ON THE SPACES OF λ^m-BOUNDED AND λ^m-ABSOLUTELY p-SUMMABLE SEQUENCES

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Abstract. In this paper, we give the notion of λ^m-boundedness and p-absolute convergence of type λ^m and using these notions we define new sequence spaces. We examine some topological and geometric properties of these spaces. We also establish some inclusion relations concerning these spaces and characterize some matrix classes.

Keywords: BK-Spaces, triangle matrix, α-, β-, γ-duals, Schauder basis, Matrix transformations, geometric properties

1. Introduction

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called a sequence space. A sequence space X is called an FK-space if it is a complete linear metric space with continuous coordinates $\tau_k : X \to \mathbb{C}$ defined by $\tau_k(x) = x_k$ for all $x = (x_n) \in X$ and every $k \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, ...\}$. An FK-space whose topology is normable is called a BK-space. [5]

The space ℓ_∞ is BK-space with $\|x\|_\infty = \sup_k |x_k|$. Also by ℓ_p $(0 < p < \infty)$, we denote the set of p-absolutely convergent series. ℓ_p is a complete p-normed space and BK-space in the cases of $0 < p < 1$ and $1 \leq p < \infty$ with the usual p-norm and ℓ_p-norm defined by

$$\|x\|_{\ell_p} = \sum_{k=0}^{\infty} |x_k|^p \, ; \, 0 < p < 1$$

and

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} \, ; \, 1 \leq p < \infty,$$

respectively [24].

A sequence (b_k) in a normed space X is called a Schauder basis for X if for every $x \in X$ there is a unique sequence (α_k) of scalars such that

$$\lim_{n \to \infty} \left\| x - \sum_{k=0}^{n} \alpha_k b_k \right\| = 0$$

then (b_k) is called Schauder basis (or briefly basis) for X.

Let X, Y be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then A defines a matrix mapping from X into Y and it is denoted by writing $A : X \to Y$ if for every sequence $x = (x_k) \in X$, the sequence $Ax = ((Ax)_n)$, the $A-$transform of x, is in Y, where

$$(1.1) \qquad (Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$$

for all $n \in \mathbb{N}$.

$(X : Y)$ denotes the class of all matrices A such that $A : X \to Y$. Thus $A \in (X : Y)$ if and only if the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and $Ax \in Y$ for all $x \in X$.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$(1.2) \qquad X_A = \{x \in w : Ax \in X\}$$

which is a sequence space.

For the sequence spaces X, Y we define the set $S(X, Y)$ by

$$(1.3) \qquad S(X, Y) = \{z = (z_k) \in w : xz = (x_k z_k) \in Y \text{ for all } x = (x_k) \in X\}.$$

By the notation of (1.3), the $\alpha-, \beta-$ and $\gamma-$ duals of a sequence space X, which are respectively denoted by X^α, X^β and X^γ are defined by

$$X^\alpha = S(X, \ell_1), \ X^\beta = S(X, cs) \text{ and } X^\gamma = S(X, bs).$$

In this paper, the collection of all nonempty and finite subsets of $\mathbb{N}$ is denoted by $\mathcal{F}$.

Definition 1.1. ([26]) A normed X space is called uniformly convex if for any $\varepsilon \in (0, 2]$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in X$ with $\|x\| = 1, \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, then $\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta$.

Definition 1.2. ([9]) A Banach space is super-reflexive if it is isomorphic to a Banach space that is uniformly convex.

The idea of constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by many authors in

many research papers. For instance, the idea of the difference sequence spaces was introduced by Kızmaz [12] as follows,

$$X\left(\Delta\right) = \left\{x = \left(x_k\right) \in w : \left(x_k - x_{k+1}\right) \in X\right\}$$

where $X = \left\{\ell_\infty, c, c_0\right\}$. The space bv_p, containing sequences $\left(x_k\right)$ such that $\left(x_k - x_{k-1}\right)$ is in the sequence space ℓ_p, was introduced in the case $0 < p < 1$ by Altay and Başar [6].

Aydın and Başar introduced the difference sequence spaces $a_0^r\left(\Delta\right)$ and $a_c^r\left(\Delta\right)$ in [8]. Polat and Altay introduced $e_\infty^r\left(\Delta\right)$, $e_c^r\left(\Delta\right)$, $e_0^r\left(\Delta\right)$ sequence spaces in [13]. Malkowsky and Parashar using difference operator of order m studied

$$X\left(\Delta^{(m)}\right) = \left\{x = \left(x_k\right) \in w : \Delta^{(m)}x \in X\right\}$$

sequence spaces where $X = \left\{c, c_0, \ell_\infty\right\}$ in [10]. Polat and Başar defined $e_\infty^r\left(\Delta^{(m)}\right)$, $e_c^r\left(\Delta^{(m)}\right)$, $e_0^r\left(\Delta^{(m)}\right)$ sequence spaces in [14].

Mursaleen and Noman [20] examined the notion of λ-convergence and they defined spaces of λ-bounded λ-convergent and λ-null sequences, respectively, that is,

$$
\begin{aligned}
\ell_\infty^\lambda &= \left\{x \in w : \sup_n \left|\Lambda_n\left(x\right)\right| < \infty\right\} \\
c^\lambda &= \left\{x \in w : \lim_{n\to\infty} \Lambda_n\left(x\right) \text{ exists}\right\} \\
c_0^\lambda &= \left\{x \in w : \lim_{n\to\infty} \Lambda_n\left(x\right) = 0\right\}
\end{aligned}
$$

where $\Lambda_n\left(x\right) = \frac{1}{\lambda_n} \sum_{k=0}^{n} \left(\lambda_k - \lambda_{k-1}\right) x_k$ for $n \in \mathbb{N}$. They also introduced the sequence space ℓ_p^λ $\left(0 < p < \infty\right)$ in [18] as follow;

$$\ell_p^\lambda = \left\{x \in w : \sum_{n=0}^{\infty} \left|\Lambda_n\left(x\right)\right|^p < \infty\right\}, \quad \left(0 < p < \infty\right).$$

Ganie and Sheikh defined sequence spaces $c\left(\Delta_u^\lambda\right)$ and $c_0\left(\Delta_u^\lambda\right)$ in [2]. Candan introduced $c_0^\lambda(\tilde{B})$ and $c^\lambda(\tilde{B})$ sequence spaces and studied some properties of these spaces in [15]. Duyar et. al. introduced $c_0^\lambda(\hat{B})$, $c^\lambda(\hat{B})$, $\ell_\infty^\lambda(\hat{B})$, $\ell_p^\lambda(\hat{B})$ sequence spaces where $1 \leq p < \infty$ in [25]. Bişgin and Sönmez defined $c_0^\lambda\left(G^m\right)$ and $c^\lambda\left(G^m\right)$ sequence spaces in [17]. Braha and Başar defined the $A_\lambda\left(\ell_\infty\right)$, $A_\lambda\left(c\right)$ and $A_\lambda\left(c_0\right)$ sequence spaces as follows;

$$
\begin{aligned}
A_\lambda(\ell_\infty) &= \left\{x \in w : \sup_n \left|\left(A_\lambda x\right)_n\right| < \infty\right\}, \\
A_\lambda(c) &= \left\{x \in w : \exists l \in \mathbb{C} \ni \lim_n \left(A_\lambda x\right)_n = l\right\}, \\
A_\lambda(c_0) &= \left\{x \in w : \lim_n \left(A_\lambda x\right)_n = 0\right\}
\end{aligned}
$$

where $(A_\lambda x)_n = \frac{1}{\Delta \lambda_n} \sum\limits_{k=0}^{n} \left(\Delta^2 \lambda_k \right) x_k$ for $n \in \mathbb{N}$ in [22].

Some other authors have constructed new sequence spaces by using the matrix domain of an infinite matrix and have studied some topological properties. (see [7], [16], [19])

2. The notion of λ^m-boundedness and p-absolute convergence of type λ^m

In this section we give the definition of λ^m-boundedness and p-absolute convergence of type λ^m. Let us consider the strictly increasing sequence $\lambda = (\lambda_k)$ of positive reals tending to infinity, that is

(2.1) $$\lim_{k \to \infty} \lambda_k = \infty \text{ and } \lambda_{k+1} \geq m\lambda_k$$

for $k, m \in \mathbb{N}$ and $m \geq 2$. Δ^m denotes the difference operator of order m, that is,

$$\Delta^m \lambda_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} \lambda_{k-v}$$

for all $k \in \mathbb{N}$. It follows from (2.1) that $\Delta^m \lambda_k \geq 0$. We assume that any term with a negative subscript is equal to naught. e.g. $\lambda_{-1} = \lambda_{-2} = ... = \lambda_{-m} = 0$.

We call $x = (x_k) \in w$ is λ^m-bounded if $\sup\limits_{n} |\Lambda_n^m(x)| < \infty$, where

(2.2) $$\Lambda_n^m(x) = \frac{1}{\Delta^{m-1}\lambda_n} \sum_{k=0}^{n} (\Delta^m \lambda_k) x_k$$

for $n \in \mathbb{N}$. Further, we say that $\sum\limits_{k} x_k$ is p-absolutely convergent of type λ^m if $\sum\limits_{n} |\Lambda_n^m(x)|^p < \infty$, where $0 < p < \infty$.

Now, let $x \in \ell_\infty$ and we have a constant $K > 0$ such that $|x_k| \leq K$ for all $k \in \mathbb{N}$. Then, we derive that

$$\begin{aligned} |\Lambda_n^m(x)| &\leq \frac{1}{\Delta^{m-1}\lambda_n} \sum_{k=0}^{n} (\Delta^m \lambda_k) |x_k| \\ &\leq \frac{K}{\Delta^{m-1}\lambda_n} \sum_{k=0}^{n} (\Delta^m \lambda_k) \\ &= K. \end{aligned}$$

This means x is λ^m-bounded. Hence we can give the following result.

Lemma 2.1 Every bounded sequence is λ^m-bounded.

Lemma 2.2 If

$$(2.3) \qquad \frac{\Delta^{m-2}\lambda_n}{\Delta^{m-1}\lambda_n} \in \ell_\infty$$

for $n \in \mathbb{N}$ and $m \geq 2$ then every λ^{m-1}-bounded sequence is λ^m-bounded.

Proof. Let take $x \in w$ is a λ^{m-1}-bounded sequence. If (2.3) holds for all $n \in \mathbb{N}$ and $m \geq 2$, then we have that

$$|\Lambda_n^m(x)| \leq \frac{\Delta^{m-2}\lambda_n}{\Delta^{m-1}\lambda_n}\left|\Lambda_n^{m-1}(x)\right| + \frac{\Delta^{m-2}\lambda_{n-1}}{\Delta^{m-1}\lambda_n}\left|\Lambda_{n-1}^{m-1}(x)\right|.$$

This means that λ^{m-1}-bounded sequence is also λ^m-bounded. As a result of this theorem we have the following property.

Remark 2.3. If (2.3) holds, every λ^r-bounded sequence is λ^m-bounded while $0 < r < m$ for $m, n \in \mathbb{N}$.

3.   The spaces $\ell_p(\lambda^m)$ $(0 < p < \infty)$ and $\ell_\infty(\lambda^m)$ of non-absolute type

In this section, we introduce the sequence space $\ell_p(\lambda^m)$ and $\ell_\infty(\lambda^m)$, where $0 < p < \infty$. We give some topological properties. $\ell_p(\lambda^m)$ and $\ell_\infty(\lambda^m)$ sequence spaces as the of all sequences such that their Λ^m-transform is in the space ℓ_p and ℓ_∞, where $0 < p < \infty$, i.e.,

$$\ell_p(\lambda^m) = \{x = (x_k) \in w : \sum_{n=0}^{\infty} |\Lambda_n^m(x)|^p < \infty\}$$

and

$$\ell_\infty(\lambda^m) = \{x = (x_k) \in w : \sup_n |\Lambda_n^m(x)| < \infty\}$$

where $\Lambda_n^m(x) = \frac{1}{\Delta^{m-1}\lambda_n}\sum_{k=0}^{n}(\Delta^m\lambda_k)x_k$. We can rewrite these spaces with the notation (1.2)

$$(3.1) \qquad \ell_p(\lambda^m) = (\ell_p)_{\Lambda^m} \text{ and } \ell_\infty(\lambda^m) = (\ell_\infty)_{\Lambda^m}.$$

It is obvious by (3.1) that $\ell_p(\lambda^m)$ is a sequence space consisting of all sequences which are p-absolutely convergent of type λ^m, where $0 < p < \infty$.

Now we give the following theorem with the essential in the text.

Theorem 3.1 The following statements hold: (i) If $0 < p < 1$, then $\ell_p(\lambda^m)$ is a complete p-normed space with the p-norm

$$(3.2) \qquad \|x\|_{\ell_p(\lambda^m)} = \|\Lambda^m(x)\|_{\ell_p} = \sum_n |\Lambda_n^m(x)|^p; \quad (0 < p < 1).$$

(ii) If $1 \leq p \leq \infty$, then $\ell_p(\lambda^m)$ is a BK-space with the norm

$$(3.3) \qquad \|x\|_{\ell_p(\lambda^m)} = \left(\sum_n |\Lambda_n^m(x)|^p\right)^{\frac{1}{p}}; \quad (1 \leq p < \infty)$$

and

$$(3.4) \qquad \|x\|_{\ell_\infty(\lambda^m)} = \sup_n |\Lambda_n^m(x)|.$$

Proof. The proof is trivial from (3.1) because of Theorem 4.3.12 in [3].

It can be easily checked that the absolute property does not hold for the space $\ell_p(\lambda^m)$, that is, $\||x|\|_{\ell_p(\lambda^m)} \neq \|x\|_{\ell_p(\lambda^m)}$ for at least one sequence x in the space $\ell_p(\lambda^m)$, where $0 < p \leq \infty$.

Theorem 3.2 $\ell_p(\lambda^m) \cong \ell_p$ is isometrically isomorphic to the space ℓ_p, that is $\ell_p(\lambda^m) \cong \ell_p$ for $0 < p \leq \infty$.

Proof. Let define the transformation as $T : \ell_p(\lambda^m) \to \ell_p$, $Tx = \Lambda^m(x) \in \ell_p$ for $x \in \ell_p(\lambda^m)$. The linearity of T is trivial. Also, $x = 0$ whenever $Tx = 0$ hence T is injective. Now let $y = (y_k) \in \ell_p$ and define $x = (x_k)$ by

$$(3.5) \qquad x_k = \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\Delta^{m-1}\lambda_j}{\Delta^m \lambda_k} y_j$$

for $k \in \mathbb{N}$. From (2.2), we have

$$\begin{aligned}
\Lambda_n^m(x) &= \frac{1}{\Delta^{m-1}\lambda_n} \sum_{k=0}^{n} (\Delta^m\lambda_k) x_k \\
&= \frac{1}{\Delta^{m-1}\lambda_n} \sum_{k=0}^{n} (\Delta^m\lambda_k) \left[\frac{\Delta^{m-1}\lambda_k}{\Delta^m\lambda_k} y_k - \frac{\Delta^{m-1}\lambda_{k-1}}{\Delta^m\lambda_k} y_{k-1}\right] \\
&= \frac{1}{\Delta\lambda_n} \sum_{k=0}^{n} \left[(\Delta^{m-1}\lambda_k) y_k - (\Delta^{m-1}\lambda_{k-1}) y_{k-1}\right] \\
&= y_n
\end{aligned}$$

for $n \in \mathbb{N}$. $\Lambda^m(x) = y$ and since $y \in \ell_p$, we have $\Lambda^m(x) \in \ell_p$. Hence $x \in \ell_p(\lambda^m)$, $Tx = y$ and T is surjective. Also for every $x \in \ell_p(\lambda^m)$, by (3.2) and (3.3) we have

$$(3.6) \qquad \|Tx\|_{\ell_p} = \|\Lambda^m(x)\|_{\ell_p} = \|x\|_{\ell_p(\lambda^m)}.$$

This means T is p-norm and norm preserving in the cases of $0 < p < 1$ and $1 \le p \le \infty$, respectively. Consequently, T is isometry.

Theorem 3.3 With the exception of the case $p = 2$, the space $\ell_p(\lambda^m)$ is not an inner product space, hence it is not a Hilbert space for $1 \le p < \infty$.

Proof. We know $\ell_2(\lambda^m)$ is a BK-space with the norm $\|x\|_{\ell_2(\lambda^m)} = \|\Lambda^m(x)\|_{\ell_2}$. This norm can be obtained from an inner product,

$$\|x\|_{\ell_2(\lambda^m)} = \langle x, x \rangle^{\frac{1}{2}} = \langle \Lambda^m(x), \Lambda^m(x) \rangle_2^{\frac{1}{2}}$$

holds for $x \in \ell_2(\lambda^m)$. Hence $\ell_2(\lambda^m)$ is a Hilbert space. $\langle ., . \rangle_2$ denotes the inner product on ℓ_2.

Now, let us define the sequences $u = (u_k)$ and $v = (v_k)$ as follows:

$$u = \left(1, 1, -\frac{\Delta^{m-1}\lambda_1}{\Delta^m \lambda_2}, 0, 0, ...\right)$$

$$v = \left(1, -\frac{\Delta^{m-1}(\lambda_1 + \lambda_0)}{\Delta^m \lambda_1}, \frac{\Delta^{m-1}\lambda_1}{\Delta^m \lambda_2}, 0, 0, ...\right).$$

We have

$$\Lambda^m(u) = (1, 1, 0, 0, ...)$$

and

$$\Lambda^m(v) = (1, -1, 0, 0, ...).$$

We also have

$$\|u + v\|_{\ell_p(\lambda^m)}^2 + \|u - v\|_{\ell_p(\lambda^m)}^2 = 8 \ne 4(2^{2/p}) = 2\left(\|u\|_{\ell_p(\lambda^m)}^2 + \|v\|_{\ell_p(\lambda^m)}^2\right); \;\; (p \ne 2).$$

Hence the norm of the space $\ell_p(\lambda^m)$ does not provide the parallelogram equality. This norm cannot be obtained from an inner product. Consequently, $\ell_p(\lambda^m)$ with $p \ne 2$ is a Banach space but it is not a Hilbert space where $1 \le p < \infty$.

We have that the domain of X_A of an infinite triangle matrix $A = (a_{nk})$ in a sequence space X has a basis if and only if X has a basis from Theorem 2.3 in [4]. Hence we have the following result:

Corollary 3.4 Let $0 < p < \infty$ and define the sequence $e_{\lambda^m}^{(k)} \in \ell_p(\lambda^m)$ for fixed $k, n \in \mathbb{N}$ by

$$\left(e_{\lambda^m}^{(k)}\right) = \begin{cases} (-1)^{n-k}\frac{\Delta^{m-1}\lambda_k}{\Delta^m \lambda_n}; & k \le n \le k+1, \\ 0; & otherwise. \end{cases}$$

$\left(e_{\lambda^m}^{(k)}\right)$ is a basis for the space $\ell_p(\lambda^m)$ and $x \in \ell_p(\lambda^m)$ has a unique representation of the form

$$x = \sum_k \Lambda_k^m(x) e_{\lambda^m}^{(k)}.$$

4. Some inclusion relations

Theorem 4.1 If $0 < p < q < \infty$, then the inclusion $\ell_p(\lambda^m) \subset \ell_q(\lambda^m)$ strictly holds.

Proof. Let $0 < p < q < \infty$. The inclusion $\ell_p(\lambda^m) \subset \ell_q(\lambda^m)$ derived by $\ell_p \subset \ell_q$. To prove that inclusion is strict we must find a sequence which is in $\ell_q(\lambda^m)$ but not in $\ell_p(\lambda^m)$. Consider the sequence $y = (y_k)$ in terms of the sequence x for $k \in \mathbb{N}$ as follow:

$$y_k = \frac{(\Delta^{m-1}\lambda_k)x_k - (\Delta^{m-1}\lambda_{k-1})x_{k-1}}{\Delta^m \lambda_k}.$$

We have that

$$
\begin{aligned}
\Lambda_n^m(y) &= \frac{1}{\Delta^{m-1}\lambda_n} \sum_{k,=0}^{n} \left[(\Delta^{m-1}\lambda_k)x_k - (\Delta^{m-1}\lambda_{k-1})x_{k-1} \right] \\
&= x_n
\end{aligned}
$$

for $n \in \mathbb{N}$. This shows $\Lambda^m(y) = x$ and $\Lambda^m(y) \in \ell_q \backslash \ell_p$. Hence, the sequence $y \in \ell_q(\lambda^m)$ and $y \notin \ell_p(\lambda^m)$. The proof is completed.

Now, let $x \in w$ and $n \geq 1$. Then from (2.2) we have

$$
\begin{aligned}
x_n - \Lambda_n^m(x) &= \frac{1}{\Delta^{m-1}\lambda_n} \sum_{i=0}^{n} \Delta^m \lambda_i (x_n - x_i) \\
&= \frac{1}{\Delta^{m-1}\lambda_n} \sum_{i=0}^{n-1} \Delta^m \lambda_i (x_n - x_i) \\
&= \frac{1}{\Delta^{m-1}\lambda_n} \sum_{i=0}^{n-1} (\Delta^m \lambda_i) \sum_{k=i+1}^{n} (x_k - x_{k-1}) \\
&= \frac{1}{\Delta^{m-1}\lambda_n} \sum_{k=1}^{n} (x_k - x_{k-1}) \sum_{i=0}^{k-1} (\Delta^m \lambda_i) \\
&= \frac{1}{\Delta^{m-1}\lambda_n} \sum_{k=1}^{n} \Delta^{m-1}\lambda_{k-1} (x_k - x_{k-1}).
\end{aligned}
$$

Hence we have that

(4.1) $$x_n - \Lambda_n^m(x) = S_n(x)$$

for $n \in \mathbb{N}$. Here the sequence $S(x) = (S_n(x))$ is defined by

(4.2) $\quad S_0(x) = 0 \quad$ and $\quad S_n(x) = \dfrac{1}{\Delta^{m-1}\lambda_n} \sum_{k=1}^{n} \left(\Delta^{m-1}\lambda_{k-1} \right) (x_k - x_{k-1}) ; (n \geq 1).$

We have that

$$S_n(x) = \frac{\Delta^{m-1}\lambda_{n-1}}{\Delta^m \lambda_n} \left[\Lambda_n^m(x) - \Lambda_{n-1}^m(x) \right]$$

for $n \in \mathbb{N}$.

Theorem 4.2 The inclusion $\ell_p(\lambda^m) \subset \ell_p$ holds if and only if $S(x) \in \ell_p$ for $x \in \ell_p(\lambda^m)$, where $0 < p \leq \infty$.

Proof. Firstly, we suppose $\ell_p(\lambda^m) \subset \ell_p$ holds, where $0 < p \leq \infty$. Let take any $x = (x_k) \in \ell_p(\lambda^m)$. By the hypothesis $x \in \ell_p$. From the equality given by (4.1) we have

$$\|S(x)\|_{\ell_p} \leq \|x\|_{\ell_p} + \|\Lambda^m(x)\|_{\ell_p} = \|x\|_{\ell_p} + \|x\|_{\ell_p(\lambda^m)} < \infty.$$

This shows that $S(x) \in \ell_p$.

Conversely, let $x \in \ell_p(\lambda^m)$ where $0 < p < \infty$. By the hypothesis that $S(x) \in \ell_p$. By (4.1) we have that

$$\|x\|_{\ell_p} \leq \|S(x)\|_{\ell_p} + \|\Lambda^m(x)\|_{\ell_p} = \|S(x)\|_{\ell_p} + \|x\|_{\ell_p(\lambda^m)} < \infty.$$

Hence $x \in \ell_p$ and the inclusion $\ell_p(\lambda^m) \subset \ell_p$ holds.

Theorem 4.3

(i) $\ell_p \cap \ell_p(\lambda^m) \neq \varnothing$.

(ii) If the inclusion $\ell_p \subset \ell_p(\lambda^m)$ holds, then $\frac{1}{\Delta^{m-1}\lambda} \in \ell_p$ for $0 < p < \infty$.

Proof.

(i) We can give an example. $(\lambda_1 - m\lambda_0, -\lambda_0, 0, ...) \in \ell_p \cap \ell_p(\lambda^m)$ for $0 < p < \infty$.

(ii) We assume that $\ell_p \subset \ell_p(\lambda^m)$ holds where $0 < p < \infty$. Let $x = e^{(0)} = (1, 0, 0, ...) \in \ell_p$. Then $x \in \ell_p(\lambda^m)$ and $\Lambda^m(x) \in \ell_p$. Hence, we derive that

$$(\Delta^m \lambda_0)^p \sum_n \left(\frac{1}{\Delta^{m-1}\lambda_n}\right)^p = \sum_n |\Lambda_n^m(x)|^p < \infty.$$

This means $\frac{1}{\Delta^{m-1}\lambda} \in \ell_p$.

Theorem 4.4 If $\frac{\Delta^{m-2}\lambda_n}{\Delta^{m-1}\lambda_n} \in \ell_p$ for $n \in \mathbb{N}$ and $m \geq 2$, the inclusion $\ell_p(\lambda^{m-1}) \subset \ell_p(\lambda^m)$ holds, where $0 < p < \infty$.

Proof. It can be seen by using Minkowski inequality, so we omitted it.

The following remark can be obtained from Theorem 4.4 and Lemma 2.2.

Remark 4.5. If $\frac{\Delta^{m-2}\lambda_n}{\Delta^{m-1}\lambda_n} \in \ell_p$, then $\ell_p(\lambda^r) \subset \ell_p(\lambda^m)$ holds while $0 < r < m$ for $m, n \in \mathbb{N}$ where $0 < p \leq \infty$.

5. $\alpha-$, $\beta-$ and $\gamma-$dual of $\ell_p(\lambda^m)$

In this section, we give the theorems determining the α-, β- and γ-duals of the space $\ell_p(\lambda^m)$ where $0 < p < \infty$. We begin the lemmas given in [21], which are needed in the proof of the theorems 5.4-5.6.

Lemma 5.1.

(*i*) $A \in (\ell_p : \ell_1)$ if and only if

$$\sup_{F \in \mathcal{F}} \sum_n \left| \sum_{k \in F} a_{nk} \right|^q < \infty$$

where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(*ii*) $A \in (\ell_1 : \ell_1)$ if and only if

$$\sup_k \sum_n |a_{nk}| < \infty.$$

Lemma 5.2.

(*i*) $A \in (\ell_p : c)$ if and only if

(5.1) $$\lim_{n \to \infty} a_{nk} \text{ exists}$$

and

(5.2) $$\sup_n \sum_k |a_{nk}|^q < \infty$$

for all $k \in \mathbb{N}$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(*ii*) $A \in (\ell_1 : c)$ if and only if (5.1) and

(5.3) $$\sup_{n,k} |a_{nk}| < \infty.$$

Lemma 5.3.

(*i*) $A \in (\ell_p : \ell_\infty)$ if and only if (5.2), where $1 < p < \infty$.

(*ii*) $A \in (\ell_p : \ell_\infty)$ if and only if (5.3), where $p = 1$.

Theorem 5.4. Define the sets $B(q)$ and C

$$B(q) = \left\{ a = (a_n) \in w : \sum_k \left| \frac{\Delta^{m-1} \lambda_k}{\Delta^m \lambda_k} a_k \right|^q < \infty \right\}$$

$$C = \left\{ a = (a_n) \in w : \sup_k \left| \frac{\Delta^{m-1} \lambda_k}{\Delta^m \lambda_k} a_k \right| < \infty \right\}.$$

Then $\{\ell_p (\lambda^m)\}^\alpha = B(q)$ and $\{\ell_1 (\lambda^m)\}^\alpha = C$.

Proof. Let $a = (a_n) \in w$ and $1 < p < \infty$. We define the matrix $B = (b_{nk})$ by

$$b_{nk} = \begin{cases} (-1)^{n-k} \frac{\Delta^{m-1}\lambda_k}{\Delta^m \lambda_n} a_n; & n-1 \leq k \leq n \\ 0; & k < n-1 \text{ or } k > n \end{cases}$$

and from (3.5) we have

$$(5.4) \qquad a_n x_n = \sum_{k=n-1}^{n} (-1)^{n-k} \frac{\Delta^{m-1}\lambda_k}{\Delta^m \lambda_n} a_n y_k = (By)_n .$$

Thus, we observe by (5.4) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in \ell_p(\lambda^m)$ if and only if $By \in \ell_1$ whenever $y = (y_k) \in \ell_p$. This means that $a = (a_k) \in \{\ell_p(\lambda^m)\}^\alpha$ if and only if $B \in (\ell_p : \ell_1)$. We obtain from Lemma 5.1 $a \in \{\ell_p(\lambda^m)\}^\alpha$ if and only if

$$(5.5) \qquad \sup_{F \in \mathcal{F}} \sum_k \left| \sum_{n \in F} b_{nk} \right|^q < \infty.$$

On the other hand, for $F \in \mathcal{F}$ we have that

$$\sum_{n \in F} b_{nk} = \begin{cases} 0; & k \notin F \text{ and } k+1 \notin F \\ \frac{\Delta^{m-1}\lambda_k}{\Delta^m \lambda_k} a_k; & k \in F \text{ and } k+1 \notin F \\ -\frac{\Delta^{m-1}\lambda_k}{\Delta^m \lambda_{k+1}} a_{k+1}; & k \notin F \text{ and } k+1 \in F \\ \left(\frac{a_k}{\Delta^m \lambda_k} - \frac{a_{k+1}}{\Delta^m \lambda_{k+1}} \right)(\Delta^{m-1}\lambda_k); & k \in F \text{ and } k+1 \in F \end{cases} .$$

Hence, we deduce that (5.4) holds if and only if

$$\sum_k \left| \frac{\Delta^{m-1}\lambda_k}{\Delta^m \lambda_k} \right|^q < \infty.$$

This leads us to the consequence that $\{\ell_p(\lambda^m)\}^\alpha = B(q)$, where $1 < p < \infty$.

Similarly, we have from (5.5) that $a = (a_k) \in \{\ell_1(\lambda^m)\}^\alpha$ if and only if $B \in (\ell_1 : \ell_1)$ which can equivalently be written as

$$(5.6) \qquad \sup_k \sum_n |b_{nk}| < \infty.$$

Also we have that

$$\sum_n |b_{nk}| = \sum_{n=k}^{k+1} \left| \frac{\Delta^{m-1}\lambda_k}{\Delta^m \lambda_n} a_n \right|$$

for $k \in \mathbb{N}$. Thus, we conclude that (5.6) holds if and only if

$$\sup_k \left| \frac{\Delta^{m-1}\lambda_k}{\Delta^m \lambda_k} a_k \right| < \infty.$$

This shows that $\{\ell_1(\lambda^m)\}^\alpha = C$ and this completes the proof.

Theorem 5.5. Define the set

$$H(q) = \left\{ a = (a_k) \in w : \sum_k \left| \bar{\Delta}\left(\frac{a_k}{\Delta^m \lambda_k}\right)(\Delta^{m-1}\lambda_k)\right|^q < \infty \right\}$$

where

$$\bar{\Delta}\left(\frac{a_k}{\Delta^m \lambda_k}\right) = \frac{a_k}{\Delta^m \lambda_k} - \frac{a_{k+1}}{\Delta^m \lambda_{k+1}}.$$

Then $\{\ell_p(\lambda^m)\}^\beta = C \cap H(q)$ for $1 < p < \infty$ and $\{\ell_1(\lambda^m)\}^\beta = C$.

Proof. Let us consider the equation

$$
\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} a_k \left(\sum_{j=k-1}^{k} (-1)^{k-j} \frac{\Delta^{m-1}\lambda_j}{\Delta^m \lambda_k} y_j \right) \\
&= \sum_{k=0}^{n-1} (\Delta^{m-1}\lambda_k) \bar{\Delta}\left(\frac{a_k}{\Delta^m \lambda_k}\right) y_k + \frac{\Delta^{m-1}\lambda_n}{\Delta^m \lambda_n} a_n y_n \\
&= (Ty)_n
\end{aligned}
$$

where $n \in \mathbb{N}$ and $T = (t_{nk})$ is the matrix defined by

$$
t_{nk} = \begin{cases}
\bar{\Delta}\left(\frac{a_k}{\Delta^m \lambda_k}\right)(\Delta^{m-1}\lambda_k), & k < n \\
\frac{\Delta^{m-1}\lambda_n}{\Delta^m \lambda_n} a_n, & k = n \\
0, & k > n.
\end{cases}
$$

It can clearly be seen that the columns of the matrix T belong to space c, since

$$\lim_n t_{nk} = \bar{\Delta}\left(\frac{a_k}{\Delta^m \lambda_k}\right)(\Delta^{m-1}\lambda_k)$$

for all $k \in \mathbb{N}$. Hence we deduce from the first equality given above in the proof with Lemma 5.2 that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in \ell_p(\lambda^m)$ if and only if $Ty \in c$ whenever $y = (y_k) \in \ell_p$. This yields that $a = (a_k) \in \{\ell_p(\lambda^m)\}^\beta$ if and only if $T \in (\ell_p : c)$, where $1 \le p < \infty$. Firstly, we prove for $1 < p < \infty$. We have from (5.2) that

$$\sum_k \left| \bar{\Delta}\left(\frac{a_k}{\Delta^m \lambda_k}\right)(\Delta^{m-1}\lambda_k)\right|^q < \infty$$

and

(5.7) $$\sup_n \left| \frac{\Delta^{m-1}\lambda_n}{\Delta^m \lambda_n} a_n \right| < \infty.$$

This leads us to the consequence that $\{\ell_p(\lambda^m)\}^\beta = C \cap H(q)$.

Then for $p = 1$, from (5.3), we have that (5.7) and

$$(5.8) \qquad \sup_k \left| \bar{\Delta}\left(\frac{a_k}{\Delta^m \lambda_k} \right) \Delta^{m-1}\lambda_k \right| < \infty$$

hold. But the condition (5.8) is redundant, since it is clearly seen from (5.7). Hence $\{\ell_1(\lambda^m)\}^\beta = C$.

Theorem 5.6. $\{\ell_1(\lambda^m)\}^\gamma = C$ and $\{\ell_p(\lambda^m)\}^\gamma = C \cap H(q)$, where $1 < p < \infty$.

Proof. This can be proved similarly to the proof of Theorem 5.5 with Lemma 5.3 instead of Lemma 5.2.

6. Some matrix mappings

In this section, we give some results which characterize various matrix transformations between the sequence space $\ell_p(\lambda^m)$ and sequence spaces ℓ_∞, c, c_0, ℓ_p where $1 \le p < \infty$. For any infinite matrix $A = (a_{nk})$, we shall write for brevity reasons that

$$\tilde{a}_{nk} = \left(\Delta^{m-1}\lambda_k \right) \bar{\Delta}\left(\frac{a_{nk}}{\Delta^m \lambda_k} \right) = \left(\Delta^{m-1}\lambda_k \right) \left(\frac{a_{nk}}{\Delta^m \lambda_k} - \frac{a_{n,k+1}}{\Delta^m \lambda_{k+1}} \right)$$

for all $k, n \in \mathbb{N}$. Further, let $x, y \in w$ be connected by the relation $y = \Lambda^m(x)$. Then,

we have by equality given in the proof of Theorem 5.5 that

$$\sum_{k=0}^{l} a_{nk}x_k = \sum_{k=0}^{l-1} \tilde{a}_{nk}y_k + \frac{\Delta^{m-1}\lambda_l}{\Delta^m \lambda_l} a_{nl}y_l$$

for $l, n \in \mathbb{N}$. Now we consider the following conditions:

$$(6.1) \qquad \left(\frac{\Delta^{m-1}\lambda_k}{\Delta^m \lambda_k} a_{nk} \right)_{k=0}^{\infty} \in \ell_\infty,$$

$$(6.2) \qquad \sup_{n,k} |\tilde{a}_{nk}| < \infty,$$

$$(6.3) \qquad \sup_n \sum_k |\tilde{a}_{nk}|^q < \infty,$$

$$(6.4) \qquad \lim_n \tilde{a}_{nk} = \alpha_k \text{ for every } k \in \mathbb{N},$$

$$(6.5) \qquad \lim_n \tilde{a}_{nk} = 0 \text{ for every } k \in \mathbb{N},$$

$$(6.6) \qquad \sup_k \sum_n |\tilde{a}_{nk}|^p < \infty.$$

By Theorem 5.5 with the results by Stieglitz and Tietz [21], we get the following result from the above conditions.

Teorem 6.1.

(i) $A = (a_{nk}) \in (\ell_1(\lambda^m), \ell_\infty)$ if and only if (6.1) and (6.2).

(ii) $A = (a_{nk}) \in (\ell_p(\lambda^m), \ell_\infty)$ if and only if (6.1) and (6.3), where $1 < p < \infty$.

(iii) $A = (a_{nk}) \in (\ell_p(\lambda^m), c)$ if and only if (6.1), (6.3) and (6.4), where $1 < p < \infty$.

(iv) $A = (a_{nk}) \in (\ell_1(\lambda^m), c)$ if and only if (6.1), (6.2) and (6.4).

(v) $A = (a_{nk}) \in (\ell_1(\lambda^m), c_0)$ if and only if (6.1), (6.2) and (6.5).

(vi) $A = (a_{nk}) \in (\ell_p(\lambda^m), c_0)$ if and only if (6.1), (6.3) and (6.5), where $1 < p < \infty$.

(vii) $A = (a_{nk}) \in (\ell_1(\lambda^m), \ell_p)$ if and only if (6.1) and (6.6), where $1 \le p < \infty$.

7.   Some geometric properties

Theorem 7.1. $\ell_1(\lambda^m)$ is not uniformly convex.

Proof. Let take $x, y \in \ell_1(\lambda^m)$ and define such as

$$x = \left(1, \frac{\lambda_0}{\Delta^{m-1}(\lambda_0 - \lambda_1)}, 0, ...\right)$$

and

$$y = \left(0, \frac{\Delta^{m-1}\lambda_1}{\Delta^{m-1}(\lambda_0 - \lambda_1)}, \frac{\Delta^{m-1}\lambda_1}{\Delta^m \lambda_2}, 0, ...\right).$$

Then $\|x\|_{\ell_1(\lambda^m)} = \|y\|_{\ell_1(\lambda^m)} = 1$, $\|x - y\|_{\ell_1(\lambda^m)} = 2 > \varepsilon$ and $\left\|\frac{1}{2}(x + y)\right\|_{\ell_1(\lambda^m)} = 1$. Hence $\ell_1(\lambda^m)$ is not uniformly convex.

Corollary 7.2. $\ell_2(\lambda^m)$ is uniformly convex and reflexive.

Proof. $\ell_2(\lambda^m)$ is a Hilbert space hence we have from Proposition 7.1.1 in [26] that $\ell_2(\lambda^m)$ is uniformly convex and from Theorem 5.5.1 given in [26] it is reflexive.

Corollary 7.3. $\ell_p(\lambda^m)$ is super-reflexive, where $1 < p < \infty$.

Proof. We have from Theorem 3.2 that $\ell_p(\lambda^m)$ is isomorphic to ℓ_p and it is well known that ℓ_p is uniformly convex in [1], where $1 < p < \infty$. Hence by Definition 1.2 we have the desired result.

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