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BOUNDARY VALUE PROBLEMS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

Ali Benlabbes, Maamar Benbachir and Mustapha Lakrib

Abstract. Sufficient conditions are given for the existence of solutions for the following boundary value problem for a nonlinear fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), \ t \in J = [0,1], \ 2 < \alpha \leq 3 \\ D_{0^+}^{\alpha-1} u\,(0) = 0, \ D_{0^+}^{\alpha-2} u\,(1) = 0, \ u(1) = 0 \end{cases}$$

where f is a given function and $D_{0^+}^{\alpha}$ is the standard Riemman fractional derivative operator of order α. The results are proved using Banach contraction principle and Krasnoselskii's cone fixed point theorem.
Keywords: Differential equation; Boundary value problem; Nonlinear fractional differential equation; Riemman fractional derivative operator

1. Introduction

Since the introduction of the different types of fractional derivatives, differential equations of fractional order have proved to be valuable tools in the modeling of many physical and chemical processes and in engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, or electromagnetics. The attention drawn to the theory of fractional integration and differentiation and applications is evident from an increased number of recent publications (see [3, 5, 7, 8, 9, 14, 15, 16, 17, 18, 19] and the references therein).

Usually, the fundamental tool used in the literature to prove the existence of positive solutions for boundary value problems for ordinary and fractional differential equations, difference equations, and dynamic equations on time scales, is the theory of fixed point (see, for example, [1, 2, 4, 6, 10, 11, 12, 13] and the works cited below).

In [18], by use of some fixed point arguments, Zhang proved the existence of solutions for the following nonlinear fractional boundary value problem involving

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Caputo's derivative:

$$\begin{cases} D_t^{\alpha} u + f(t, u(t)) = 0, \ \ 0 < t < 1, \ 1 < \alpha < 2 \\ u(0) = \nu, \quad u(1) = \rho, \quad \nu\rho \neq 0. \end{cases}$$

In another paper, by use of a fixed point theorem in cones, Zhang in [19] studied the existence and multiplicity of positive solutions of the nonlinear fractional boundary value problem:

$$\begin{cases} D_t^{\alpha} u + f(t, u(t)) = 0, \ 0 < t < 1, \ 1 < \alpha < 2, \\ u(0) + u^{'}(0) = 0, \quad u(1) + u'(1) = 0, \end{cases}$$

where D_t^{α} is the Caputo's fractional derivative.

In [5], Benchohra, Hamani, Ntouyas and Ouahab, by means of the Banach fixed point theorem and the nonlinear alternative of Leray–Schauder type, proved the existence of solutions for the first order boundary value problem for a fractional order differential equation:

$$\begin{cases} D_t^{\alpha} u = f(t, u(t)) = 0, \ 0 < t < 1, \ 0 < \alpha < 1, \\ au(0) + bu(1) = c, \end{cases}$$

where D_t^{α} is the Caputo's fractional derivative, f is a continuous function and a, b, c are real constants with $a + b \neq 0$.

In [3], Bai and Lü considered the nonlinear fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u + f(t, u(t)) = 0, \ 0 < t < 1, \ 1 < \alpha < 2, \\ u(0) = u(1) = 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional differential operator of order α. They obtained the existence of positive solutions by means of some fixed point theorems on cone.

In [8], Delbosco and Rodino proved existence and uniqueness for some classes of nonlinear fractional differential equations of the form:

$$D^s u = f(t, u), \quad 0 < s < 1,$$

where $f : [0, a] \times \mathbb{R} \to \mathbb{R}$, $0 < a \leq +\infty$ is a given continuous function. In this paper, the authors used the Banach contraction principle.

By using Krasnoselskii's fixed point theorem in cones, El-Shahed in [9] proved the existence and non-existence of positive solutions for the following nonlinear fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u + \lambda a(t) f(u(t)) = 0, \ 0 < t < 1, \ 2 < \alpha \leq 3, \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional differential operator of order α.

In [16], Saadi and Benbachir obtained sufficient conditions for the existence and non-existence of positive solutions for the following nonlinear fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u + a(t) f(u(t)) = 0, \ 0 < t < 1, \ 2 < \alpha \leq 3, \\ u(0) = u'(0) = 0, \ u'(1) - \mu u'(\eta) = \lambda, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional differential operator of order α, $f : [0, \infty) \longrightarrow [0, \infty)$ and $a : (0, 1) \longrightarrow [0, \infty)$ are continuous functions, $\eta \in (0, 1)$, $\mu \in \left[0, 1/\eta^{\alpha-2}\right)$ and $\lambda \in [0, \infty)$ are some fixed constants. By use of Guo-Krasnoselskii's fixed point theorem and Schauder's fixed point theorem, the existence of positive solutions to this problem is obtained in case when either f is superlinear or sublinear.

Recently, in [17], Saadi, Benmezai and Benbachir gave some sufficient conditions for the existence of positive solutions to the nonlinear fractional semi-positone boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u + f(t, u(t)) = 0, 0 < t < 1, 2 < \alpha \leq 3, \\ u(0) = u'(0) = 0, \ u'(1) = \mu u'(\eta), \end{cases}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville differential operator of order α and the nonlinear term $f : [0, 1] \times [0, +\infty) \to \mathbb{R}$ satisfies a L^1-Carathéodory condition. By use of a fixed point index theorem, the existence of at least two positive solutions is obtained.

Motivated by [17], this paper deals with the existence of solutions for the following nonlinear fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), \ t \in J, \ 2 < \alpha \leq 3 \\ D_{0^+}^{\alpha-1} u(0) = 0, \ D_{0^+}^{\alpha-2} u(1) = 0, \ u(1) = 0. \end{cases} \tag{1.1}$$

The rest of this paper is organized as follows. In section 2, we present some preliminaries and lemmas. Section 3 is devoted to the existence of solutions to problem (1.1). In section 4 an example is treated illustrating our results.

2. Elementary definitions and lemmas

In this section [11], we introduce notations, definitions and preliminary facts which are used throughout the paper. We denote by $C(J, \mathbb{R})$ the Banach space of all continuous functions from $J = [0, 1]$ into $\mathbb{R}$ with the norm $\|u\|_\infty := \sup_{t \in J} |u(t)|$.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u \in \mathbf{L}^1([a,b],[0,\infty))$ is defined by

$$I_a^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)\, ds,$$

where Γ is the gamma function.

When $a = 0$, we write $I_0^\alpha u(t) = (u * \varphi_\alpha)(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t < 0$, and $\varphi_\alpha \longrightarrow \delta$ as $\alpha \longrightarrow 0$, where δ is the delta function.

Definition 2.2. For a function u given in $[0,+\infty)$, the expression

$$D_{0^+}^\alpha u(t) = \frac{1}{\Gamma(\alpha-n)} \left(\frac{d}{dt}\right)^n \int_0^s (t-s)^{n-\alpha-1} u(s)\, ds$$

is called the Riemann-Liouville fractional derivative of order α, where $n = [\alpha] + 1$, and $[\alpha]$ denote the integer part of number α.

Remark 2.1.

- For $\alpha < 0$, we use the convention that $D_{0^+}^\alpha u(t) = I_0^{-\alpha} u(t)$, for $t \geq 0$.
- For $\beta \in [0,\alpha)$, we have $D_{0^+}^\beta I_{0^+}^\alpha u = I_{0^+}^{\alpha-\beta} u$, with $D_{0^+}^\alpha I_{0^+}^\alpha u = u$.
- For $\lambda > -1, \lambda \neq \alpha-1, \alpha-2, ...\alpha-n$, we have, for $t \geq 0$,

(2.1) $$D_{0^+}^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha} \quad \text{and} \quad D_{0^+}^\alpha t^{\alpha-i} = 0,\ \forall i = 1,2,..,n.$$

Lemma 2.1. *Let $\alpha > 0$. The general solution to the homogeneous equation*

$$D_{0^+}^\alpha u(t) = 0,$$

in $C(J,\mathbb{R}) \cap \mathbf{L}^1(J,\mathbb{R})$ is

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n},\ t \in J.$$

Lemma 2.2. *Let $n-1 < \alpha < n$ and $u \in C(J,\mathbb{R})$. We have*

$$I_{0^+}^\alpha D_{0^+}^\alpha u(t) = u(t) - c_1 t^{\alpha-1} - c_2 t^{\alpha-2} - ... - c_n t^{\alpha-n},\ t \in J.$$

Let us now present the fundamental tools on which the proofs of our main results are based.

Definition 2.3. Let $(\mathbb{X}, \|\cdot\|)$ be a normed space. A contraction of $\mathbb{X}$ is a mapping $P : \mathbb{X} \longrightarrow \mathbb{X}$ that satisfies

$$\forall x_1, x_2 \in \mathbb{X} : \ \|P(x_1) - P(x_2)\| \leq \beta \|x_1 - x_2\|,$$

for some real number $\beta < 1$.

Theorem 2.1. (Banach fixed point theorem) *Every contraction mapping on a complete metric space has a unique fixed point.*

Theorem 2.2. (Krasnselskii's fixed point theorem) *Let $\mathcal{B}$ be a closed convex non-empty subset of a Banach space $\mathbb{X}$. Suppose that P_1, P_2 map $\mathcal{B}$ into $\mathbb{X}$ such that*

1. $P_1 u + P_2 v \in \mathcal{B}, \forall u, v \in \mathcal{B}$,
2. *P_1 is a contraction mapping,*
3. *P_2 is continuous and $P_2(\mathcal{B})$ is contained in a compact set.*

Then there exists $u \in \mathcal{B}$ such that $P_1 u + P_2 u = u$.

3. Existence of solutions

Let us consider the following problem

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), \ t \in J = [0,1], \ 2 < \alpha \leq 3, \\ D_{0^+}^{\alpha-1} u(0) = 0, \ D_{0^+}^{\alpha-2} u(1) = 0, \ u(1) = 0. \end{cases} \tag{3.1}$$

Definition 3.1. A function $u \in C(J, \mathbb{R})$ is said to be a solution to problem (3.1) if u satisfies the fractional differential equation

$$D_{0^+}^{\alpha} u(t) = f(t, u(t)), \ t \in J,$$

and the conditions

$$D_{0^+}^{\alpha-1} u(0) = 0, \ D_{0^+}^{\alpha-2} u(1) = 0, \ u(1) = 0.$$

Lemma 3.1. *Let $2 < \alpha < 3$ and let $h : J \longrightarrow \mathbb{R}$ be a continuous function. A function u is a solution to the fractional integral equation*

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) \, ds$$

if and only if it is a solution to the initial value problem

$$\begin{cases} D_{0^+}^{\alpha} u(t) = h(t), \ t \in J \\ u(0) = u_0. \end{cases}$$

Lemma 3.2. *Let $2 < \alpha < 3$ and let $h : J \longrightarrow \mathbb{R}$ be a continuous function. A function u is a solution of the initial value problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) = h(t), \ t \in J, \\ D_{0^+}^{\alpha-1} u(0) = 0, \ D_{0^+}^{\alpha-2} u(1) = 0, \ u(1) = 0, \end{cases} \tag{3.2}$$

if and only if it is a solution to the fractional integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds + \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)}\int_0^1 (1-s)\,h(s)\,ds - \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} h(s)\,ds. \tag{3.3}$$

Proof. Let u be a solution to problem (3.2). By Lemma 2.2 we have

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)ds. \tag{3.4}$$

If we consider the boundary conditions, we have

$$D_{0^+}^{\alpha-1}u(0) = 0 \Rightarrow c_1 = 0,$$

$$D_{0^+}^{\alpha-2}u(1) = 0 \Rightarrow c_2 = -\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)\,h(s)ds$$

and

$$u(1) = 0 \Rightarrow c_3 = \frac{1}{\Gamma(\alpha)}\int_0^1 \left[(1-\alpha)(1-s) - (1-s)^{\alpha-1}\right] ds.$$

Replace then c_2 and c_3 in (3.4) to get

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds + \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)}\int_0^1 (1-s)\,h(s)\,ds - \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} h(s)\,ds,$$

that is, u is a solution to problem (3.3). Conversely if u is a solution to problem (3.3), it can be written as follows

$$u(t) = I_0^{\alpha}h(t) + \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)} I_0^2 h(1) - t^{\alpha-3} I_0^{\alpha} h(1),$$

so that, by (2.1) we deduce that: $D_{0^+}^{\alpha}u(t) = h(t)$, $D_{0^+}^{\alpha-1}u(0) = I_0^1 h(0) = 0$, $D_{0^+}^{\alpha-2}u(1) = I_0^2 h(1) - I_0^2 h(1) = 0$ and $u(1) = I_0^{\alpha}h(1) - I_0^{\alpha}h(1) = 0$, which finishes to prove that u is a solution to problem (3.2). □

The first main result of this paper states that problem (3.1) admits a unique solution. It reads as follows.

Theorem 3.1. *Let $f : J \times [0,\infty) \longrightarrow \mathbb{R}$ be a continuous function such that:* $(\mathbf{H}_1)$ *There exists a constant $k > 0$ such that*

$$\left| f(t, u(t) - f(t, u^*(t))) \right| < k\,|u(t) - u^*(t)|, \forall u, u^* \in \mathbb{R}, \forall t \in J.$$

If the following condition is satisfied

$$k\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha-1)}\right)<1 \tag{3.5}$$

then problem (3.1) *has a unique solution on* J.

Proof. We transform problem (3.1) into a fixed point problem. For this, consider the operator $P : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ defined by

$$\begin{aligned} Pu(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds \\ &\quad + \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)}\int_0^1 (1-s) f(s,u(s))ds \\ &\quad - \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,u(s))ds. \end{aligned} \tag{3.6}$$

It is easy to see that fixed points of P are solutions to problem (3.1). Now, according to Theorem 2.1, it is enough to prove that P is a contraction. Let $u \in C(J,\mathbb{R})$ and $t \in J$. We have

$$\begin{aligned} & |Pu_1(t) - Pu_2(t)| \\ \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,u_1(s)) - f(s,u_2(s))\right| ds \\ & + \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)}\int_0^1 (1-s)\left|f(s,u_1(s)) - f(s,u_2(s))\right| ds \\ & + \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f(s,u_1(s)) - f(s,u_2(s))\right| ds \\ \leq\ & \frac{k\|u_1-u_2\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds + k\|u_1-u_2\|_\infty \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)}\int_0^1 (1-s)\, ds \\ & + k\|u_1-u_2\|_\infty \frac{t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} ds \\ \leq\ & \frac{k\|u_1-u_2\|_\infty}{\alpha\Gamma(\alpha)}t^\alpha + \frac{k\|u_1-u_2\|_\infty t^{\alpha-3}(1-t)}{2\Gamma(\alpha-1)} + \frac{k\|u_1-u_2\|_\infty t^{\alpha-3}}{\alpha\Gamma(\alpha)} \\ \leq\ & \left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha-1)}\right)k\|u_1-u_2\|_\infty . \end{aligned}$$

from which we deduce that

$$\|Pu_1 - Pu_2\|_\infty \leq \left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha-1)}\right)k\|u_1-u_2\|_\infty .$$

By (3.5), this proves that P is a contraction. As a consequence of Theorem 2.1, P has a unique fixed point which is a unique solution of problem (3.1). □

Now, we will consider the following condition that we use in the second main result of this work, that is Theorem 3.2 below.

($\mathbf{H}_2$) There exists a constant $\beta > 0$ such that

$$\left|f(t,u)\right| \leq \beta, \quad \forall t \in J,\ \forall x \geq 0.$$

Theorem 3.2. *Suppose that* ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$) *are satisfied. If there exists* $\delta > 0$ *such that*

$$\beta \frac{\alpha^2 - \alpha + 2}{\Gamma(\alpha+1)} \leq \delta, \tag{3.7}$$

then problem (3.1) *has at least one solution on* J.

Proof. We will use Theorem 2.2 to prove the conclusion of Theorem 3.2. For this, consider again the operator P defined in (3.6) and define operators P_1 and P_2 as follows: For $u \in C(J,\mathbb{R})$ and $t \in J$,

$$P_1u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s))\,ds + \frac{t^{\alpha-3}}{\Gamma(\alpha-1)} \int_0^1 (1-s) f(s,u(s))\,ds,$$

and

$$P_2u(t) = -\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^1 (1-s) f(s,u(s))\,ds - \frac{t^{\alpha-3}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,u(s))\,ds.$$

• let $\mathcal{B} = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \leq \delta\}$. We will prove that $P_1u_1 + P_2u_2 \in \mathcal{B}$, for any $u_1, u_2 \in \mathcal{B}$. Let $u_1, u_2 \in \mathcal{B}$ and $t \in J$. We have

$$\begin{aligned}
&|P_1u_1(t) + P_2u_2(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f(s,u_1(s))\right| ds + \frac{t^{\alpha-3}}{\Gamma(\alpha-1)} \int_0^1 (1-s) \left|f(s,u_1(s))\right| ds \\
& + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^1 (1-s) \left|f(s,u_2(s))\right| ds + \frac{t^{\alpha-3}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left|f(s,u_2(s))\right| ds. \\
\leq\ & \frac{\beta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{\beta}{\Gamma(\alpha-1)} \int_0^1 (1-s)\,ds \\
& + \frac{\beta}{\Gamma(\alpha-1)} \int_0^1 (1-s)\,ds + \frac{\beta}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds, \\
\leq\ & \frac{\beta}{\Gamma(\alpha)} \left(\frac{t^\alpha}{\alpha} + \alpha - 1 + \frac{1}{\alpha}\right) \leq \frac{\beta}{\Gamma(\alpha)} \left(\frac{\alpha^2 - \alpha + 2}{\alpha}\right)
\end{aligned}$$

from which and by condition (3.7) we deduce that

$$\|P_1u_1 - P_2u_2\|_\infty \leq \frac{\beta}{\Gamma(\alpha)} \left(\frac{\alpha^2 - \alpha + 2}{\alpha}\right) \leq \delta.$$

which finish to prove that $Pu_1 + Pu_2 \in \mathcal{B}$.

• Now, we will prove that P_1 is a contraction mapping on $C(J,\mathbb{R})$. Let $u_1, u_2 \in C(J,\mathbb{R})$ and $t \in J$. We have

$$\begin{aligned}
&|P_1u_1(t) - P_1u_2(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,u_1(s)) - f(s,u_2(s))\right| ds \\
&+\frac{t^{\alpha-3}}{\Gamma(\alpha-1)}\int_0^1 (1-s)\left|f(s,u_1(s)) - f(s,u_2(s))\right| ds. \\
\leq\ & \frac{k\|u_1-u_2\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds + t^{\alpha-3}\frac{k\|u_1-u_2\|_\infty}{\Gamma(\alpha-1)}\int_0^1 (1-s)\,ds \\
\leq\ & \frac{k\|u_1-u_2\|_\infty}{\Gamma(\alpha+1)}t^{\alpha} + t^{\alpha-3}\frac{k\|u_1-u_2\|_\infty}{2\Gamma(\alpha-1)} \\
\leq\ & k\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)}\right)\|u_1-u_2\|_\infty
\end{aligned}$$

from which we deduce that

$$\|P_1u_1 - P_1u_2\|_\infty \leq k\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)}\right)\|u_1-u_2\|_\infty .$$

Using the condition (3.5), we conclude that P_1 is a contraction. • We will prove that P_2 is continuous. Let $(u_n)_n$ be a sequence such that $u_n \longrightarrow u$ in $C(J,\mathbb{R})$. For $t \in J$, we have

$$\begin{aligned}
&|P_2u_n(t) - P_2u(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha-1)}\left[\int_0^1 \left((1-s)-(1-s)^{\alpha-1}\right)\left|f(s,u_n(s)) - f(s,u(s))\right| ds\right] \\
\leq\ & \frac{1}{\Gamma(\alpha-1)}k\|u_n-u\|\int_0^1 \left((1-s)-(1-s)^{\alpha-1}\right) \\
\leq\ & \frac{\alpha-2}{2\alpha\Gamma(\alpha-1)}k\|u_n-u\|.
\end{aligned}$$

Therefore

$$\|P_2u_n - P_2u\| \longrightarrow 0 \text{ as } \|u_n-u\| \longrightarrow 0.$$

• Finally, we will prove the compactness of P_2. From the Ascoli-Arzela Theorem, it is sufficient to prove that for each bounded subset $\mathcal{B}$ of $C(J,\mathbb{R})$, the set $P\mathcal{B}$ is bounded and is equicontinuous. Let $\mathcal{B}$ be a bounded subset of $C(J,\mathbb{R})$. ⋄ We prove that $P_2\mathcal{B}$ is a bounded subset of $C(J,\mathbb{R})$. Let $u \in \mathcal{B}$ and $t \in J$. We have

$$\begin{aligned}
|P_2u(t)| \leq\ & \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)\left|f(s,u(s))\right| ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f(s,u(s))\right| ds \\
\leq\ & \frac{\beta}{\Gamma(\alpha-1)}\int_0^1 (1-s)\,ds + \frac{\beta}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\,ds \\
\leq\ & \beta\left(\frac{1}{2\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha+1)}\right),
\end{aligned}$$

from which we deduce that

$$\|P_2u\|_\infty \leq \beta\left(\frac{1}{2\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha+1)}\right)=cte.$$

⋄ Now, we prove that $P_2\mathcal{B}$ is equicontinuous. Let $u \in \mathcal{B}$ and $t_1, t_2 \in J$ with $t_1 < t_2$. We have

$$\begin{aligned}
&|P_2u(t_1)-P_2u(t_2)| \\
\leq\ & \frac{t_1^{\alpha-2}-t_2^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1(1-s)\left|f(s,u(s))\right|ds \\
&+\frac{t_2^{\alpha-3}-t_2^{\alpha-3}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left|f(s,u(s))\right|ds \\
\leq\ & \beta\frac{t_1^{\alpha-2}-t_2^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1(1-s)\,ds+\beta\frac{t_2^{\alpha-3}-t_2^{\alpha-3}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\,ds \\
\leq\ & \beta\frac{t_1^{\alpha-2}-t_2^{\alpha-2}}{2\Gamma(\alpha-1)}+\beta\frac{t_2^{\alpha-3}-t_2^{\alpha-3}}{\Gamma(\alpha+1)},
\end{aligned}$$

from which we deduce the desired property. Finally as a consequence of Theorem 2.2 we deduce that P has at least one fixed point, which is a solution to problem (3.1). □

4. Example

Let $f:[0,1]\times[0,+\infty)\to\mathbb{R}$ be defined by

$$f(t,x)=\frac{x}{3e^t+2}. \tag{4.1}$$

For $t\in[0,1]$ and $x\geq 0$ we have

$$\left|f(t,x)-f(t,y)\right|=\frac{1}{3e^t+2}\left|x-y\right|\leq\frac{1}{5}\left|x-y\right|$$

which proves that f is a contraction.

Now, to apply Theorem 3.1 we should verify that

$$\frac{1}{5}\left(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{2\Gamma(\alpha-1)}\right)<1,$$

which is the case since we have:

For $\alpha=\frac{14}{5}$ we get $\frac{1}{5}\left(\frac{2}{\Gamma(\frac{19}{5})}+\frac{1}{2\Gamma(\frac{9}{5})}\right)=0.192\,58<1$, and

for $\alpha=\frac{7}{3}$ we get $\frac{1}{5}\left(\frac{2}{\Gamma(\frac{10}{5})}+\frac{1}{2\Gamma(\frac{4}{3})}\right)=0.511\,98<1$.

In conclusion, by Theorem 3.1 we conclude that for f given by (4.1) problem (3.1) admits at least one solution.

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Ali Benlabbes
Faculty of Sciences and Technology
Bechar University
Bechar, Algeria.
alibenlabbes@hotmail.fr

Maamar Benbachir
Faculty of Sciences and Technology
Khemis Miliana University
Ain Defla, Algeria.
mbenbachir2001@gmail.com

Mustapha Lakrib
Laboratory of Mathematics
Djillali Liabès University
Sidi Bel Abbès, Algeria.
m.lakrib@univ-sba.dz