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LOCAL EXISTENCE AND SUFFICIENT CONDITIONS OF THE NON-GLOBAL SOLUTION FOR WEIGHTED DAMPED WAVE EQUATIONS *

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Abstract. In this paper we study the following Cauchy problem of the weighted damped wave equation with nonlinear memory

$$u_{tt} - \Delta u + g(x) |u|^{m-1} u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau, .)|^p d\tau$$

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N,$$

in the multi-dimensional real space $\mathbb{R}^n$. where, $m > 1$, $p > 1$, $0 < \gamma < 1$ and Δ is the usual Laplace operator and g is a positive smooth function which will be specified later. Firstly, we will prove the existence and uniqueness of the local solution theorem and, secondly, the nonexistence of the global solutions theorem is established.

Keywords: Damped wave equation; weak solution; test function; fractional derivative

1. Introduction

In 2008, Cazenave and al [6] generalized some results obtained by Fujita [5] in 1966 when they studied the following equation

$$\partial_t u(t,x) - \Delta u(t,x) = \int_0^t (t-\tau)^{-\gamma} |u|^{p-1} u(\tau,x) d\tau \tag{1.1}$$

where $0 \leq \gamma < 1$ and $u_0 \in C_0(\mathbb{R}^N)$. Their results are the following. Let

$$p_\gamma = 1 + \frac{4-2\gamma}{(N-2+2\gamma)_+} \text{and } p^* = \max\left(p_\gamma, \frac{1}{\gamma}\right)$$
$$\text{with } (N-2+2\gamma)_+ = \max(N-2+2\gamma, 0),$$

then

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1. If $\gamma \neq 0, p \leq p^*$ and $u_0 > 0$, then the solution u of (1.1) blows up in finite time.
2. If $\gamma \neq 0, p > p^*$ and $u_0 \in L_{q^*}(\mathbb{R}^N)$ where $q^* = \frac{(p-1)N}{4-2\gamma}$ with $\|u_0\|_{L_{q^*}}$ small enough, then u exists globally.

   In particular, they proved that the critical exponent in Fujita's sense p^* is not the one predicted by scaling. But this is not a surprising result since it is well known that scaling is efficient only for parabolic equations and not for pseudo-parabolic ones. To show this, it is sufficient to note that, formally, equation (1.1) is equivalent to

   $$D^{\alpha}_{0|t} u_t - D^{\alpha}_{0|t} \Delta u = \Gamma(\alpha)|u|^{p-1}u,$$

   where $\alpha = 1 - \gamma$ and $D^{\alpha}_{0|t}$ is the fractional derivative operator of order α ($\alpha \in]0,1[$) of Riemann-Liouville defined by

   $$D^{\alpha}_{0|t} u = \frac{d}{dt} I^{1-\alpha}_{0|t} u.$$

3. In the case of $\gamma = 0$, Souplet [21] has showed that a non-zero positive solution blows up in finite time.

After that, precisely in 2013, M. Berbiche and A. Hakem [8] thought about generalizing the above results for problems more difficult than those they addressed in the study of the following problem

$$\partial_t^2 u(t,x) - \Delta u(t,x) + |u|^{m-1} \partial_t u(t,x) = \int_0^t (t-\tau)^{-\gamma} |u(\tau,x)|^p \, d\tau, \tag{1.2}$$

which describes a damped wave equation with nonlinear memory and the damping is not linear, either. Specifically, they proved that if $p > m > 1$ and the initial datum satisfies

$$\int_{\mathbb{R}^N} u_0(x)dx > 0, \int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x)dx > 0, \int_{\mathbb{R}^N} u_1(x)dx > 0,$$

and if

$$N \leq \left(\frac{2\,(m + (1-\gamma)\,p)}{(p - 1 + (1-\gamma)\,(m-1))}, \frac{2\,(1 + (2-\gamma)\,p)}{\left(\frac{(p-1)(2-\gamma)}{p-m} + \gamma - 1\right)(p-1)} \right) \quad \textbf{or} \quad p \leq \frac{1}{\gamma},$$

then the solution of the equation (1.2) with such initial data u_0 and u_1 does not exist globally in time.

In this paper we would like to obtain similar results for a problem that is more general than (1.2), namely, a weighted damped wave equation with nonlinear memory, which reads

$$\begin{cases} \partial_t^2 u(t,x) - \Delta u(t,x) + g(x)\,|u|^{m-1} \partial_t u(t,x) = \displaystyle\int_0^t (t-\tau)^{-\gamma} |u(\tau,x)|^p \, d\tau, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{1.3}$$

where the weight $g(x)$ is stationary (independent of time).

Firstly, our purpose in this paper is to explore the local existence and uniqueness of the solution for the equation (1.3) by using the fixed point theorem. Secondly, we shall make full use of the test function method considered by Fino [7], Berbiche and Hakem [8], Pohozaev and Tesei [15], Mitidieri and Pohozaev ([16], [17]) and by Zhang [12] to prove blow-up results of the solution to the equation (1.3).

Remark 1.1. Throughout this paper, the constants will be denoted C and are different from one place to another.

2. Notations and Preliminary results

For a multi-index $\alpha = (\alpha_1, \alpha_1, ..., \alpha_N) \in \mathbb{N}^N$ we denote by

1. $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_N$ the length of the multi-index α.
2. $\alpha!$ for the factorial of α: $\alpha! = \alpha_1!\alpha_2!...\alpha_N!$
3. For all $\beta = (\beta_1, ..., \beta_N) \in \mathbb{N}^N$, $x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N$, we define

$$D_x^\beta \varphi = \frac{\partial_x^{|\beta|} \varphi}{\partial_{x_1}^{\beta_1} ... \partial_{x_N}^{\beta_N}}, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

   where $\mathcal{D}(\Omega)$ is the set of $\mathcal{C}^\infty$ functions with compact supports included in Ω.

We denote by $H^s(\mathbb{R}^N)$ the Sobolev space defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N), \quad \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^N), \quad i = 1, 2, ..., N \right\},$$

where the derivation is considered in the distribution sense. We also need some results as Sobolev's embedding theorems. We need the following Lemmas.

Lemma 2.1. *([4]) If $s > N/2$ then one has*

$$H^s(\mathbb{R}^N) \subset \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),$$

where the inclusion is continuous. Indeed, there exists a constant $C > 0$ such that

$$\forall u \in H^s(\mathbb{R}^N), \quad \|u\|_{L^\infty(\mathbb{R}^N)} \leq C \|u\|_{H^s(\mathbb{R}^N)}.$$

Lemma 2.2. *([8]) Assume that $s_1, s_2 \geq s > N/2$, then for all $u \in H^{s_1}(\mathbb{R}^N)$ and $v \in H^{s_2}(\mathbb{R}^N)$ there exists a positive constant C independent of u and v such that*

$$\|uv\|_{H^s(\mathbb{R}^N)} \leq C \|u\|_{H^{s_1}(\mathbb{R}^N)} \|v\|_{H^{s_2}(\mathbb{R}^N)}.$$

Lemma 2.3. *([8]) Let $s, N \geq 1$ such that $N \geq s - 1$, then for all nonnegative functions u belonging to $L^\infty(\mathbb{R}^N) \cap H^{s-1}(\mathbb{R}^N)$ and $n \in \mathbb{N}^*$ we have*

1. $u^n \in H^{s-1}(\mathbb{R}^N)$.

2. There exists a constant $C > 0$ *such that*

$$\|u^n\|_{H^{s-1}(\mathbb{R}^N)} \leq C \|u\|^{n-1}_{L^\infty(\mathbb{R}^N)} \|u\|_{H^{s-1}(\mathbb{R}^N)}.$$

The next lemma is an immediate consequence of (proposition 3.7 pp. 11 in [4]). Invoking the fact that f is bounded with all its derivatives as it is mentioned in Lemma 2.4 and by using the below Leibnitz formula (see formula 3.23 p.11 in [4]),

$$D^\alpha(fu) = \sum_{\beta+\gamma=\alpha} C^\beta_\alpha (D^\beta f)(D^\gamma u), \quad \textit{where } C^\beta_\alpha = \frac{\alpha!}{\beta!(\alpha-\beta)!} \tag{2.1}$$

Lemma 2.4. *Let* $s \in \mathbb{N}^*,\ u \in H^{s-1}(\mathbb{R}^N)$ *and* f *be a real valued bounded function with all its derivatives, then one has*

1. $fu \in H^{s-1}(\mathbb{R}^N)$.

2. Denoting $C_f = \sup_{x\in\mathbb{R}^N}\left(\sup_{|\alpha|\leq s-1} |D^\alpha_x f(x)|\right)$ *then* $\|fu\|_{H^{s-1}(\mathbb{R}^N)} \leq C_f \|u\|_{H^{s-1}(\mathbb{R}^N)}$.

3. Well-posedness of the problem

3.1. Introduction and statement of our problem

In this section we will prove the theorem of the existence and uniqueness of solutions to the following Cauchy problem, which described a weighted damped wave equation with nonlinear memory

$$(P)\begin{cases} u_{tt} - \Delta u + g(x)|u|^{m-1}u_t = \displaystyle\int_0^t (t-\tau)^{-\gamma}|u|^p d\tau \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

with $m > 1,\ p > 1, 0 < \gamma < 1,\ \Delta$ is the usual Laplace operator and g is a function which will be specified later.

3.2. Main result

Now we are able to state the main result and its proof concerning the local existence and uniqueness of the solution to *problem (P).*

Theorem 3.1. *Let $N \geq 1, s > \frac{N}{2}$, $m, p \in (1, \infty)$ such that $m, p > s-1$. Assume that $g \in C^s(\mathbb{R}^N)$ such that g is positive and satisfies*

$$\forall \beta = (\beta_1, ..., \beta_N) \in \mathbb{N}^N, \quad |\beta| \leq s \text{ such that: } D_x^\beta g = \frac{\partial^{|\beta|} g}{\partial_{x_1}^{\beta_1} ... \partial_{x_N}^{\beta_N}} = \mathcal{O}(1),$$

uniformly with respect to $x \in \mathbb{R}^N$, then for any $(u_0, u_1) \in H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$, the problem (P) admits a unique solution

$$u \in \mathcal{C}([0,T], H^s(\mathbb{R}^N)) \cap \mathcal{C}^1([0,T], H^{s-1}(\mathbb{R}^N)).$$

Where T depends only on $\|u_0\|_{H^s(\mathbb{R}^N)} + \|u_1\|_{H^{s-1}(\mathbb{R}^N)}$.

The main tool for the proof of *Theorem 3.1* is the fixed point theorem. For this reason, we need a suitable functional space and a contraction mapping. To do this, we define for some $T > 0$ and $M > 0$ the following functional spaces.

$$\begin{aligned} X_T &= \mathcal{C}([0,T], H^s(\mathbb{R}^N)) \cap \mathcal{C}^1([0,T], H^{s-1}(\mathbb{R}^N)), \\ E_T &= L^\infty([0,T], H^s(\mathbb{R}^N)) \cap W^{1,\infty}([0,T], H^{s-1}(\mathbb{R}^N)), \\ E_{T,M} &= \left\{ u \in E_T; \sup_{t \in [0,T]} \left(\|u\|_{H^s(\mathbb{R}^N)} + \|u_t\|_{H^{s-1}(\mathbb{R}^N)} \right) \leq M \right\}, \end{aligned}$$

and we put by definition

$$X_{T,M} := X_T \cap E_{T,M}.$$

Remark 3.1. It is easy to remark that one has $X_T \subset E_T$ and $X_{T,M} \subset E_{T,M}$ for all $T > 0$ and $M > 0$.

Still denoting

$$\begin{aligned} P_\alpha(g, \partial_t)u &= -g(x)|u|^{m-1}u_t + \int_0^t (t-\tau)^{-\gamma} |u(\tau, .)|^p d\tau \\ &= -g(x)|u|^{m-1}u_t + \Gamma(\alpha) I_{0|t}^\alpha (|u|^p), \end{aligned} \tag{3.1}$$

where $\alpha = 1 - \gamma$ and $I_{0|t}^\alpha$ is the Riemann-Liouville fractional integral of order α ($\alpha \in]0,1[$) defined by (See [18])

$$I_{0|t}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \tag{3.2}$$

and Γ is the usual Euler's Gamma function. First of all, we will state some results to prove *Theorem 3.1.*

Lemma 3.1. *Let $P_\alpha(g,\partial_t)$ be the operator defined by (3.1), then for all $t\in[0,T]$ and $u\in E_T$ one has*

$$\begin{aligned}\int_0^t P_\alpha(g,\partial_t)u(\tau,.)d\tau &= \frac{g(x)}{m}\Big(|u(0,.)|^{m-1}u(0,.)-|u(t,.)|^{m-1}u(t,.)\Big)\\ &+ \frac{1}{\alpha}\int_0^t (t-\tau)^\alpha |u(\tau,.)|^p d\tau.\end{aligned}$$

Proof. We have

$$\begin{aligned}\int_0^t P_\alpha(g,\partial_t)u(\tau,.)d\tau &= -g(x)\int_0^t |u(\tau,.)|^{m-1}u_t(\tau,.)d\tau\\ &+\int_0^t\int_0^\tau (\tau-s)^{\alpha-1}|u(s,.)|^p dsd\tau = -g(x)I+J.\end{aligned}\tag{3.3}$$

For I, it is enough to note that for all $u\in E_T$, we have

$$\partial_t(|u(t,.)|^{m-1}u(t,.)) = m|u(t,.)|^{m-1}u_t(t,.),\tag{3.4}$$

and for J we just use Fubini's theorem (Theorem 1.1.7, pp. 8 in [3]) to calculate the integral with respect to τ and find

$$J=\frac{1}{\alpha}\int_0^t (t-s)^\alpha |u(s,.)|^p ds.\tag{3.5}$$

Combining *(3.4)* and *(3.5)* into *(3.3)* we get

$$\begin{aligned}\int_0^t P_\alpha(g,\partial_t)u(\tau,.)d\tau &= -\frac{1}{m}g(.)\Big(|u(t,.)|^{m-1}u(t,.)-|u(0,.)|^{m-1}u(0,.)\Big)\\ &+ \frac{1}{\alpha}\int_0^t (t-\tau)^\alpha |u(\tau,.)|^p d\tau.\end{aligned}$$

This completes the proof of *Lemma 3.1.* □

Thanks to *Lemma 3.1* for estimating the norm in $H^{s-1}(\mathbb{R}^N)$ of $P_\alpha(g,\partial_t)u$ for all $u\in E_{T,M}$ as in the following.

Lemma 3.2. *Let P_α be the operator defined by* (3.1) *and*

$$M=C\left(\|u_0\|_{H^s(\mathbb{R}^N)}+\|u_1\|_{H^{s-1}(\mathbb{R}^N)}\right).$$

Then there exist two positive constants C_1 and C_2 depending only on T and M such that for all $u\in E_{T,M}$ one has

$$\int_0^t \|P_\alpha(g,\partial_t)u(\tau,.)\|_{H^{s-1}(\mathbb{R}^N)}d\tau \le C_1M^m + C_2T^{\alpha+1}M^p.$$

Proof. Since

$$\|u\|_{H^{s-1}(\mathbb{R}^N)} = \sum_{|\beta|\le s-1} \|D_x^\beta u\|_{L^2(\mathbb{R}^N)},$$

then, using *Lemma 3.1*, we get for all $\beta \in \mathbb{N}^N$ such that $|\beta| \le s-1$ and $x \in \mathbb{R}^N$

$$\begin{aligned} D_x^\beta \int_0^t P_\alpha(g,\partial_t)u(\tau,x)d\tau = & \frac{1}{m} D_x^\beta \left[g(x)|u(0,x)|^{m-1}u(0,x)\right] \\ & - \frac{1}{m} D_x^\beta \left(g(x)|u(t,x)|^{m-1}u(t,x)\right) \\ & + \frac{1}{\alpha} D_x^\beta \int_0^t (t-\tau)^\alpha \left|u(\tau,x)\right|^p d\tau. \end{aligned}$$

Applying Lebesgue's dominated convergence theorem (Theorem 1.1.4, pp. 3 in [3]) and Leibnitz's formula (2.1) we obtain

$$\begin{aligned} \int_0^t \|P_\alpha(g,\partial_t)u(\tau,.)\|_{H^{s-1}(\mathbb{R}^N)} d\tau \quad \le & \quad \left\|g\left|u(0,.)\right|^{m-1} u(0,.)\right\|_{H^{s-1}(\mathbb{R}^N)} \\ + & \quad \left\|g\left|u(t,.)\right|^{m-1} u(t,.)\right\|_{H^{s-1}(\mathbb{R}^N)} \\ + & \quad \int_0^t (t-\tau)^\alpha \left\|\left|u(\tau,.)\right|^p\right\|_{H^{s-1}(\mathbb{R}^N)} d\tau. \end{aligned}$$

By Sobolev's embedding theorem (*Lemma 2.1*) and *Lemma 2.4* we find

$$\begin{aligned} \int_0^t \|P_\alpha(g,\partial_t)u(\tau,.)\|_{H^{s-1}(\mathbb{R}^N)} d\tau \quad \le & \quad C\left\|g\left|u(0,.)\right|^{m-1} u(0,.)\right\|_{H^{s-1}(\mathbb{R}^N)} \\ + & \quad C\left\|g\left|u(t,.)\right|^{m-1} u(t,.)\right\|_{H^{s-1}(\mathbb{R}^N)} \\ + & \quad C\int_0^t (t-\tau)^\alpha \left\|\left|u(\tau,.)\right|^p\right\|_{H^{s-1}(\mathbb{R}^N)} d\tau \\ \le & \quad C\|u\|^m_{H^s(\mathbb{R}^N)} + C\sup_{0\le t\le T}\|u(t,.)\|^m_{H^s(\mathbb{R}^N)} \\ + & \quad \sup_{0\le t\le T}\|u(t,.)\|^p_{H^s(\mathbb{R}^N)} \sup_{0\le t\le T}\int_0^t (t-\tau)^\alpha d\tau \\ \le & \quad C_1 M^m + C_2 T^{\alpha+1} M^p. \end{aligned}$$

Hence the proof is completed. □

Remark 3.2. Let $T > 0$. Suppose that $u \in X_T$ is a solution of the following Cauchy problem

$$(P2) \quad \begin{cases} u_{tt}(t,x) - \Delta u(t,x) = P_\alpha(g,\partial_t)v(t,x), & t>0,\ x\in\mathbb{R}^n \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n \\ u_t(0,x) = u_1(x), & x\in\mathbb{R}^n. \end{cases}$$

Then for all $v \in E_T$ the mapping Φ defined by $\Phi(v) = u$ is well defined.

Proof. Note that since g is bounded then for any $v \in E_T$ we have by Sobolev's embedding theorem that $P_\alpha(g, \partial_t)v \in L^\infty\left([0,T]; H^s\left(\mathbb{R}^N\right)\right)$ which implies the existence and uniqueness of such u in E_T by the theory of mixed Cauchy problems for linear wave equations. □

Now using *Theorem 3.1* and *Lemma 3.2* we obtain the following results.

Proposition 3.1. *If $v \in E_{T,M}$ then $\Phi(v) = u \in X_{T,M}$.*

Proof. Assume that $v \in E_{T,M}$ and by choosing

$$M = C_0\left(\|u_0\|_{H^s(\mathbb{R}^N)} + \|u_t\|_{H^{s-1}(\mathbb{R}^N)}\right), \quad C_0 > 0,$$

it follows by using *Lemma 3.2* and the theory of linear wave equations

$$\begin{aligned}\sup_{0\le t\le T}\left(\|u(t,.)\|_{H^s} + \|u_t(t,.)\|_{H^{s-1}}\right) &\le C(1+T)\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right)\\ &\quad + C(1+T)\left(\int_0^t \|P_\alpha(g,\partial_t)v(\tau,.)\|_{H^{s-1}}d\tau\right)\\ &\le C(T,M)M,\end{aligned}$$

where

$$C(T,M) = C(1+T)\left(\frac{1}{C_0} + CM^{m-1} + CT^{\alpha+1}M^{p-1}\right).$$

Since we can find $T_1 > 0$ such that $\forall T \in [0,T_1]$, $C(T,M) \le 1$, we deduce that

$$\sup_{0\le t\le T}\left(\|u\|_{H^s(\mathbb{R}^N)} + \|u_t\|_{H^{s-1}(\mathbb{R}^N)}\right) \le M.$$

This ends the proof. □

Proposition 3.2. *The mapping Φ is a contraction from $X_{T,M}$ into $X_{T,M}$.*

Proof. Since the function $(x,y) \mapsto |x|^{m-1}y$ is not Lipschitz continuous with respect to $(x,y) \in \mathbb{R}^2$ for $1 < m < 2$, we cannot apply the mean value theorem directly. To overcome this obstacle, we will use the linearity and we modify the technique as it has been done by Berbiche and Hakem [8], Katayama [13], MD. Abu Naim [14] and by Lions and Strauss [19]. Let v_1, $v_2 \in E_{T,M}$ such that $v_1(0,x) = v_2(0,x) = u_0(x), x \in \mathbb{R}^N$ and let w, ν be solutions for the following problems, respectively

$$(P.1.1)\quad \begin{cases} w_{tt}(t,x) - \Delta w(t,x) = \displaystyle\int_0^t (t-s)^{\alpha-1}|v_1(s,x)|^p\,ds, & t>0, x \in \mathbb{R}^n\\ w(0,x) = u_0(x) & x \in \mathbb{R}^n\\ w_t(0,x) = u_1(x) + \frac{g(x)}{m}|u_0|^{m-1}u_0 & x \in \mathbb{R}^n,\end{cases}$$

and

$$(P.1.2)\begin{cases} \nu_{tt}(t,x)-\Delta\nu(t,x)=\displaystyle\int_0^t (t-s)^{\alpha-1}\left|v_2(s,x)\right|^p ds, & t>0, x\in\mathbb{R}^n\\ \nu(0,x)=u_0(x) & x\in\mathbb{R}^n\\ \nu_t(0,x)=u_1(x)+\frac{g(x)}{m}\left|u_0\right|^{m-1}u_0 & x\in\mathbb{R}^n. \end{cases}$$

Denote also by $\tilde{w}$ and $\tilde{\nu}$ the solutions of the following problems, respectively

$$(P.2.1)\quad\begin{cases} \tilde{w}_{tt}(t,x)-\Delta\tilde{w}(t,x)=-\frac{g(x)}{m}\left|v_1(t,x)\right|^{m-1}v_1(t,x), & t>0, x\in\mathbb{R}^n\\ \tilde{w}(0,x)=0 & x\in\mathbb{R}^n\\ \tilde{w}_t(0,x)=0 & x\in\mathbb{R}^n, \end{cases}$$

and

$$(P.2.2)\begin{cases} \tilde{\nu}_{tt}(t,x)-\Delta\tilde{\nu}(t,x)=-\frac{g(x)}{m}\left|v_2(t,x)\right|^{m-1}v_2(t,x), & t>0, x\in\mathbb{R}^n\\ \tilde{\nu}(0,x)=0 & x\in\mathbb{R}^n\\ \tilde{\nu}_t(0,x)=0 & x\in\mathbb{R}^n. \end{cases}$$

Remark 3.3. The fact that g is bounded with all its derivatives and $v_i\in E_{T,M}$ implies that for $i=1,2$ we have

$$\int_0^t (t-s)^{\alpha-1}\left|v_i(s,x)\right|^p ds,\quad \frac{g(x)}{m}\left|v_i(t,x)\right|^{m-1}v_i(t,x)\quad\text{and}$$
$$\frac{g(x)}{m}\left|v_i(t,x)\right|^{m-1}\partial_t v_i(t,x)\in L^\infty\left([0,T]\,;H^{s-1}(\mathbb{R}^N)\right),$$

consequently, by Sobolev's embedding theorem, we deduce that

$$w,\nu\in E_{T,M}$$

and

$$\tilde{w},\tilde{\nu}\in C\left([0,T]\,;H^{s+1}(\mathbb{R}^N)\right)\cap C^1\left([0,T]\,;H^{s}(\mathbb{R}^N)\right)\cap C^2\left([0,T]\,;H^{s-1}(\mathbb{R}^N)\right).$$

Then we have the following results:

Proposition 3.3. *Denoting $\bar{w}=w+\tilde{w}_t$, then $\bar{w}$ is solution for the problem*

$$(\bar{P}_1)\quad\begin{cases} \bar{w}_{tt}(t,x)-\Delta\bar{w}(t,x)=P_\alpha(g,\partial_t)v_1(t,x), & t>0, x\in\mathbb{R}^n\\ \bar{w}(0,x)=u_0(x) & x\in\mathbb{R}^n\\ \bar{w}_t(0,x)=u_1(x) & x\in\mathbb{R}^n. \end{cases}$$

Proof. One can remark that

$$\text{for all } v_1\in E_T \text{ and } t\in[0,T],\quad \partial_t\left(\left|v_1\right|^{m-1}v_1\right)=m\left|v_1\right|^{m-1}\partial_t v_1,$$

hence a simple computation shows that

$$\begin{aligned}
\partial_t^2 \bar{w} - \Delta \bar{w} &= \partial_t^2 \left(w + \tilde{w}_t \right) - \Delta \left(w + \tilde{w}_t \right) \\
&= w_{tt}(t,x) - \Delta w(t,x) - \partial_t \left(\tilde{w}_{tt}(t,x) - \Delta \tilde{w}(t,x) \right) \\
&= \int_0^t (t-s)^{\alpha-1} |v_1(t,x)|^p ds - g(x) \left| v_1(t,x) \right|^{m-1} v_1(t,x) \\
&= P_\alpha(g, \partial_t) v_1(t,x).
\end{aligned}$$

It is easy to see that the initial conditions are satisfied too as follows. We have firstly

$$\bar{w}(0,x) = w(0,x) + \tilde{w}_t(0,x) = u_0(x) \quad \textit{for all} \quad x \in \mathbb{R}^N.$$

Secondly, we have

$$\tilde{w}_{tt}(t,x) - \Delta \tilde{w}(t,x) = -\frac{g(x)}{m} \left| v_1(t,x) \right|^{m-1} v_1(t,x), \quad t \geq 0, \ x \in \mathbb{R}^N,$$

then for $t = 0$, we get

$$\tilde{w}_{tt}(0,x) - \Delta \tilde{w}(0,x) = -\frac{g(x)}{m} \left| v_1(0,x) \right|^{m-1} v_1(0,x), \tag{3.6}$$

and since $\Delta \tilde{w}(0,x) = 0$ because $\tilde{w}(0,x) = 0$ and $v_1(0,x) = u_0(x)$, we obtain from (3.6)

$$\tilde{w}_{tt}(0,x) = -\frac{g(x)}{m} \left| u_0(x) \right|^{m-1} u_0(x). \tag{3.7}$$

Now, using the formula (3.7) we find

$$\begin{aligned}
\bar{w}_t(0,x) &= w_t(0,x) + \tilde{w}_{tt}(0,x) \\
&= u_1(x) + \frac{g(x)}{m} \left| u_0(x) \right|^{m-1} u_0(x) - \frac{g(x)}{m} \left| u_0(x) \right|^{m-1} u_0(x) \\
&= u_1(x).
\end{aligned}$$

This completes the proof of *Proposition 3.3.* $\square$

Lemma 3.3. *Denoting $\bar{\nu} = \nu + \tilde{\nu}_t$, then $\bar{\nu}$ is solution for the following problem*

$$(\bar{P}_2) \quad \begin{cases} \bar{\nu}_{tt}(t,x) - \Delta \bar{\nu}(t,x) = P_\alpha(g, \partial_t) v_2(t,x), & t > 0, x \in \mathbb{R}^n \\ \bar{\nu}(0,x) = u_0(x), & x \in \mathbb{R}^n \\ \bar{\nu}_t(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases}$$

Proof. The proof is similar to the proof of Proposition 3.3. $\square$

Now by *Proposition 3.3, Lemma 3.3 and the definition of* Φ, we have the following corollary:

Corollary 3.1. *One has* $\Phi(v_1) = w + \tilde{w}_t$ *and* $\Phi(v_2) = \nu + \tilde{\nu}_t$.

Proof. An immediate consequence of the definition of Φ, *Proposition 3.3*, *Lemma 3.3* and the uniqueness of the solution to linear wave equations. $\square$

In order to prove that Φ is a contraction mapping into $E_{T,M}$, we also need the following.

Proposition 3.4. *Denoting* $w^* = w - \nu$, *then there exists a constant* $C > 0$ *such that*

$$\|w^*(t,.)\|_{H^s} + \|w^*_t(t,.)\|_{H^{s-1}} \leq C(1+T)\,T^{\alpha+1}M^{p-1}\sup_{0\leq t\leq T}\|v_1(t,.) - v_2(t,.)\|_{H^s}.$$

Proof. First of all, we show that w^* is a solution for the following homogenous Cauchy problem:

$$\begin{cases} \partial_t^2 w^* - \Delta w^* = \displaystyle\int_0^t (t-s)^{-\gamma}\left(|v_1(s,.)|^p - |v_2(s,.)|^p\right) ds \\ w^*(0,x) = w^*_t(0,x) = 0. \end{cases}$$

To do this, it is enough to note that firstly

$$\begin{aligned} \partial_t^2 w^* - \Delta w^* &= w_{tt} - \Delta w - (\nu_{tt} - \Delta\nu) \\ &= \int_0^t (t-s)^{-\gamma}|v_1(s,x)|^p\,ds - \int_0^t (t-s)^{-\gamma}|v_2(s,x)|^p\,ds \\ &= \int_0^t (t-s)^{-\gamma}\left(|v_1(s,.)|^p - |v_2(s,.)|^p\right) ds, \end{aligned}$$

and secondly

$$w^*(0,x) = w(0,x) - \nu(0,x) = u_0(0,x) - u_0(0,x) = 0.$$

In the same way, we prove that $w^*_t(0,x) = 0$, hence, by the theory of linear wave equations, we get

$$\|w^*(t,.)\|_{H^s} + \|w^*_t(t,.)\|_{H^{s-1}} \leq C(1+T)\int_0^t\int_0^\tau (\tau-s)^{-\gamma}\||v_1(s,.)|^p - |v_2(s,.)|^p\|_{H^{s-1}}\,ds\,d\tau.$$

Applying Fubini's theorem, we arrive at

$$\begin{aligned} &\|w^*(t,.)\|_{H^s} + \|w^*_t(t,.)\|_{H^{s-1}} \\ &\leq C(1+T)\int_0^t\left(\int_s^t (\tau-s)^{-\gamma}d\tau\right)\||v_1(s,.)|^p - |v_2(s,.)|^p\|_{H^{s-1}}ds \\ &\leq C(1-\gamma)^{-1}(1+T)\int_0^t (t-s)^{1-\gamma}\||v_1(s,.)|^p - |v_2(s,.)|^p\|_{H^{s-1}}ds. \end{aligned}$$

By the mean value theorem and Sobolev's embedding theorem, we obtain

$$\begin{aligned}
&\|w^*(t,.)\|_{H^s} + \|w_t^*(t,.)\|_{H^{s-1}} \\
&\le C(1-\gamma)^{-1}(1+T)\sup_{0\le t\le T}\left(\|v_1(t,.)\|_{H^s}^{p-1} + \|v_2(t,.)\|_{H^s}^{p-1}\right) \\
&\times \sup_{0\le t\le T}\|v_1(t,.)-v_2(t,.)\|_{H^s}\sup_{0\le t\le T}\int_0^t (t-s)^{1-\gamma}\,ds \\
&\le C(1+T)T^{-\gamma}M^{p-1}\sup_{0\le t\le T}\|v_1(t,.)-v_2(t,.)\|_{H^s}.
\end{aligned}$$

This ends the proof. □

Proposition 3.5. *Denoting $\tilde{w}^* = \tilde{w} - \tilde{\nu}$, then there exists a constant $C > 0$ such that, for all $t \in [0,T]$ we have*

$$\|\tilde{w}^*(t,.)\|_{H^s} + \|\tilde{w}_t^*(t,.)\|_{H^{s-1}} \le C(1+T)TM^{m-1}\sup_{0\le t\le T}\|v_1(t,.)-v_2(t,.)\|_{H^s}.$$

Proof. It is easy to show that $\tilde{w}^*$ is a solution to the following Cauchy problem

$$\begin{cases} \partial_t^2\tilde{w}^* - \Delta\tilde{w}^* = -\frac{g}{m}\left(|v_1(s,.)|^{m-1}v_1 - |v_2(s,.)|^{m-1}v_2\right) \\ \tilde{w}^*(0,x) = \tilde{w}_t^*(0,x) = 0, \end{cases}$$

hence, by the theory of linear wave equations, we find

$$\begin{aligned}
&\|\tilde{w}^*(t,.)\|_{H^s} + \|\tilde{w}_t^*(t,.)\|_{H^{s-1}} \\
&\le C(1+T)\int_0^t \left\|\frac{g}{m}|v_1(s,.)|^{m-1}v_1 - \frac{g}{m}|v_2(s,.)|^{m-1}v_2\right\|_{H^{s-1}} ds,
\end{aligned}$$

by using *Lemma.2.4*, we arrive at

$$\begin{aligned}
&\|\tilde{w}^*(t,.)\|_{H^s} + \|\tilde{w}_t^*(t,.)\|_{H^{s-1}} \\
&\le C(1+T)\int_0^t \left\||v_1(s,.)|^{m-1}v_1 - |v_2(s,.)|^{m-1}v_2\right\|_{H^{s-1}} ds.
\end{aligned}$$

Taking into account the mean value theorem, we get

$$\begin{aligned}
&\|\tilde{w}^*(t,.)\|_{H^s} + \|\tilde{w}_t^*(t,.)\|_{H^{s-1}} \le \\
&C(1+T)\int_0^t \left(\|v_1(s,.)\|_{H^s}^{m-1} + \|v_2(s,.)\|_{H^s}^{m-1}\right)\|v_1(s,.)-v_2(s,.)\|_{H^s}\,ds \\
&\le C(1+T)\sup_{0\le t\le T}\left(\|v_1(t,.)\|_{H^s}^{m-1} + \|v_2(t,.)\|_{H^s}^{m-1}\right) \\
&\times \sup_{0\le t\le T}\|v_1(t,.)-v_2(t,.)\|_{H^s}\sup_{0\le t\le T}\int_0^t ds \\
&\le C(1+T)TM^{m-1}\sup_{0\le t\le T}\|v_1(t,.)-v_2(t,.)\|_{H^s}.
\end{aligned} \tag{3.8}$$

This completes the proof. □

Now we are able to estimate the H^s-norm of the quantity $\Phi(v_1)-\Phi(v_2)$ and show that Φ is a contraction mapping as follows. For all $v_1,\ v_2 \in E_{T,M}$ and $t\in[0,T]$, we have

$$\begin{aligned}
\left\|\Phi\left(v_1\right)-\Phi\left(v_2\right)\right\|_{H^s} &= \left\|w\left(t,.\right)+\tilde{w}_t\left(t,.\right)-\nu\left(t,.\right)-\tilde{\nu}_t\left(t,.\right)\right\|_{H^s}\\
&\leq \left\|w-\nu\right\|_{H^s}+\left\|\tilde{w}_t-\tilde{\nu}_t\right\|_{H^s}\\
&\leq \left\|w^*\left(t,.\right)\right\|_{H^s}+\left\|w^*_t\left(t,.\right)\right\|_{H^{s-1}}+\left\|\tilde{w}^*\left(t,.\right)\right\|_{H^s}+\left\|\tilde{w}^*_t\left(t,.\right)\right\|_{H^{s-1}}\\
&\leq C\left(1+T\right)T^{\alpha+1}M^{p-1}\sup_{0\leq t\leq T}\left\|v_1\left(t,.\right)-v_2\left(t,.\right)\right\|_{H^s}\\
&\quad + C\left(1+T\right)TM^{m-1}\sup_{0\leq t\leq T}\left\|v_1\left(t,.\right)-v_2\left(t,.\right)\right\|_{H^s}\\
&\leq C\left(1+T\right)\left[T^{\alpha+1}M^{p-1}+TM^{m-1}\right]\sup_{0\leq t\leq T}\left\|v_1\left(t,.\right)-v_2\left(t,.\right)\right\|_{H^s},
\end{aligned}$$

hence

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left\|\Phi(v_1)-\Phi(v_2)\right\|_{H^s}\leq\\
&C\left(1+T\right)\left[T^{\alpha+1}M^{p-1}+TM^{m-1}\right]\sup_{0\leq t\leq T}\left\|v_1(t,.)-v_2(t,.)\right\|_{H^s}.
\end{aligned}\tag{3.9}$$

Since it is possible to find $T_1>0$ satisfying

$$C\left(1+T\right)\left[T^{\alpha+1}M^{p-1}+TM^{m-1}\right]<1,\quad \forall T\in\left[0,T_1\right],$$

we deduce from (3.9) that

$$\sup_{0\leq t\leq T}\left\|\Phi\left(v_1\right)-\Phi\left(v_2\right)\right\|_{H^s}\leq k\sup_{0\leq t\leq T}\left\|v_1\left(t,.\right)-v_2\left(t,.\right)\right\|_{H^s},\ k\in\left]0,1\right[.\tag{3.10}$$

Now, using *Remark 3.1* and *Proposition 3.1*, we easily show that

$$\Phi(X_{T,M})\subset X_{T,M}.$$

Finally, define a sequence $(u^{(n)})_n$ as follows

$$\left\{\begin{array}{l} u^{(0)}(t,x)=u(0,x)=u_0(x),\\ u^{(n)}(t,x)=\Phi(u^{(n-1)})(t,x).\end{array}\right.\tag{3.11}$$

By (3.10), for all $T>0$ there exists some $\bar{u}\in C\left([0,T],H^s\right)$ such that $u^{(n)}\to\bar{u}$ in $C\left([0,T],H^s\right)$ as $n\to\infty$. The aim now is to show that this $\bar{u}$ belongs to X_T and is a solution to *problem* (1.3). Since $u^{(n)}\in X_{T,M}$, then $\left(u^{(n)}\right)_n$ and $\left(u_t^{(n)}\right)_n$ has a weak convergent subsequence $\left(u^{(n_k)}\right)_k$ (resp. $(u_t^{(n_k)})_k$) in $L^\infty\left([0,T];H^s\right)$ (resp. in $L^\infty\left([0,T];H^{s-1}\right)$). Since $\left(u^{(n)}\right)_n$ converges to $\bar{u}$ in $C\left([0,T],H^s\right)$, the above subsequence converges weakly to $\bar{u}$ in $L^\infty\left([0,T];H^s\right)$ (resp. in $L^\infty\left([0,T];H^{s-1}\right)$), as a consequence, we see that $\bar{u} \in L^{\infty}([0,T];H^s)$ and $\bar{u}_t \in L^{\infty}\left([0,T];H^{s-1}\right)$, this means that $\bar{u} \in E_{T,M}$ and then $\Phi(\bar{u}) \in X_{T,M}$. Applying (3.10) we get for $k \in]0,1[$

$$\sup_{0\le t\le T}\left\|\Phi\left(u^{(n)}\right)-\Phi\left(\bar{u}\right)\right\|_{H^s} \le k \sup_{0\le t\le T}\left\|u^{(n)}(t,.)-\bar{u}(t,.)\right\|_{H^s}. \tag{3.12}$$

Since the right-hand side of (3.12) goes to 0 as $n\to+\infty$, then $\Phi\left(u^{(n)}\right)$ converges to $\Phi\left(\bar{u}\right)$ in $C\left([0,T],H^s\right)$. Passing to the limit in $u^{(n)}=\Phi\left(u^{(n-1)}\right)$ as $n\to+\infty$ and using the fact that $u_n^{(n)}\to\bar{u}$ in $C\left([0,T],H^s\right)$, we obtain

$$\Phi(\bar{u})=\bar{u}\in X_{T,M}.$$

This $\bar{u}$ is apparently the desired solution. The uniqueness of such a solution in $X_{T,M}$ follows immediately from the formula (3.10). This achieves the proof of Theorem 3.1. □

4. Blow-up results of solutions for problem (1.3)

In this section, we will investigate the blow-up results of *problem* (1.3).

4.1. Notations and definitions

Definition 4.1. Let $u_0 \in L^1_{loc}\left(\mathbb{R}^N\right)\cap L^m_{loc}\left(\mathbb{R}^N\right)$ and $u_1 \in L^1_{loc}\left(\mathbb{R}^N\right)$ be given. We say that u is a weak solution to the problem (1.3) if $u \in L^p\left((0,T),L^p_{loc}\left(\mathbb{R}^N\right)\right)\cap L^m\left((0,T),L^m_{loc}\left(\mathbb{R}^N\right)\right)$ and satisfies the following formula

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)\varphi(t,x)dtdx+\int_{\mathbb{R}^n}u_1(x)\varphi(0,x)dx\\
&-\int_{\mathbb{R}^n}u_0(x)\varphi_t(0,x)dx+\frac{1}{m}\int_{\mathbb{R}^n}g(x)\left|u_0\right|^{m-1}(x)u_0(x)\varphi(0,x)dx\\
&=\int_0^T\int_{\mathbb{R}^n}u(t,x)\varphi_{tt}(t,x)dtdx-\int_0^T\int_{\mathbb{R}^n}u(t,x)\Delta\varphi(t,x)dtdx,\\
&-\frac{1}{m}\int_0^T\int_{\mathbb{R}^n}g(x)\left(\left|u\right|^{m-1}u\right)(t,x)\varphi_t(t,x)dtdx
\end{aligned} \tag{4.1}$$

for all non-negative test functions $\varphi \in C^2([0,T]\times\mathbb{R}^N)$ such that $\varphi_t\left(T,.\right)=\varphi\left(T,.\right)=0$ and $\alpha=1-\gamma$.

The main result of this section is the following theorem.

Theorem 4.1. *Let $0<\gamma<1$ and $p,m\in\mathbb{R}$ such that $p>m>1$. Assume that*

$$\int_{\mathbb{R}^n}u_0(x)dx>0,\quad \int_{\mathbb{R}^n}g(x)\left|u_0(x)\right|^m dx>0,\quad \int_{\mathbb{R}^n}u_1(x)dx>0. \tag{4.2}$$

Then the solution of the Cauchy problem (1.3) does not exist globally in time if one of the following conditions is fulfilled:

1.

$$N \leq \min\left(\frac{2\left((1-\gamma)p+m\right)}{(1-\gamma)\left(m-1\right)+(p-1)}, -\frac{2p(2-\gamma)+2}{\left(\frac{(2-\gamma)(p-1)}{m-p}+(1-\gamma)\right)(p-1)}\right)$$

2. $p < \frac{1}{\gamma}$ *or* $p = \frac{1}{\gamma}$ *and moreover* $\frac{N-2}{N} < \gamma < 1$.

Proof. The theorem (4.1) will be proved by absurd, so we suppose that u is a global non-trivial weak solution to the problem (1.3). To prove Theorem 4.1 we also need some results that we will give in the following section. □

4.2. Preliminary results

Since the principle of the method is the right choice of the test function, we choose it as follows

$$\varphi(t,x) = D^{\alpha}_{t|T}\psi(t,x) = \varphi_1^{r}(x)D^{\alpha}_{t|T}\varphi_2(t), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, \tag{4.3}$$

where $D^{\alpha}_{t|T}$ is the right fractional derivative operator in the sense of Riemann-Liouville defined by

$$D^{\alpha}_{t|T}v(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{v(s)}{(s-t)^{\alpha}}ds, \tag{4.4}$$

and the functions φ_1 and φ_2 are given by

$$\varphi_1(x) = \phi\left(\frac{x^2}{T^{\theta}}\right) \quad \text{and} \quad \varphi_2(t) = \sup\left\{0, \left(1-\frac{t}{T}\right)^{\beta}\right\}, \tag{4.5}$$

with $\beta \in \mathbb{R}_+^*$, θ is a nonnegative parameter which will be specified later and ϕ is a cut-off non-increasing function satisfying

$$\phi(s) = \begin{cases} 1 & \text{if} \quad 0 \leq s \leq 1 \\ 0 & \text{if} \quad \quad\; s \geq 2 \end{cases}, \quad 0 \leq \phi \leq 1 \text{ everywhere and } \phi'(s) \leq \frac{C}{s}. \tag{4.6}$$

We will also use the fractional version of the integration by parts (See [18])

$$\int_0^t f(t)D^{\alpha}_{t|T}g(t)dt = \int_0^t \left(D^{\alpha}_{0|t}f(t)\right)g(t)dt, \tag{4.7}$$

for all $f, g \in \mathcal{C}([0,T])$ such that $D^{\alpha}_{0|t}\left(f(t)\right)$ and $D^{\alpha}_{t|T}g(t)$ exist and are continuous, and the following identity

$$\left(D^{\alpha}_{0|t} \circ I^{\alpha}_{0|t}\right)(u) = u \text{ for all } u \in L^q\left([0,T]\right) \tag{4.8}$$

and also the bellow identity (see [18])

$$(-1)^n \partial_t^n D_{t|T}^{\alpha} u(t) = D_{t|T}^{\alpha+n} u(t),\ n \in \mathbb{N},\ \alpha \in]0,1[\tag{4.9}$$

which occurs for all $u \in C^n\,[0,T]\,; T > 0$, where ∂_t^n is the $n-$times ordinary derivative with respect to t that will be useful in this paper.

A simple and immediate computation leads to

Proposition 4.1. *Given $\beta > 0$. Let φ_2 be the function defined by*

$$\varphi_2(t) = \left(1 - \frac{t}{T}\right)_+^{\beta},$$

then, for all $\alpha \in]0,1[\,$, we have

$$\begin{aligned} D_{t|T}^{\alpha}\varphi_2(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} T^{-\beta}(T-t)_+^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} T^{-\alpha}\left(1-\frac{t}{T}\right)_+^{\beta-\alpha} \end{aligned}$$

$$\begin{aligned} D_{t|T}^{\alpha+1}\varphi_2(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)} T^{-\beta}(T-t)_+^{\beta-\alpha-1} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)} T^{-\alpha-1}\left(1-\frac{t}{T}\right)_+^{\beta-\alpha-1} \end{aligned}$$

$$\begin{aligned} D_{t|T}^{\alpha+2}\varphi_2(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha-1)} T^{-\beta}(T-t)_+^{\beta-\alpha-2} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha-1)} T^{-\alpha-2}\left(1-\frac{t}{T}\right)_+^{\beta-\alpha-2} \end{aligned}$$

Proof. The proof of *Proposition 4.1* is a simple and immediate verification. We get from the formula (4.4)

$$D_{t|T}^{\alpha}\varphi_2(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{\varphi_2(s)}{(s-t)^{\alpha}}ds.$$

Using Euler's change of the variable

$$s \mapsto y = \frac{s-t}{T-t}$$

1. We have firstly

$$\begin{aligned}
D^{\alpha}_{t|T}\varphi_2(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{(1-\frac{s}{T})^{\beta}}{(s-t)^{\alpha}}ds \\
&= \frac{T^{-\beta}}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\left((T-t)^{\beta-\alpha+1}\int_0^1 y^{-\alpha}(1-y)^{\beta}dy\right) \\
&= \frac{(\beta-\alpha+1)\,\mathcal{B}(1-\alpha,\beta+1)}{\Gamma(1-\alpha)}T^{-\beta}(T-t)^{\beta-\alpha} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha}(T-t)^{\beta-\alpha},
\end{aligned}$$

   where $\mathcal{B}$ is the famous *Béta function defined by*

$$\mathcal{B}(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}\,dt.$$

2. We directly apply the formula (4.9) to show that

$$\forall t \in [0,T] : D^{\alpha+1}_{t|T}\varphi_2(t) = -\partial_t D^{\alpha}_{t|T}\varphi_2(t) \text{ et } D^{\alpha+2}_{t|T}\varphi_2(t) = \partial_t^2 D^{\alpha}_{t|T}\varphi_2(t).$$

Hence the proof is completed. □

4.3. Treatment of the weak formulation (4.1)

4.3.1. Treatment of the left-hand side

Using the parts integration formula (4.7) and the identity (4.8) we get

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)\varphi(t,x)dtdx = \int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)D^{\alpha}_{0|T}\psi(t,x)dtdx \\
&= \int_0^T\int_{\mathbb{R}^n} D^{\alpha}_{0|T}I^{\alpha}_{0|T}(|u|^p)\psi(t,x)dtdx = \int_0^T\int_{\mathbb{R}^n} |u|^p\,\psi(t,x)dtdx.
\end{aligned} \tag{4.10}$$

For the 2^{nd} term of the left-hand side of the equality (4.1), we use *Proposition 4.1*. We easily obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} u_1(x)\varphi(0,x)dx &= \int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x)\; D^{\alpha}_{t|T}\varphi_2(t)\Big|_{t=0}dx \\
&= C_1T^{-\alpha}\int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x)dx,
\end{aligned} \tag{4.11}$$

since

$$D^{\alpha}_{t|T}\varphi_2(t)\Big|_{t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha} = C_1T^{-\alpha}.$$

For the third term, noting that

$$\varphi_t(t,x) = \frac{\partial \varphi}{\partial t}(t,x) = \varphi_1^r(x) D_{t|T}^{\alpha+1}\varphi_2(t),$$

using therefore *Proposition 4.1* we get the following estimate

$$\int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)dx = C_2 T^{-\alpha-1}\int_{\mathbb{R}^n} u_0(x)\varphi_1^r(x)dx, \tag{4.12}$$

since

$$D_{0|T}^{\alpha+1}\varphi_2(t)\Big|_{t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)}T^{-\alpha-1} = C_2 T^{-\alpha-1}.$$

Hence by *Proposition 4.1*, the following estimate will be obtained for the 4^{th} term of the left-hand side of the weak formulation(4.1).

$$\begin{aligned} &\int_{\mathbb{R}^n} g(x)\left|u_0\right|^{m-1}(x)u_0(x)\varphi(0,x)dx \\ &= C_1 T^{-\alpha}\int_{\mathbb{R}^n} g(x)\left|u_0\right|^{m-1}u_0(x)(x)\varphi_1^r(x)dx. \end{aligned} \tag{4.13}$$

4.3.2. Treatment of the right-hand side

Making use of (4.9), it can be seen that

$$\varphi_{tt}(t,x) = \varphi_1^r(x)\partial_t^2 D_{t|T}^{\alpha}\varphi_2(t) = \varphi_1^r(x) D_{t|T}^{\alpha+2}\varphi_2(t),$$

one can deduce that

$$\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_{tt}(t,x)dtdx = \int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_1^r(x)D_{t|T}^{\alpha+2}\varphi_2(t)dtdx. \tag{4.14}$$

Similarly, as

$$\varphi_t(t,x) = \varphi_1^r(x)\partial_t D_{t|T}^{\alpha}\varphi_2(t) = -\varphi_1^r(x)D_{t|T}^{\alpha+1}\varphi_2(t),$$

we get

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}^n} g(x)\left|u\right|^{m-1}u(t,x)\varphi_t(t,x)dtdx = \\ -&\int_0^T\int_{\mathbb{R}^n} g(x)\left|u\right|^{m-1}u(x)\varphi_1^r(x)D_{t|T}^{\alpha+1}\varphi_2(t)dtdx. \end{aligned} \tag{4.15}$$

Finally, for the third term of the right-hand side of the formulation (4.1), we use the following identity

$$\Delta(\varphi_1^r) = r\varphi_1^{r-1}\Delta\varphi_1 + r\left(r-1\right)\varphi_1^{r-2}\left|\nabla\varphi_1\right|^2,$$

to obtain

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} u(t,x)\Delta\varphi(t,x)dtdx = \\
&\int_0^T\int_{\mathbb{R}^n} u(t,x)(r\varphi_1^{r-1}\Delta\varphi + r(r-1)\varphi^{r-2}|\nabla\varphi_1|^2)D^{\alpha}_{t|T}\varphi_2(t)dtdx.
\end{aligned}
\tag{4.16}
$$

Inserting the formulas (4.10), (4.11), (4.12), (4.13), (4.14), (4.15) et (4.16) in the formula (4.1) we deduce

$$
\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n} |u|^p\,\psi(t,x)dtdx + C_1T^{-\alpha}\int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x)dx \\
&+ C_2T^{-\alpha-1}\int_{\mathbb{R}^n} u_0(x)\varphi_1^r(x)dx + \frac{C_1}{m}T^{-\alpha}\int_{\mathbb{R}^n} g(x)\,|u_0|^{m-1}(x)\varphi_1^r(x)dx \\
&= \int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_1^r(x)D^{\alpha+2}_{t|T}\varphi_2(t)dtdx \\
&- \int_0^T\int_{\mathbb{R}^n} u(t,x)\left[r\varphi_1^{r-1}\Delta\varphi_1 + r(r-1)\varphi_1^{r-2}\,|\nabla\varphi_1|^2\right](x).D^{\alpha}_{t|T}\varphi_2(t)dtdx \\
&+ \frac{1}{m}\int_0^T\int_{\mathbb{R}^n} g(x)\,|u|^m(x)\varphi_1^r(x)D^{\alpha+1}_{t|T}\varphi_2(t)dtdx.
\end{aligned}
\tag{4.17}
$$

The fact that $\varphi_1^r \le 1$ and

$$
\left|r\varphi_1^{r-1}\Delta\varphi_1 + r\,(r-1)\,\varphi_1^{r-2}\,|\nabla\varphi_1|^2\right| \le \varphi_1^{r-2}\left(|\Delta\varphi_1| + |\nabla\varphi_1|^2\right),
$$

allow us to get from the formula (4.17) the following inequality

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n} |u|^p\,\psi(t,x)dtdx + CT^{-\alpha}\int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x)dx \\
&+ CT^{-\alpha-1}\int_{\mathbb{R}^n} u_0(x)\varphi_1^r(x)dx + CT^{-\alpha}\int_{\mathbb{R}^n} g(x)\,|u_0|^m(x)\varphi_1^r(x)dx \\
&\le C\int_0^T\int_{\mathbb{R}^n} |u(t,x)|\,\varphi_1^r(x)\left|D^{\alpha+2}_{t|T}\varphi_2(t)\right|dtdx \\
&+ C\int_0^T\int_{\mathbb{R}^n} |u(t,x)|\,\varphi_1^{r-2}\left[|\Delta\varphi_1| + |\nabla\varphi_1|^2\right]\left|D^{\alpha}_{t|T}\varphi_2(t)\right|dtdx \\
&+ C\int_0^T\int_{\mathbb{R}^n} |u|^m\,g(x)\varphi_1^r(x)\left|D^{\alpha+1}_{t|T}\varphi_2(t)\right|dtdx,
\end{aligned}
\tag{4.18}
$$

for some constant $C > 0$. Applying $\varepsilon-$*Young inequality*

$$
AB \le \varepsilon A^p + C(\varepsilon)B^q,\ pq = p + q,\ p, q > 1,
$$

to terms of the right-hand side of the inequality (4.18), we find

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} |u(t,x)| \, \varphi_1^r(x) \left| D_{t|T}^{\alpha+2} \varphi_2(t) \right| dtdx = \\
&\int_0^T \int_{\mathbb{R}^n} u(t,x) \psi^{\frac{1}{p}} \psi^{-\frac{1}{p}} \varphi_1^r(x) \left| D_{t|T}^{\alpha+2} \varphi_2(t) \right| dtdx \\
&\leq \varepsilon \int_0^T \int_{\mathbb{R}^n} |u|^p \, \psi dtdx + C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dtdx.
\end{aligned} \tag{4.19}$$

Similarly, we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} |u| \, \varphi_1^{r-2} \left(|\Delta \varphi_1| + |\nabla \varphi_1|^2 \right) \left| D_{t|T}^{\alpha} \varphi_2 \right| dtdx \leq \\
&\varepsilon \int_0^T \int_{\mathbb{R}^n} |u|^p \, \psi dtdx + C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \mu(\varphi_1) \, \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha} \varphi_2 \right|^q dtdx,
\end{aligned} \tag{4.20}$$

with $\mu(\varphi_1) = |\Delta \varphi_1|^q + |\nabla \varphi_1|^{2q}$. For the third term of the right-hand side, we get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} |g| \, |u|^m \, \varphi_1^r(x) \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right| dtdx = \\
&\int_0^T \int_{\mathbb{R}^n} |u|^m \, \psi^{\frac{m}{p}} \psi^{-\frac{m}{p}} \, |g| \, \varphi_1^r(x) \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right| dtdx \\
&\leq \varepsilon \int_0^T \int_{\mathbb{R}^n} |u|^p \, \psi dtdx \\
&+ C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \varphi_1^r \, |\varphi_2|^{-\frac{m}{p-m}} \, |g| \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right|^{\frac{p}{p-m}} dtdx.
\end{aligned} \tag{4.21}$$

Using (4.2) and the fact that

$$\int_{\mathbb{R}^n} u_i(x) \varphi_1^r(x) dx > 0, i = 0, 1 \text{ and } \int_{\mathbb{R}^n} g(x) \, |u_0|^m (x) \varphi_1^r(x) dx > 0, \tag{4.22}$$

we deduce from (4.18), (4.19), (4.20) and (4.21), for ε small enough

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} |u|^p \, \psi(t,x) dtdx \leq C \left(\int_0^T \int_{\mathbb{R}^n} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dtdx \right. \\
&+ \int_0^T \int_{\mathbb{R}^n} \mu(\varphi_1) \, \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha} \varphi_2 \right|^q dtdx \\
&\left. + \int_0^T \int_{\mathbb{R}^n} \varphi_1^r \, |\varphi_2|^{\frac{m}{p-m}} \, |g| \left| D_{t|T}^{\alpha+1} \varphi_2(t) \right|^{\frac{p}{p-m}} dtdx \right) \\
&\leq C \left(I_1 + I_2 + I_3 \right),
\end{aligned} \tag{4.23}$$

for some positive constant C. Now, to estimate the integrals I_1, I_2 and I_3, at this stage we consider the scaled variables

$$x = T^{\frac{\theta}{2}} y \quad \text{and} \quad t = T\tau, \tag{4.24}$$

where θ is the parameter which appears in the form of φ_1 (page 643) and noting that I_1, I_2 and I_3 are null outside Ω_T such that

$$\Omega_T := \left\{ x \in \mathbb{R}^N, \quad |x|^2 \leq 2T^\theta \right\} = \mathrm{supp}\varphi_1.$$

By Fubini's theorem (*Theorem 1.1.7 pp. 8 in [3]*), it is easy to see that

$$\int_0^T \int_{\Omega_T} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dtdx =$$
$$\left(\int_{\Omega_T} \varphi_1^r dx \right) \left(\int_0^T \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dt \right) = J_{11} J_{12}.$$

We have

$$J_{11} = \int_{\Omega_T} \varphi_1^r(x) dx = T^{\frac{N\theta}{2}} \int_0^2 \phi^r(y^2) dy = CT^{\frac{N\theta}{2}}, \tag{4.25}$$

then using *Proposition 4.1,* we get

$$J_{12} = \int_0^T \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dt = CT^{1-(\alpha+2)\frac{p}{p-1}}. \tag{4.26}$$

Combining (4.25) and (4.26) into (4.25) we obtain

$$\int_0^T \int_{\Omega_T} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dtdx = CT^{-(\alpha+2)\frac{p}{p-1}+\frac{N\theta}{2}+1}. \tag{4.27}$$

In the same way we have

$$\begin{aligned} &\int_0^T \int_{\Omega_T} \mu(\varphi_1) \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha} \varphi_2 \right|^q dtdx = \\ &\left(\int_{\Omega_T} \mu(\varphi_1) \varphi_1^{r-2q} dx \right) \left(\int_0^T \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha} \varphi_2 \right|^q dt \right) = J_{21} J_{22}. \end{aligned} \tag{4.28}$$

So, if we replace q by its value $\frac{p}{p-1}$ we get

$$J_{21} = \int_{\Omega_T} \left(|\Delta\varphi_1|^{\frac{p}{p-1}} + |\nabla\varphi_1|^{2\frac{p}{p-1}} \right) \varphi_1^{r-2\frac{p}{p-1}} dx = CT^{-\theta\frac{p}{p-1}+\frac{N\theta}{2}}, \tag{4.29}$$

and

$$J_{22} = \int_0^T \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha} \varphi_2 \right|^{\frac{p}{p-1}} dt = CT^{-\alpha\frac{p}{p-1}+1}. \tag{4.30}$$

Plugging (4.29) and (4.30) into (4.28) we find

$$\int_0^T \int_{\Omega_T} \mu\left(\varphi_1\right) \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} \left| D^{\alpha}_{t|T} \varphi_2 \right|^q dtdx = CT^{-(\alpha+\theta)\frac{p}{p-1}+\frac{N\theta}{2}+1}. \tag{4.31}$$

For the third term we have

$$\begin{gathered} \int_0^T \int_{\Omega_T} \varphi_1^r \left|\varphi_2\right|^{-\frac{m}{p-m}} |g| \left| D^{\alpha+1}_{t|T} \varphi_2(t) \right|^{\frac{p}{p-m}} dtdx = \\ \left(\int_{\Omega_T} |g| \, \varphi_1^r dx \right) \left(\int_0^T \left|\varphi_2\right|^{-\frac{m}{p-m}} \left| D^{\alpha+1}_{t|T} \varphi_2(t) \right|^{\frac{p}{p-m}} dt \right) = J_{31} J_{32}. \end{gathered} \tag{4.32}$$

By the mean value theorem, we get for J_{31}

$$\begin{aligned} J_{31} &\leq |g(\xi)| \int_{\Omega_T} \varphi_1^r dx \quad \text{for some } \xi \in \Omega_T \\ &= T^{\frac{N\theta}{2}} |g(\xi)| \int_0^2 \phi^r(y^2) dy = CT^{\frac{N\theta}{2}}. \end{aligned} \tag{4.33}$$

Since g is bounded, we have

$$J_{32} = \int_0^T \left|\varphi_2\right|^{-\frac{m}{p-m}} \left| D^{\alpha+1}_{t|T} \varphi_2(t) \right|^{\frac{p}{p-m}} dt = CT^{-(\alpha+1)\frac{p}{p-m}+1}. \tag{4.34}$$

Hence inserting (4.33) and (4.34) in 4.32 we obtain

$$\int_0^T \int_{\Omega_T} \varphi_1^r \left|\varphi_2\right|^{-\frac{m}{p-m}} |g| \left| D^{\alpha+1}_{tT} \varphi_2(t) \right|^{\frac{p}{p-m}} dtdx = CT^{-(\alpha+1)\frac{p}{p-m}+1+\frac{N\theta}{2}}. \tag{4.35}$$

Finally, we replace (4.27),(4.31) and (4.35) into (4.23) we get

$$\begin{gathered} \int_0^T \int_{\Omega_T} |u|^p \, \psi(t,x) dtdx \leq \\ C \left(T^{-(\alpha+2)\frac{p}{p-1}+\frac{N\theta}{2}+1} + T^{-(\alpha+\theta)\frac{p}{p-1}+\frac{N\theta}{2}+1} + T^{-(\alpha+1)\frac{p}{p-m}+1+\frac{N\theta}{2}} \right). \end{gathered} \tag{4.36}$$

Now, since θ is arbitrary and it must only be nonnegative, we choose it as

$$\theta = \frac{(p-1)\,(\alpha+1)}{p-m} - \alpha > 0 \quad \text{since } p > m. \tag{4.37}$$

This choice of θ allows us to have

$$-(\alpha+\theta)\frac{p}{p-1} + \frac{N\theta}{2} + 1 = -(\alpha+1)\frac{p}{p-m} + 1 + \frac{N\theta}{2}. \tag{4.38}$$

Then by (4.38) we get from (4.36)

$$\int_0^T \int_{\Omega_T} |u|^p \, \psi(t,x) dtdx \leq CT^{\sigma}, \tag{4.39}$$

where

$$\sigma = \max\left(-(\alpha+2)\frac{p}{p-1}+\frac{N\theta}{2}+1, -(\alpha+1)\frac{p}{p-m}+1+\frac{N\theta}{2}\right).$$

Then we distinguish two principal cases:

Case 1: If $\sigma \leq 0$

This case itself is divided into two subcases as follows:

1. *i.* Subcase of $\sigma < 0$

In this case we have $-(\alpha+2)\frac{p}{p-1}+\frac{N\theta}{2}+1<0$ and $-(\alpha+1)\frac{p}{p-m}+1+\frac{N\theta}{2}<0$, so the condition $-(\alpha+2)\frac{p}{p-1}+\frac{N\theta}{2}+1<0$ implies

$$N < -\frac{2p(\alpha+1)+2}{\left(\frac{(\alpha+1)(p-1)}{m-p}+\alpha\right)(p-1)},$$

and the condition $-(\alpha+1)\frac{p}{p-m}+1+\frac{N\theta}{2}<0$ implies

$$N < \frac{2\left(\alpha p+m\right)}{\alpha\left(m-1\right)+\left(p-1\right)},$$

where we have replaced θ by its value. This means that

$$N < \min\left(\frac{2\left(\alpha p+m\right)}{\alpha\left(m-1\right)+\left(p-1\right)}, -\frac{2p(\alpha+1)+2}{\left(\frac{(\alpha+1)(p-1)}{m-p}+\alpha\right)(p-1)}\right). \tag{4.40}$$

If we come back and replace α by its value $1-\gamma$ in (4.40) we get

$$N < \min\left(\frac{2\left((1-\gamma)p+m\right)}{\left(1-\gamma\right)\left(m-1\right)+\left(p-1\right)}, -\frac{2p(2-\gamma)+2}{\left(\frac{(2-\gamma)(p-1)}{m-p}+(1-\gamma)\right)(p-1)}\right). \tag{4.41}$$

Then if the condition (4.40) (or equivalently 4.41) is satisfied, we pass to the limit as $T\to+\infty$ in the formula (4.39) and we get

$$\lim_{T\to+\infty}\int_0^T\int_{\Omega_T}|u|^p\,\psi(t,x)dtdx=0.$$

Using the dominated convergence theorem of Lebesgue (Theorem 1.1.4 page 3 in [3]), the continuity of u with respect to t and x and the fact that

$$\lim_{T\to+\infty}\psi(t,x)=1, \tag{4.42}$$

we get

$$\int_0^{+\infty}\int_{\mathbb{R}^N}|u|^p\,dtdx=0,$$

which implies that $u \equiv 0$ and this is a contradiction.

1. ii. Subcase of $\sigma = 0$.

Firstly, taking the limit as $T \to \infty$ in (4.39) with the consideration $\sigma = 0$, we see that

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p \, dtdx < +\infty.$$

This means that $u \in L^p\left((0, +\infty); L^p(\mathbb{R}^N)\right)$ and from which we get

$$\lim_{R\to\infty} \int_0^{+\infty} \int_{\Delta_R} |u|^p \, \psi dtdx = 0, \tag{4.43}$$

where

$$\Delta_R := \left\{x \in \mathbb{R}^N : R^\theta < |x|^2 \le 2R^\theta\right\} \text{ and } \theta \text{ is defined by (4.37).}$$

Now fixing arbitrarily R in $]0, T[$ for some $T > 0$ and taking in this time

$$\varphi_1(x) = \phi\left(\frac{|x|^2}{T^{\frac{\theta}{2}} R^{-\frac{\theta}{2}}}\right),$$

where ϕ is the function defined by (4.6). Using Hölder's inequality

$$\int_X uvd\mu \le \left(\int_X u^p d\mu\right)^{\frac{1}{p}} \left(\int_X v^q d\mu\right)^{\frac{1}{q}}; \; u \in L^p(X), \; v \in L^q(X), \, p, q > 1, pq = p+q,$$

instead of the Young's one to estimate the integral I_2 in (4.23) on the set

$$\Omega_{TR^{-1}} := \left\{x \in \mathbb{R}^N : |x|^2 \le 2T^\theta R^{-\theta}\right\} = supp\varphi_1,$$

and noting that $\Delta_{TR^{-1}} \subset \Omega_{TR^{-1}}$ and the support of $\Delta\varphi_1$ is contained in $\Delta_{TR^{-1}}$ where

$$\Delta_{TR^{-1}} := \left\{x \in \mathbb{R}^N : (TR^{-1})^\theta < |x|^2 \le 2(TR^{-1})^\theta\right\},$$

and θ is always given by (4.37), we get

$$\begin{aligned}
&\int_0^T \int_{\Omega_{TR^{-1}}} |u| \, \varphi_1^{r-2} \left[|\Delta\varphi_1|^2 + |\nabla\varphi_1|^2\right] \left|D^\alpha_{t|T}\varphi_2\right| dtdx \le \\
&\left(\int_0^T \int_{\Delta_{TR^{-1}}} |u|^p \, \psi dtdx\right)^{\frac{1}{p}} \times \\
&\left(\int_0^T \int_{\Delta_{TR^{-1}}} \psi^{\frac{q}{p}} \varphi_1^{r-2q} \left(|\Delta\varphi_1|^q + |\nabla\varphi_1|^{2q}\right) \left|D^\alpha_{t|T}\varphi_2\right|^q dtdx\right)^{\frac{1}{q}}.
\end{aligned} \tag{4.44}$$

Recalling the integrals I_1, I_3 in page (648) and $\tilde{I}_2$ such that

$$\tilde{I}_2 := \left(\int_0^T \int_{\Delta_{TR^{-1}}} \psi^{\frac{q}{p}} \varphi_1^{r-2q} \left(|\Delta\varphi_1|^q + |\nabla\varphi_1|^{2q}\right) \left|D^{\alpha}_{t|T}\varphi_2\right|^q dtdx\right)^{\frac{1}{q}}.$$

To estimate them, we use the scaled variables $x = T^{\frac{\theta}{2}} R^{-\frac{\theta}{2}} y$, and $t = T\tau$ on the set $\Omega_{TR^{-1}}$. We get firstly

$$I_1 + I_3 \le C\left(T^{1-(\alpha+2)\frac{p}{p-1}+N\frac{\theta}{2}} + T^{1-(\alpha+1)\frac{p}{p-m}+N\frac{\theta}{2}}\right) R^{-N\theta}, \tag{4.45}$$

and using the hypothesis $\sigma = 0$ we obtain from (4.45)

$$I_1 + I_3 \le CR^{-N\theta/2}. \tag{4.46}$$

Computing the integral $\tilde{I}_2$ using the same scaled variables and the same form of the function φ_1 and using (4.46) we get from (4.23)

$$\begin{aligned} \int_0^T \int_{\Omega_{TR^{-1}}} |u|^p \,\psi dtdx &\le CR^{-N\theta/2} \\ &\quad + CR^{\theta-\frac{N\theta}{2q}} \left(\int_0^T \int_{\Delta_{TR^{-1}}} |u|^p \,\psi dtdx\right)^{\frac{1}{p}}. \end{aligned} \tag{4.47}$$

Now taking the limit as $T \to +\infty$ in (4.47), using (4.43) and (4.42) we get

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p \, dtdx \le CR^{-N\theta/2},$$

which means that necessarily $R \to +\infty$. This contradicts our hypothesis. Noting that condition $\sigma = 0$ is equivalent to

$$\begin{aligned} N &= \frac{2\left((1-\gamma)p + m\right)}{(1-\gamma)(m-1) + (p-1)} \quad \text{or} \\ N &= -\frac{2p(2-\gamma)+2}{\left(\frac{(2-\gamma)(p-1)}{m-p} + (1-\gamma)\right)(p-1)}. \end{aligned} \tag{4.48}$$

Then by (4.41) and (4.48) we have

$$N \le \min\left(\frac{2\left((1-\gamma)p+m\right)}{(1-\gamma)(m-1)+(p-1)}, -\frac{2p(2-\gamma)+2}{\left(\frac{(2-\gamma)(p-1)}{m-p} + (1-\gamma)\right)(p-1)}\right).$$

The second main case is

Case 2: if $p \le \frac{1}{\gamma}$.

Even this case is divided into two subcases as follows:

2. i. Subcase of $p < \frac{1}{\gamma}$.

In this case we recall (4.23) and we take $\varphi_1(x) = \phi\left(\frac{|x|^2}{R^\theta}\right)$ where ϕ is the function defined by (4.6) and R is a fixed positive number. Trying to estimate the integrals I_1, I_2 and I_3 (page(648)) with respect to x on the set

$$\Sigma_R = \left\{x \in \mathbb{R}^N : |x| \leq 2R^{\theta/2}\right\} = supp\varphi_1.$$

Using the scaled variables $x = R^{\frac{\theta}{2}}y, \quad t = T\tau$ for the first integral we find

$$\begin{aligned}
&\int_0^T \int_{\Sigma_R} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} \left|D_{t|T}^{\alpha+2}\varphi_2\right|^{\frac{p}{p-1}} dtdx = \\
&\left(\int_{\Sigma_R} \varphi_1^r dx\right)\left(\int_0^T \varphi_2^{-\frac{1}{p-1}} \left|D_{t|T}^{\alpha+2}\varphi_2\right|^{\frac{p}{p-1}} dt\right) \\
&= \left(R^{N\theta/2}\int_0^1 \phi^r(y^2)dy\right) \times \\
&\left(T^{1-(\alpha+2)\frac{p}{p-1}}\int_0^T (1-\tau)^{-\frac{\beta}{p-1}+(\beta-\alpha-2)\frac{p}{p-1}} d\tau\right) \\
&= CR^{\frac{N\theta}{2}}T^{1-(\alpha+2)\frac{p}{p-1}}.
\end{aligned} \tag{4.49}$$

In the same way, we have

$$\begin{aligned}
&\int_0^T \int_{\Sigma_R} \mu(\varphi_1)\varphi_1^{r-2q}\varphi_2^{-\frac{1}{p-1}} \left|D_{t|T}^{\alpha}\varphi_2\right|^q dtdx = \\
&\left(\int_{\Sigma_T} \mu(\varphi_1)\varphi_1^{r-2q}dx\right)\left(\int_0^T \varphi_2^{-\frac{1}{p-1}} \left|D_{t|T}^{\alpha}\varphi_2\right|^q dt\right) \\
&= CR^{\frac{N\theta}{2}-\theta\frac{p}{p-1}}T^{1-\alpha\frac{p}{p-1}}.
\end{aligned} \tag{4.50}$$

We deduce by using the mean value theorem that

$$\begin{aligned}
&\int_0^T \int_{\Sigma_R} \varphi_1^r |\varphi_2|^{-\frac{m}{p-m}} |g| \left|D_{t|T}^{\alpha+1}\varphi_2(t)\right|^{\frac{p}{p-m}} dtdx = \\
&\left(\int_{\Sigma_T} |g|\,\varphi_1^r dx\right)\left(\int_0^T |\varphi_2|^{-\frac{m}{p-m}} \left|D_{t|T}^{\alpha+1}\varphi_2(t)\right|^{\frac{p}{p-m}} dt\right) \\
&= CR^{\frac{N\theta}{2}}T^{1-(\alpha+1)\frac{p}{p-m}}.
\end{aligned} \tag{4.51}$$

Using the formulas (4.49), (4.50) and (4.51) we arrive at

$$\begin{aligned}
\int_0^T \int_{\Sigma_R} |u|^p \psi(t,x)dtdx = {} & CR^{\frac{N\theta}{2}}\left(T^{1-(\alpha+2)\frac{p}{p-1}} + T^{1-(\alpha+1)\frac{p}{p-m}}\right) \\
& + CR^{\left(\frac{N}{2}-\frac{p}{p-1}\right)\theta}T^{1-\alpha\frac{p}{p-1}}.
\end{aligned} \tag{4.52}$$

Firstly, passing to the limit in (4.52) as $T \to +\infty$ and using the fact that $\lim_{T\to+\infty} \psi(t,x) = 1$, we get

$$\int_0^{+\infty} \int_{\Sigma_R} |u|^p \, dtdx = 0, \tag{4.53}$$

Secondly, taking the limit in (4.53) as $R \to +\infty$ we obtain

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p \, dtdx = 0, \quad \text{whereupon} \quad u \equiv 0.$$

which is a contradiction.

2. ii. Subcase of $p = \frac{1}{\gamma}$

In this case we further assume that

$$\frac{N}{2} - \frac{p}{p-1} < 0, \tag{4.54}$$

which is equivalent to $\alpha < \frac{2}{N}$ or $\frac{N-2}{N} < \gamma < 1$ since $\alpha = 1 - \gamma$. Under this assumption, we have

$$\begin{gathered} 1 - (\alpha+2)\,\frac{p}{p-1} = -\frac{2}{\alpha} < 0; \quad 1 - \alpha\frac{p}{p-1} = 0; \\ 1 - (\alpha+1)\,\frac{p}{p-m} = \frac{m\alpha - m - \alpha}{m\alpha - m + 1} < 0. \end{gathered} \tag{4.55}$$

Hence, passing to the limit as $T \to \infty$ in (4.52) and the fact that (4.55) and (4.42) are fulfilled, we obtain

$$\int_0^{\infty} \int_{\Sigma_R} |u|^p \, dtdx = CR^{\left(\frac{N}{2} - \frac{p}{p-1}\right)\theta}. \tag{4.56}$$

Finally, passing to the limit as $R \to \infty$ in (4.56), using the condition (4.54) and the fact that $\theta > 0$, we get

$$\int_0^{\infty} \int_{\mathbb{R}^N} |u|^p \, dtdx = 0, \quad \text{whereupon} \quad u \equiv 0.$$

This is exactly the desired contradiction. The proof of *Theorem 4.1* is achieved.

5. Acknowledgements

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