. Hence condition (b) is satisfied.

Now let $p = (x, y),\ q = (u, v) \in A_0 \times A_0$ such that $q \preceq_1 p$, then (2.23) reduce to $\psi\big(d_1(Tp, Tq))\big) \leq \varphi\big(d_1(p, q)\big) - \theta\big(d_1(p, q)\big) + L\ m(p, q)$, where

$$m(p,q) = \min\{d_1(p, Tp) - d_1(A^*, B^*),\ d_1(p, Tq) - d_1(A^*, B^*), d_1(q, Tp)) - d_1(A^*, B^*),$$
$$d_1(q, Tq) - d_1(A^*, B^*)\}.$$

By the hypothesis, there exist $(x_0, y_0),\ (x_1, y_1) \in A_0 \times A_0$ such that

$$d(x_1, F(x_0, y_0)) = d(A, B) \text{ and } d(y_1, F(y_0, x_0)) = d(A, B) \text{ with } (x_0, y_0) \preceq_1 (x_1, y_1),$$

then $z_0 = (x_0, y_0),\ z_1 = (x_1, y_1) \in A_0^*$ such that $d_1(z_1, Tz_0) = d_1(A^*, B^*)$ with $z_0 \preceq_1 z_1$.

Hence T satisfies all the hypotheses of Theorem 2.1 and hence T has a best proximity point in A_0^*, so that there exists an element $z^* = (x^*, y^*) \in A_0^*$ such that
$d_1(z^*, Tz^*) = d_1(A^*, B^*)$; which implies that,
$d_1((x^*, y^*), T(x^*, y^*)) = d_1(A \times A, B \times B)$. That is,
$d(x^*, F(x^*, y^*)) + d(y^*, F(y^*, x^*)) = d(A, B) + d(A, B)$, from which we have

$$d(x^*, F(x^*, y^*)) = d(A, B) \text{ and } d(y^*, F(y^*, x^*)) = d(A, B).$$

Therefore $(x^*, y^*) \in A_0^*$ is coupled best proximity point of F. □

In following theorem we prove uniqueness of coupled best proximity point under certain additional hypothesis.

Theorem 2.4. *In addition to the hypotheses of Theorem 2.3, assume the following.*
Condition (H_1): *for every* $(x, y), (x^*, y^*) \in A_0 \times A_0$ *there exists* (u, v) *such that* (u, v) *is comparable to* (x, y) *and* (x^*, y^*).
Then F *has a unique coupled best proximity point.*

Proof. By the proof of Theorem 2.3, the set of coupled best proximity points of F is non-empty. Suppose $(x, y),\ (x^*, y^*) \in A_0 \times A_0$ are two distinct coupled best proximity points of F. i.e.,
$d(x, F(x, y)) = d(y, F(y, x)) = d(x^*, F(x^*, y^*)) = d(y^*, F(y^*, x^*)) = d(A, B)$.
From this it follows that

$$d(x, F(x, y)) + d(y, F(y, x)) = 2d(A, B) \quad \text{and}$$
$$d(x^*, F(x^*, y^*)) + d(y^*, F(y^*, x^*)) = 2d(A, B).$$

From this and by the definition of d_1, we obtain

$$\left.\begin{aligned} d_1\big((x, y), (F(x, y), F(y, x))\big) &= d_1(A^*, B^*) \\ d_1\big((x^*, y^*), (F(x^*, y^*), F(y^*, x^*))\big) &= d_1(A^*, B^*). \end{aligned}\right\} \tag{2.29}$$

By Lemma 1.1, the pair (A^*, B^*) has P-property and hence, we have

$$d_1\big((x, y), (x^*, y^*)\big) = d_1\big((F(x, y), F(y, x)), (F(x^*, y^*), F(y^*, x^*))\big). \tag{2.30}$$

Let $p = (x, y)$ and $q = (x^*, y^*)$, hence equation (2.30) reduce to

$$d_1\big(p, q\big) = d_1\big(Tp, Tq\big). \tag{2.31}$$

Suppose that p and q are two distinct best proximity points of T.
Case (i): p is comparable to q. i.e., either $p \preceq_1 q$ or $q \preceq_1 p$.
We assume, without loss of generality, that $q \preceq_1 p$. Since $q \preceq_1 p$, by using the inequality (2.6), we get

$$\begin{aligned} \psi(d_1(p, q)) =& \psi(d_1(Tp, Tq)) \\ &\leq \varphi(d_1(p, q)) - \theta(d_1(p, q)) + L \min \{d_1(p, Tp) - d_1(A^*, B^*), \\ & d_1(q, Tq) - d_1(A^*, B^*), d_1(p, Tq) - d_1(A^*, B^*), d_1(q, Tq) - d_1(A^*, B^*)\} \end{aligned}$$

$$= \varphi(d_1(p,q)) - \theta(d_1(p,q)).$$

Since $(x,y) \neq (x^*,y^*)$, it follows that $p \neq q$. Hence on taking the upper limit in the above inequality, we get

$$\psi(d_1(p,q)) - \overline{\lim}\varphi(d_1(p,q)) + \underline{\lim}\theta(d_1(p,q)) \leq 0,$$

a contradiction. Hence $p = q$. i.e., $x = x^*$ and $y = y^*$.
Case (ii): p is not comparable to q.

By the hypothesis, there exists $r = (u,v) \in A_0 \times A_0$ such that r is comparable to p and q. Now, we set $r_0 = (u_0,v_0) = (u,v) = r$. Suppose that either

$$p \preceq_1 r_0 \text{ or } r_0 \preceq p. \tag{2.32}$$

We assume, without loss of generality, that

$$r_0 \preceq_1 p. \tag{2.33}$$

Since $F(A_0 \times A_0) \subseteq B_0$ and $(u_0,v_0) = r_0 = r = (u,v) \in A_0 \times A_0$, we have $F(u_0,v_0),\ F(v_0,u_0) \in B_0$. Therefore, there exists $r_1 = (u_1,v_1) \ \in A_0 \times A_0$ such that

$$d_1\big((u_1,v_1),(F(u_0,v_0),F(v_0,u_0))\big) = d_1(r_1,Tr_0) = d_1(A^*,B^*). \tag{2.34}$$

By Lemma 1.2, T is proximally increasing on A_0 and hence from (2.33), (2.34) and (2.29), we have

$$r_1 \preceq_1 p. \tag{2.35}$$

By using the P-property of the pair (A^*,B^*), (2.29), and (2.34), we get

$$d_1(p,r_1) = d_1(Tp,Tr_0).$$

On continuing this process we can construct a sequence $\{r_n\}$ in $A_0 \times A_0$ such that

$$d_1(p,r_{n+1}) = d_1(Tp,Tr_n) \text{ and } r_n \preceq_1 p \text{ for all } n \geq 0. \tag{2.36}$$

Since $r_n \preceq_1 p$ for all $n \geq 0$, by the inequality (1.5), we have

$$\begin{aligned}\psi(d_1(p,r_{n+1})) &= \psi(d_1(Tp,Tr_n))\\ &\leq \varphi(d_1(p,r_n)) - \theta(d_1(p,r_n)) + L \min\{d_1(p,Tp) - d_1(A^*,B^*),\\ &\quad d_1(r_n,Tr_n) - d_1(A^*,B^*), d_1(p,Tr_n) - d_1(A^*,B^*), d_1(r_n,Tp) - d_1(A^*,B^*)\}\end{aligned}$$

$$= \varphi(d_1(p,r_n)) - \theta(d_1(p,r_n)) \leq \varphi(d_1(p,r_n)). \tag{2.37}$$

From (2.37), it follows that $\psi(d_1(p,r_{n+1})) \leq \varphi(d_1(p,r_n))$. Hence by (2.1), we have $d_1(p,r_{n+1}) \leq d_1(p,r_n)$. Therefore $\{d_1(p,r_n)\}$ is a decreasing sequence of non-negative real numbers. Hence there exists $s \geq 0$ such that

$$\lim_{n\to\infty} d_1(p,r_n) = s. \tag{2.38}$$

Suppose $s > 0$. On taking the upper limit on both sides of (2.37), we have
$\psi(s) \leq \overline{\lim}\varphi(d_1(p, r_n)) + \overline{\lim}(-\theta(d_1(p, r_n)))$. i.e.,
$\psi(s) - \overline{\lim}\varphi(d_1(p, r_n)) + \underline{\lim}\theta(d_1(p, r_n)) \leq 0$,
which is a contradiction. Hence $s = 0$.

Similarly, we can show that $\lim_{n\to\infty} d_1(q, r_n) = 0$. Hence by the uniqueness of limit, it follows that $p = q$. i.e., $x = x^*$ and $y = y^*$. $\square$

3. Corollaries and Examples

If ψ is the the identity mapping and $\theta(t) = 0$ for all $t \in [0, \infty)$ in Theorem 2.1, we have the following.

Corollary 3.1. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let (A, B) be a pair of non-empty closed subsets of X such that A_0 is non-empty closed and (A, B) satisfies the P-property. Let $T : A \to B$ be a mapping such that $T(A_0) \subseteq B_0$ and T is proximally increasing on A_0. Suppose that there exist $L \geq 0$ and $\varphi \in \Theta$ such that for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \to t > 0$, $\overline{\lim}\varphi(x_n) < t$ and for all $x, y \in A_0$ with $x \succeq y$,*

$$d(Tx, Ty) \leq \varphi(d(x, y)) + L \min \{d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), d(x, Ty) - d(A, B), d(y, Tx) - d(A, B)\}.$$

Assume that either

(a) T is continuous or

(b) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \geq 0$.

Also, suppose that there exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \preceq x_1$.
Then T has a best proximity point in A_0.

If $\psi(t) = \varphi(t)$ for all $t \in [0, \infty)$ in Theorem 2.1, we have the following.

Corollary 3.2. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let (A, B) be a pair of non-empty closed subsets of X such that A_0 is non-empty closed and (A, B) satisfies the P-property. Let $T : A \to B$ be a mapping such that $T(A_0) \subseteq B_0$ and T is proximally increasing on A_0. Suppose that there exist $L \geq 0$, $\psi \in \Psi$ and $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \to t > 0$, $\underline{\lim}\theta(x_n) > 0$ and for all $x, y \in A_0$ with $x \succeq y$,*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \theta(d(x, y)) + L \min \{d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), d(x, Ty) - d(A, B), d(y, Tx) - d(A, B)\}.$$

Assume that either

(a) *T is continuous or*

(b) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \geq 0$.*

Also, suppose that there exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \preceq x_1$.
Then T has a best proximity point in A_0.

If ψ and φ are identity mappings and $\theta(t) = (1-k)t$, where $0 \leq k < 1$ in Theorem 2.1, we have the following.

Corollary 3.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let (A, B) be a pair of non-empty closed subsets of X such that A_0 is non-empty closed and (A, B) satisfies the P-property. Let $T : A \to B$ be a mapping such that $T(A_0) \subseteq B_0$ and T is proximally increasing on A_0. Suppose that there exist $L \geq 0$ and $k \in [0.1)$ such that for all $x, y \in A_0$ with $x \succeq y$,*

$$d(Tx, Ty) \leq k\ d(x, y) + L\ \min\ \{d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), \\ d(x, Ty) - d(A, B), d(y, Tx) - d(A, B)\}.$$

Assume that either

(a) *T is continuous or*

(b) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \geq 0$.*

Also, suppose that there exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \preceq x_1$.
Then T has a best proximity point in A_0.

Remark 3.1. Theorem 1.1 follows as a corollary to Theorem 2.1 by choosing $L = 0$ in (1.5).

If L=0 in Theorem 2.3, we have the following coupled best proximity result.

Corollary 3.4. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let (A, B) be a pair of nonempty closed subsets of X such that A_0 is nonempty closed and (A, B) satisfies the P-property. Let $F : A \times A \to B$ be a mapping such that $F(A_0, A_0) \subseteq B_0, F$ has proximal mixed monotone property on $A_0 \times A_0$. Suppose that there exist $\psi \in \Psi,\ \varphi, \theta \in \Theta$ and $L \geq 0$ such that (2.1) and (2.2) are satisfied and for all $(x, y), (u, v) \in A_0 \times A_0$ with $(x, y) \succeq (u, v)$,*

$$\psi\big(d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))\big) \leq \varphi\big(d(x, u) + d(y, v)\big) - \theta\big(d(x, u) + d(y, v)\big)$$

Suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) if $\{x_n\}$ is a nondecreasing sequence such that $x_n \to x$, then $x_n \preceq x$ for all $n \geq 0$;

(ii) if $\{y_n\}$ is a non-increasing sequence such that $y_n \to y$, then $y \preceq y_n$ for all $n \geq 0$.

Also, suppose that there exist (x_0, y_0), $(x_1, y_1) \in A_0 \times A_0$ such that $d(x_1, F(x_0, y_0)) = d(A, B)$ and $d(y_1, F(y_0, x_0)) = d(A, B)$ with $(x_0, y_0) \preceq (x_1, y_1)$. Then F has a coupled best proximity point in $A_0 \times A_0$; that is, there exists an element $(x^*, y^*) \in A_0 \times A_0$ such that $d(x^*, F(x^*, y^*)) = d(A, B)$ and $d(y^*, F(y^*, x^*)) = d(A, B)$.

The following example is in support of Theorem 2.1.

Example 3.1. Let $X = [0, \infty) \times [0, \infty)$, with the metric $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We define a partial order $\preceq$ on X by
$\preceq := \{((x_1, x_2), (y_1, y_2)) \in X \times X | x_1 = y_1, x_2 = y_2\} \cup \{((x_1, x_2), (y_1, y_2)) \in X \times X | x_1, y_1 \in \{0\}, x_2, y_2 \in (0, \frac{7}{6}] \text{ and } x_2 \geq y_2\}$.
Let $A = \{0\} \times [0, 2]$, $B = \{2\} \times [0, 2]$. We define $T : A \to B$ by

$$T(0, x) = \begin{cases} (2, \frac{x}{2}) & \text{if } x \in [0, 1] \\ (2, \frac{3}{2}x - 1) & \text{if } x \in [1, 2]. \end{cases}$$

Clearly $d(A, B) = 2$, $A_0 = A$, $B_0 = B$, $T(A_0) \subseteq B_0$ and T is continuous. Now, we choose $x_0 = (0, 1)$ and $x_1 = (0, \frac{1}{2})$, then $d(x_1, Tx_0) = d(A, B)$ and $x_0 \preceq x_1$.

Easily it can be shown that the pair (A, B) satisfies the P-property.

Now, we show that T is proximally increasing on A_0. In this case, let $(0, x), (0, y), (0, u)$ and $(0, v) \in A_0$ such that

$$\left.\begin{array}{l} (0, y) \preceq (0, v) \\ d((0, x), T(0, y)) = 2 \\ d((0, u), T(0, v)) = 2. \end{array}\right\}$$

Case (i): $y, v \in (0, 1]$.

Since $d((0, x), T(0, y)) = d((0, x), (1, \frac{y}{2})) = 1$, we have

$$x = \frac{y}{2} \in (0, \frac{1}{2}]. \tag{3.1}$$

From $d((0, u), T(0, v)) = d((0, u), (1, \frac{v}{2})) = 1$, we obtain

$$u = \frac{v}{2} \in (0, \frac{1}{2}]. \tag{3.2}$$

Since $(0, y) \preceq (0, v)$, from (3.1) and (3.2), we obtain $x = \frac{y}{2} \geq \frac{v}{2} = u$ and hence $(0, x) \preceq (0, u)$.

Case (ii): $y,\ v \in [1, \frac{7}{6}]$.

By the fact that $d((0, x), T(0, y)) = d((0, x), (2, \frac{3}{2}y - 1)) = 2$, it follows that

$$x = \frac{3y - 2}{2} \in [\frac{1}{2}, \frac{3}{4}]. \tag{3.3}$$

From $d((0,u),T(0,v)) = d((0,u),(2,\frac{3}{2}v-1)) = 2$, we get

$$u = \frac{3v-2}{2} \in [\frac{1}{2},\frac{3}{4}]. \tag{3.4}$$

Since $(0,y) \preceq (0,v)$, i.e., $y \geq v$, from (3.3) and (3.4), we obtain $x = \frac{3y-2}{2} \geq \frac{3v-2}{2} = u$ and hence $(0,x) \preceq (0,u)$.

Case (iii): $v \in (0,1]$ and $y \in (1,\frac{7}{6}]$.

By the fact that $d((0,x),T(0,y)) = d((0,x),(2,\frac{3}{2}y-1)) = 2$, we get

$$x = \frac{3y-2}{2} \in [\frac{1}{2},\frac{3}{4}]. \tag{3.5}$$

From $d((0,u),T(0,v)) = d((0,u),(2,\frac{3}{2}v-1)) = 2$, we have

$$u = \frac{v}{2} \in (0,\frac{1}{2}]. \tag{3.6}$$

Since $(0,y) \preceq (0,v)$, from (3.5) and (3.6), we obtain $x = \frac{3y-2}{2} \geq \frac{v}{2} = u$ and hence $(0,x) \preceq (0,u)$.

Hence T is proximally increasing on A_0.

Now, we define functions $\psi,\ \varphi,\theta : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \begin{cases} t & \text{if } t \in [0,1] \\ t^2 & \text{if } t \geq 1 \end{cases} \quad \varphi(t) = \begin{cases} t & \text{if } t \in [0,1] \\ \frac{t^2}{2} & \text{if } t \geq 1 \end{cases} \quad \text{and} \quad \theta(t) = \begin{cases} \frac{t}{2} & \text{if } t \in [0,1] \\ \frac{t^2}{4} & \text{if } t \geq 1. \end{cases}$$

Let $(0,x),\ (0,y) \in A_0$ with $(0,x) \succeq (0,y)$. Here we consider only the non-trivial cases. For this purpose, let $x,\ y \in [1,\frac{7}{6}]$.

Now, we consider

$$\begin{aligned}
\psi\big(d(T(0,x),T(0,y))\big) &= \psi\big(d(2,\tfrac{3}{2}x-1),(2,\tfrac{3}{2}y-1)\big) = \psi\big(\tfrac{3}{2}(y-x)\big) \\
&= \tfrac{3}{2}(y-x) = (y-x) - \tfrac{(y-x)}{2} + (y-x) \leq (y-x) - \tfrac{(y-x)}{2} + 1(\tfrac{1}{4}) \\
&= \varphi(d(0,x),(0,y)) - \theta(d(0,x),(0,y)) \\
&\qquad + 1 \min\{1-\tfrac{x}{2}, 1-\tfrac{y}{2}, y+1-\tfrac{3}{2}x, x+1-\tfrac{3}{2}y\} \\
&= \varphi(d(0,x),(0,y)) - \theta(d(0,x),(0,y)) \\
&\qquad + L \min\{d((0,x),T(0,x)) - d(A,B), d((0,y),T(0,y)) - d(A,B), \\
&\qquad d((0,y),T(0,x)) - d(A,B), d((0,x),T(0,y)) - d(A,B)\}.
\end{aligned}$$

Hence T,ψ,φ and θ satisfy all the conditions of Theorem 2.1, and T has two best proximity points $(0,0)$ and (0,2).

Here we observe that for elements $(0,2)$ and $(0,0)$, there does not exist an element $u \in X$ such that u is comparable to both $(0,2)$ and $(0,0)$ so that 'condition H' fails to hold in Theorem 2.2.

Remark 3.2. If $L = 0$ in Example 3.1, the inequality (1.5) fails to hold for any $\psi \in \Psi$, φ and $\theta \in \Theta$. For this purpose, we choose $x = (0,\frac{7}{6})$ and $y = (0,1)$.

$$\begin{aligned}
\psi(d(Tx,Ty)) = \psi(d(T(0,\frac{7}{6}),T(0,1))) = \psi(d((2,\frac{3}{4}),(2,\frac{1}{2}))) = \psi(\frac{1}{4}) = \frac{1}{4} \not\leq \frac{1}{12} \\
= \varphi(d(0,\frac{7}{6}),(0,1)) - \theta(d(0,\frac{7}{6}),(0,1)).
\end{aligned}$$

Hence Theorem 2.1 is a generalization of Theorem 1.1.

The following example is in support of Theorem 2.2.

Example 3.2. Let $X = A \cup B$, where
$A = \{(\frac{1}{2^n}, 1):\ n = 1, 2, ...\} \cup \{(0,1), (1,1), (\frac{5}{4}, 1), (\frac{3}{2}, 1), (\frac{7}{6}, 1)\}$ and
$B = \{(\frac{1}{2^n}, 0):\ n = 1, 2, ...\} \cup \{(0,0), (1,0), (\frac{5}{4}, 0), (\frac{3}{2}, 0)\}$. We define a metric d on $X \times X$ by $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

We define a partial order $\preceq$ on X by

$$\preceq := \ \{(x,y) \in X \times X |\ x = (x_1, x_2) \text{ and } y = (y_1, y_2) \text{ with } x_1 = y_1 \text{ and } x_2 = y_2\}$$

$$\cup \ \{\big((\frac{1}{2^n}, 1), (0,1)\big), \big((\frac{1}{2^n}, 1), (\frac{1}{2^m}, 1)\big):\ n,\ m = 1,\ 2, ...,\ \text{with } m > n\}$$

$$\cup \{\big((\frac{5}{4}, 1), (0,1)\big), \big((1,1), (0,1)\big), \big((\frac{5}{4}, 1), (1,1)\big), \big((\frac{5}{4}, 1), (\frac{1}{2}, 1)\big), \big), \big((\frac{3}{2}, 1), (0,1)\big)\},$$

where $(x_1, x_2) \preceq (y_1, y_2)$ if and only if $x_1 \geq y_1$ and $x_2 \geq y_2$ for all $(x_1, x_2),\ (y_1, y_2) \in X$.

We define $T : A \to B$ by

$$T(0,1) = T(1,1) = (0,0),$$

$$T(\frac{5}{4}, 1) = T(\frac{3}{2}, 1) = T(\frac{7}{6}, 1) = (\frac{1}{2}, 0) \ \text{ and}$$
$$T(\frac{1}{2^n}, 1) = (\frac{1}{2^{n+1}}, 0), \ \text{where} \ \ n = 1, 2, ... \ .$$

Clearly, we see that $d(A,B) = 1$,
$A_0 = \{(\frac{1}{2^n}, 1):\ n = 1, 2, ...\} \cup \{(0,1), (1,1), (\frac{5}{4}, 1), (\frac{3}{2}, 1)\}$,
$B_0 = B,\ A_0 \subseteq A$ and $T(A_0) \subseteq B_0$.

Now, we choose $x_0 = (\frac{5}{4}, 1)$ and $x_1 = (\frac{1}{2}, 1)$, such that $d((\frac{1}{2}, 1), T(\frac{5}{4}, 1)) = d(A,B)$ and $(\frac{5}{4}, 1) \preceq (\frac{1}{2}, 1)$. Clearly the pair (A,B) has the P-property.

We now show that T is proximally increasing on A_0. In this case, let $(x,1), (y,1)$, $(u,1)$ and $(v,1) \in A_0$ such that

$$\left.\begin{array}{l} (y,1) \preceq (v,1) \\ d((x,1), T(y,1)) = 1 \\ d((u,1), T(v,1)) = 1. \end{array}\right\}$$

Case (i): $(\frac{1}{2^n}, 1) \preceq (0,1)$ where $n = 1, 2, ...$.

If $d((x,1), T(\frac{1}{2^n}, 1)) = d((x,1), (\frac{1}{2^{n+1}}, 0)) = 1$, we have

$$x = \frac{1}{2^{n+1}}. \tag{3.7}$$

If $d((u,1), T(0,1)) = d((u,1), (0,0)) = 1$, we get

$$u = 0. \tag{3.8}$$

From (3.7) and (3.8), we obtain $(x,1) = (\frac{1}{2^{n+1}}, 1) \preceq (0,1) = (u,1)$.

Case (ii): $(\frac{1}{2^n}, 1) \preceq (\frac{1}{2^m}, 1)$, where $n, m = 1, 2, ...$.

If $d((x,1), T(\frac{1}{2^n}, 1)) = d((x,1), (\frac{1}{2^{n+1}}, 0)) = 1$, we have

$$x = \frac{1}{2^{n+1}}. \tag{3.9}$$

If $d((u,1), T(\frac{1}{2^m},1)) = d((u,1),(\frac{1}{2^{m+1}},0)) = 1$, we have

$$u = \frac{1}{2^{m+1}}. \tag{3.10}$$

From (3.9) and (3.10), we obtain $(x,1) = (\frac{1}{2^{n+1}},1) \preceq (\frac{1}{2^{m+1}},1) = (u,1)$.
Case (iii): $(\frac{5}{4},1) \preceq (0,1)$.

If $d((x,1), T(\frac{5}{4},1)) = d((x,1),(\frac{1}{2},0)) = 1$, we have

$$x = \frac{1}{2}. \tag{3.11}$$

If $d((u,1), T(0,1)) = d((u,1),(0,0)) = 1$, we have

$$u = 0. \tag{3.12}$$

From (3.11) and (3.12), we obtain $(x,1) = (\frac{1}{2},1) \preceq (0,1) = (u,1)$.

In similar way, if we consider all comparable elements of A_0, we can show that T is proximally increasing on A_0.

We now define functions $\psi,\ \varphi, \theta : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \ t^2 \quad \text{if } t \geq 0 \qquad \varphi(t) = \begin{cases} \frac{t}{2} & \text{if } t \in [2,3] \\ \frac{t^2}{2} & \text{otherwise} \end{cases} \quad \text{and} \quad \theta(t) = \begin{cases} \frac{t}{4} & \text{if } t \in [2,3] \\ \frac{t^2}{4} & \text{otherwise.} \end{cases}$$

With these $\psi,\ \varphi$, and θ, it is easy to see that T satisfies the inequality (1.5) with $L = 1$. Also, it is trivial to see that condition (H) holds. Hence T satisfies all the hypotheses of Theorem 2.2 and $(0,1)$ is the unique best proximity point of T in A_0.

Furthermore, we observe that the inequality (1.5) fails to hold when $L = 0$, $x = (\frac{5}{4},1)$ and $y = (1,1)$ for any $\psi \in \Psi$, φ and $\theta \in \Theta$. Hence Theorem 1.2 is not applicable. Therefore Theorem 2.2 is a generalization of Theorem 1.2.

REFERENCES

1. A. Abkar, M. Gabeleh, *Global optimal solutions of noncyclic mappings in metric spaces.* J. Optim. Theory Appl. **153** (2012), 298–305.
2. A. Abkar, M. Gabeleh, *Best proximity points of non-self mappings.* **21** (2013), 287–295.
3. M.A. Al-Thaga, N. Shahzad, *Convergence and existence results for best proximity points.* Nonlinear Anal. **70** (2009), 3665–3671.
4. G. V. R. Babu and P. D. Sailaja, *A fixed point theorem of generalized weak contractive maps in orbitally complete metric spaces.* Thai Journal of Mathematics. **9** (1), (2011), 1–10.
5. S.S. Basha, *Best proximity points, optimal solutions.* J. Optim. Theory Appl. **151** (2011), 210–216.
6. S. S. Basha, *Discrete optimization in partially ordered sets.* J. Glob. Optim. **54** (2012), 511–517.
7. J. Caballero, J. Harjani, K. Sadarangani, *A best proximity point theorem for Geraghty-contractions,* Fixed Point Theory and Appl., **2012**, Article ID 231 (2012), 9-pages.

8. B. S Choudhury, N. Metiya, M. Postolache and P. Konar, *A discussion on best proximity point and coupled best proximity point in partially ordered metric space.* Fixed Point Theory and Applications (2015) 2015:170 DOI 10.1186/s13663-015-0423-1, 17 pages.
9. B.S Choudhury, P. Maity, P. Konar, *A global optimality result using non-self mappings.* Opsearch **51** (2014), 312–320.
10. B. S. Choudhury, P. Maity, P. Konar, *A global optimality result using Geraghty type contraction.* Int. J. Optim. Control, Theor. Appl. **4**, (2014), 99–104.
11. A.A. Eldred, P. Veeramani, *Existence and convergence of best proximity points.* J. Math. Anal. Appl. **323** (2006), 1001–1006.
12. M. Gabeleh, *Best proximity point theorems via proximal non-self mappings.* J. Optim. Theory, Appl. **164** (2015), 565–576.
13. T. Gnana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications.* Nonlinear Anal. TMA **65** (2006), 1379–1393.
14. E. Karapinar, *On best proximity point of ψ-Geraghty contractions.* Fixed Point Theory Appl. 2013, Article ID 200 (2013), 9 pages.
15. S. Karpagam, S. Agrawal, *Best proximity points for cyclic Meir-Keeler contraction maps.* Nonlinear Anal. **74** (2011), 1040–1046.
16. P. Kumam, V. Pragadeeswarar, M. Marudai, K. Sitthithakerngkiet, *Coupled best proximity points in ordered metric spaces.* Fixed Point Theory and Appl. **2014** (1), (2014), 13 pages.
17. M.A. Kutbi, S. Chandok, W. Sintunavarat, *Optimal solutions for nonlinear proximal CN-contraction mapping in metric space.* J. Inequal. Appl., **2014**, Article ID 193 (2014), 9 pages.
18. V. Sankar Raj, *A best proximity point theorem for weakly contractive non-self-mappings.* Nonlinear Anal. **74** (2011), 4804–4808.
19. W. Sintunavarat, P. Kumam, *Coupled best proximity point theorem in metric spaces.* Fixed Point Theory and Appl. **2012** (2012), Article ID 93, 16 pages.

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