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THREE-DIMENSIONAL ALMOST α-PARA-KENMOTSU MANIFOLDS SATISFYING CERTAIN NULLITY CONDITIONS *

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Abstract. In this paper, we study 3-dimensional almost α-para-Kenmotsu manifolds satisfying special types of nullity conditions depending on smooth functions $\tilde{\kappa}, \tilde{\mu}$ and $\tilde{\nu}$=constant, also we present a local description of the structure of a 3-dimensional almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ with $\tilde{\kappa} + \alpha^2 \neq 0$ such that $d\tilde{\kappa} \wedge \eta = 0$.

Keywords: Almost paracontact metric manifold; almost α-para-Kenmotsu manifold; nullity distribution.

1. Introduction

The aim of this paper is to study the local description of almost α-para-Kenmotsu manifolds. Kenmotsu manifolds have been introduced and studied by K. Kenmotsu in 1972 [10], and the geometry of almost Kenmotsu manifolds have been investigated in many aspects [5]-[7]. Most of the results contained in [5]-[6] can be easily generalized to the class of almost α-Kenmotsu manifolds, where α is a non-zero real number [7]. Many authors have investigated the geometry of contact metric manifolds whose characteristic vector field ξ belongs to the (κ, μ)-nullity distribution, i.e. the curvature tensor field satisfies the condition

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{1.1}$$

for some real numbers κ and μ, where $2h$ denotes the Lie derivative of φ in the direction of ξ. This new class of Riemannian manifolds was introduced in [4] as a natural generalization both of the Sasakian condition $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$ and of those contact metric manifolds satisfying $R(X,Y)\xi = 0$ which were studied by D.E. Blair in [3]. Koufogiorgos and Tsichlias found a new class of 3-dimensional contact metric manifolds that κ and μ are non-constant smooth functions[11]. They generalized (κ, μ)-contact metric manifolds for dimensions greater than three on

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non-Sasakian manifolds, where the functions κ, μ are constant. Nowadays contact metric (κ, μ)-space is considered as a very important topic in contact Riemannian geometry. Following these works, P. Dacko and Z. Olszak studied almost cosymplectic (κ, μ, ν)-spaces in [12], whose almost cosymplectic structures (φ, ξ, η, g) satisfy the condition

$$R(X,Y)\xi = \eta(Y)(\kappa I + \mu h + \nu\varphi h)X - \eta(X)(\kappa I + \mu h + \nu\varphi h)Y, \tag{1.2}$$

for $\kappa, \mu, \nu \in R_\eta(M^{2n+1})$, where $R_\eta(M^{2n+1})$ is the ring of smooth functions f on M^{2n+1} for which $df \wedge \eta = 0$. Later, [8] studied the generalized almost cosymplectic (κ, μ, ν)-spaces, that is: almost α-cosymplectic (κ, μ, ν)-spaces and also pointed out that the nullity condition is invariant under D-homothetic deformation of almost cosymplectic (κ, μ, ν)-spaces in all dimensions.

The study of paracontact geometry was initiated by S. Kaneyuki and F.L. Williams in [14] and then it was continued by many other authors in many aspects, for example, a systematic study of paracontact metric manifolds, and some remarkable subclasses like para-Sasakian manifolds, was carried out by S. Zamkovoy [15], a systematic study of almost α-paracosymplectic manifolds carried by I. K. Erken, P. Dacko and C. Murathan [9], [13].The importance of paracontact geometry has been pointed out highlighting the interplays with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics. In recent years, many authors turned to the study of paracontact geometry due to an unexpected relationship between (κ, μ)-contact metric manifold and paracontact geometry was found in [2]. It was proved that any (non-Sasakian) (κ, μ)-contact metric manifold carries a canonical paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ whose Levi-Civita connection satisfies a condition formally similar to (1.1)

$$\tilde{R}(X,Y)\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y), \tag{1.3}$$

where $2\tilde{h} := L_\xi\tilde{\varphi}$ and, in this case, $\tilde{\kappa} = (1 - \frac{\mu}{2})^2 + \kappa - 2, \tilde{\mu} = 2$. In [1], the authors showed that while the values of $\tilde{\kappa}$ and $\tilde{\mu}$ change the form but (1.3) remains unchanged under D-homothetic deformations. There are differences between a (κ, μ)-contact metric manifold $(M, \varphi, \xi, \eta, g)$ and $(\tilde{\kappa}, \tilde{\mu})$-paracontact metric manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$. Namely, unlike in the contact Riemannian case, a $(\tilde{\kappa}, \tilde{\mu})$-paracontact metric manifold such that $\tilde{\kappa} = -1$ in general is not para-Sasakian. And there are $(\tilde{\kappa}, \tilde{\mu})$-paracontact metric manifold such that $\tilde{h}^2 = 0$ but with $\tilde{h} \neq 0$ in [2]. Another important difference with the contact metric manifold is that while for contact metric case $\kappa \leq 1$, $(\tilde{\kappa}, \tilde{\mu})$-paracontact metric manifold has no resriction for the constants $\tilde{\kappa}$ and $\tilde{\mu}$. There are similar results about almost α-cosymplectic κ, μ, ν-spaces and almost α-paracosymplectic κ, μ, ν-spaces [8] and [9].

Recently, in [16] V. Saltarelli studied 3-dimensional almost Kenmotsu manifolds satisfying certain nullity conditions and gave some complete local descriptions of their structure. Motivated by the unexpected relationship between almost Kenmotsu and para-Kenmotsu manifold, we study almost α-para-Kenmotsu manifold in this paper and give a complete local description of 3-dimensional almost α-para-Kenmotsu (κ, μ, ν)-spaces.

This paper is organized in the following way. In section 2, some preliminaries and properties about almost α-para-kenmotsu manifolds are given. In section 3, we give some results concerning almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-spaces. In section 4, we will give a local description of the structure of a 3-dimensional almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space with $d\tilde{\kappa} \wedge \eta = 0$. We also construct in R^3 two families of such manifolds depending on $\tilde{h}$ of $\mathfrak{h}_1$ or $\mathfrak{h}_3$ type, and in the last section we give a necessary and sufficient condition for a local structure to be an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space with $d\tilde{\kappa} \wedge \eta = 0$. All manifolds are assumed to be connected and smooth.

2. Preliminaries

In this section, we recall some basic facts about paracontact metric manifolds.

A $2n+1$-dimensional smooth manifold M is said to have an almost paracontact structure if it admits a $(1,1)$-tensor field $\tilde{\varphi}$, a vector field ξ and a 1-form η satisfying the following conditions:

(i) $\tilde{\varphi}^2 = \mathrm{Id} - \eta \otimes \xi, \quad \eta(\xi) = 1,$

(ii) the tensor field $\tilde{\varphi}$ induces an almost paracomplex structure on each fiber of $\mathcal{D}$ =Ker(η), i.e. the $\pm 1-$eigendistributions $\mathcal{D}^{\pm} := \mathcal{D}_{\tilde{\varphi}}(\pm 1)$ of $\tilde{\varphi}$ have equal dimension n.

From the definition it follows that $\tilde{\varphi}(\xi) = 0$, $\eta \circ \tilde{\varphi} = 0$ and rank$(\tilde{\varphi}) = 2n$. When the tensor field $N_{\tilde{\varphi}} := [\tilde{\varphi}, \tilde{\varphi}] - 2d\eta \otimes \xi$ vanishes identically the almost paracontact manifold is said to be normal. If an almost paracontact manifold admits a pseudo-Riemannian metric $\tilde{g}$ such that

$$\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y) \tag{2.1}$$

for any vector fields $X, Y \in \Gamma(TM)$. Then we say that $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is an almost paracontact metric manifold. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n, n+1)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{\xi, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ such that $\tilde{g}(X_i, X_j) = \delta_{ij}, \tilde{g}(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \tilde{\varphi}X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a φ-basis. The fundamental 2-form $\tilde{\Phi}$ associate with the structure is defined by $\tilde{\Phi}(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)$ for all vector fields X, Y on M. The structure is normal if the tensor field $\mathcal{N} = [\tilde{\varphi}, \tilde{\varphi}] + 2d\eta \otimes \xi$ vanishes, where $[\tilde{\varphi}, \tilde{\varphi}]$ is the Nijenhuistorsion of $\tilde{\varphi}$. For more details, we refer the reader to [15]. According to [9], an almost paracontact metric manifold$(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be an almost α-para-Kenmotsu manifold if

$$d\eta = 0, \quad d\tilde{\Phi} = 2\alpha\eta \wedge \tilde{\Phi}, \quad \alpha = const. \neq 0. \tag{2.2}$$

A normal almost α-para-Kenmotsu manifold is an α-para-Kenmotsu manifold.

Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-para-Kenmotsu manifold. Since $d\eta = 0$, the canonical distribution $\mathcal{D} = ker(\eta)$ is completely integrable. Each leaf of the foliation, determined by $\mathcal{D}$, carries an almost para-Kähler structure $(J, <,>)$

$$J\bar{X} = \tilde{\varphi}\bar{X}, \quad <\bar{X}, \bar{Y}>= \tilde{g}(\bar{X}, \bar{Y}),$$

$\bar{X}, \bar{Y}$ are vector fields tangent to the leaf. If this structure is para-Kähler, leaf is called a para-Kähler leaf. Furthermore, we have $L_\xi \eta = 0$ and $[\xi, X] \in \mathcal{D}$ for any $X \in \mathcal{D}$. Furthermore, we have $\tilde{\nabla}_\xi \varphi = 0$, so that $\tilde{\nabla}_\xi \xi = 0$ and $\tilde{\nabla}_\xi X \in \mathcal{D}$ for any $X \in \mathcal{D}$. Define $\tilde{h} = \frac{1}{2} L_\xi \tilde{\varphi}$, we get the following proposition,

Proposition 2.1. *[9] Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-paracosymplectic manifold, we have the following relations:*

$$\tilde{g}(\tilde{h}X, Y) = \tilde{g}(X, \tilde{h}Y), \; \tilde{h}\tilde{\varphi} = -\tilde{\varphi}\tilde{h}, \; \tilde{h}\xi = 0,$$

$$\tilde{\nabla}\xi = \alpha\tilde{\varphi}^2 + \tilde{\varphi}\tilde{h}, \tag{2.3}$$

$$\mathrm{tr}(\tilde{h}) = 0, \quad \mathrm{tr}(\tilde{\varphi}\tilde{h}) = 0. \tag{2.4}$$

Moreover, also in [9], it follows that the curvature properties of an almost α-para-Kenmotsu manifold,

$$\tilde{R}(X, Y)\xi = \alpha\eta(X)(\alpha Y + \tilde{\varphi}\tilde{h}Y) - \alpha\eta(Y)(\alpha X + \tilde{\varphi}\tilde{h}X) + (\tilde{\nabla}_X \tilde{\varphi}\tilde{h})Y - (\tilde{\nabla}_Y \tilde{\varphi}\tilde{h})X \tag{2.5}$$

$$(\tilde{\nabla}_X \tilde{\varphi})Y - (\tilde{\nabla}_{\tilde{\varphi}X} \tilde{\varphi})\tilde{\varphi}Y = \eta(Y)(\alpha\tilde{\varphi}X - \tilde{h}X) - 2\alpha(\tilde{g}(X, \tilde{\varphi}Y)\xi + \eta(Y)\tilde{\varphi}). \tag{2.6}$$

Finally, we recall that an almost paracontact metric manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be η-Einstein if its Ricci tensor satisfies

$$\tilde{Ric} = a\tilde{g} + b\eta \oplus \eta,$$

or equivalently

$$\tilde{Q} = aI + b\eta \otimes \xi, \tag{2.7}$$

where a and b are smooth functions on M^{2n+1}. A vector field $X \in T_pM$ is called Killing vector field if $\mathcal{L}_X \tilde{g} = 0$, that is, $\tilde{g}(\tilde{\nabla}_Y X, Z) + \tilde{g}(\tilde{\nabla}_Z X, Y) = 0$, where $Y, Z \in T_pM$ are arbitrary vector fields.

In [9], Authors showed that Ricci curvature $\tilde{S}$ in the direction of ξ is given by

$$\tilde{S}(\xi, \xi) = -2n\alpha^2 + \mathrm{tr}\tilde{h}^2. \tag{2.8}$$

We recall that the curvature tensor of a 3-dimensional pseudo-Riemannian manifold satisfies

$$\begin{aligned} \tilde{R}(X, Y)Z &= \tilde{g}(Y, Z)\tilde{Q}X - \tilde{g}(X, Z)\tilde{Q}Y + \tilde{g}(\tilde{Q}Y, Z)X - \tilde{g}(\tilde{Q}X, Z)Y \\ &\quad - \frac{\tau}{2}(\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y). \end{aligned} \tag{2.9}$$

3. Almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-spaces

Firstly, let us recall the following theorem which is exactly the same as almost Kenmotsu manifolds [9], where $\tilde{h} = 0$, it is certainly $\tilde{h}^2 = 0$.

Theorem 3.1. *Let M^{2n+1} be an almost α-para-Kenmotsu manifold with $\tilde{h} = 0$. Then M^{2n+1} is locally a warped product $M_1 \times_{f^2} M_2$, where M_2 is an almost para-Kähler manifold, M_1 is an open interval with coordinate t, and $f^2 = we^{2\alpha t}$ for some positive constant w.*

Now, we give some properties for later use.

Lemma 3.1. *Let $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-para-Kenmotsu manifold, then, for any orthonormal frame $X_i, i = 1, \cdots, 2n+1$, the following identities hold:*

$$\sum_{i=1}^{2n+1} \varepsilon_i (\tilde{\nabla}_{X_i} \tilde{\varphi}\tilde{h}) X_i = \tilde{Q}\xi + 2n\alpha^2 \xi, \tag{3.1}$$

$$\sum_{i=1}^{2n+1} \varepsilon_i (\tilde{\nabla}_{X_i} \tilde{\varphi}) X_i = 0. \tag{3.2}$$

Proof. Let $X_i (i = 1, \cdots, 2n+1)$ be an orthonormal frame. For any vector field X, putting $X = X_i$, replacing Y by $\tilde{\varphi}X$ in (2.6), taking the inner product with $X = X_i$, by using $\tilde{h}\xi = \tilde{\varphi}\xi = 0$, $\mathrm{tr}(\tilde{\varphi}\tilde{h}) = 0$, the symmetry of $\tilde{\nabla}_{X_i}\tilde{\varphi}\tilde{h}$, and the skew-symmetry of $\tilde{\varphi}$ we get

$$\begin{aligned}
&\tilde{g}(\tilde{Q}\xi, \tilde{\varphi}X) \\
= &\sum_{i=1}^{2n+1} \varepsilon_i \tilde{g}(\tilde{R}(X_i, \tilde{\varphi}X)\xi, X_i) \\
= &\sum_{i=1}^{2n+1} \varepsilon_i \{\alpha\eta(X_i)\tilde{g}(\alpha\tilde{\varphi}X - \tilde{\varphi}\tilde{h}\tilde{\varphi}X, X_i) + \tilde{g}((\tilde{\nabla}_{X_i}\tilde{\varphi}\tilde{h})\tilde{\varphi}X, X_i) - \tilde{g}((\tilde{\nabla}_{\tilde{\varphi}X}\tilde{\varphi}\tilde{h})X_i, X_i)\} \\
= &\sum_{i=1}^{2n+1} \varepsilon_i \tilde{g}((\tilde{\nabla}_{X_i}\tilde{\varphi}\tilde{h})\tilde{\varphi}X, X_i).
\end{aligned}$$

Thus the above equality reduces to

$$\tilde{\varphi}\tilde{Q}\xi = \sum_{i=1}^{2n+1} \varepsilon_i \tilde{\varphi}(\tilde{\nabla}_{X_i}\tilde{\varphi}\tilde{h})X_i,$$

Applying $\tilde{\varphi}$ to the above equality, using $\tilde{\varphi}^2 = \mathrm{Id} - \eta \otimes \xi$ and (2.8), combining with (2.4), we get $\sum_{i=1}^{2n+1} \varepsilon_i \tilde{g}((\tilde{\nabla}_{X_i}\tilde{\varphi}\tilde{h})X_i, \xi) = \mathrm{tr}\tilde{h}^2$, it follows that

$$\sum_{i=1}^{2n+1} \varepsilon_i (\tilde{\nabla}_{X_i}\tilde{\varphi}\tilde{h}) X_i = \tilde{Q}\xi + 2n\alpha^2\xi.$$

In order to obtain (3.4), we choose a $\tilde\varphi$-basis $\{E_i, \tilde\varphi E_i, \xi\}$,using (2.6) and $\tilde\nabla_\xi\tilde\varphi = 0$, we get

$$\sum_{i=1}^{2n+1}\varepsilon_i(\tilde\nabla_{X_i}\tilde\varphi)X_i = \sum_{i=1}^{n}\varepsilon_i(\tilde\nabla_{E_i}\tilde\varphi)E_i - \sum_{i=1}^{n}\varepsilon_i(\tilde\nabla_{\tilde\varphi E_i}\tilde\varphi)\tilde\varphi E_i + (\tilde\nabla_\xi\tilde\varphi)\xi = 0.$$

□

The next lemma concerns almost α-para-Kenmotsu manifolds having the canonical distribution $\mathcal{D}$ with para-Kähler leaves for which the following formula holds [9]:

$$(\tilde\nabla_X\tilde\varphi)Y = \tilde g(\alpha\tilde\varphi X + \tilde hX, Y)\xi - \eta(Y)(\alpha\tilde\varphi X + \tilde hX). \tag{3.3}$$

Lemma 3.2. *Let $(M^{2n+1}, \tilde\varphi, \xi, \eta, \tilde g)$ be an almost α-para-Kenmotsu manifold and assume that the distribution $\mathcal{D}$ has para-Kähler leaves, then, for any orthonormal frame $X_i, i = 1, \cdots, 2n+1$, we have*

$$\sum_{i=1}^{2n+1}\varepsilon_i(\tilde\nabla_{X_i}\tilde h)X_i = \tilde\varphi\tilde Q\xi. \tag{3.4}$$

Proof. Since

$$\tilde\nabla_X\tilde h\tilde\varphi Y = (\tilde\nabla_X\tilde h)\tilde\varphi Y + \tilde h(\tilde\nabla_X\tilde\varphi)Y + \tilde h\tilde\varphi\tilde\nabla_X Y, \tag{3.5}$$

$$\tilde\nabla_X\tilde\varphi\tilde hY = \tilde\varphi(\tilde\nabla_X\tilde h)Y + \tilde\varphi\tilde h(\tilde\nabla_X Y) + (\tilde\nabla_X\tilde\varphi)\tilde hY, \tag{3.6}$$

By (3.5)-(3.6) and $\tilde\varphi\tilde h = -\tilde h\tilde\varphi$, we get

$$\begin{aligned}(\tilde\nabla_X\tilde h)\tilde\varphi Y + \tilde\varphi(\tilde\nabla_X\tilde h)Y &= -\tilde h(\tilde\nabla_X\tilde\varphi)Y - (\tilde\nabla_X\tilde\varphi)\tilde hY\\ &= \eta(Y)(\alpha\tilde h\tilde\varphi X + \tilde h^2X) - \tilde g(\alpha\tilde\varphi X + \tilde hX, \tilde hY)\xi.\end{aligned} \tag{3.7}$$

Taking $X = Y = X_i$ in (3.7), summing on i and using $\mathrm{tr}(\tilde h\tilde\varphi) = 0$ and $\tilde h\xi = 0$, we get

$$\sum_{i=1}^{2n+1}\varepsilon_i\{(\tilde\nabla_{X_i}\tilde h)\tilde\varphi X_i + \tilde\varphi(\tilde\nabla_{X_i}\tilde h)X_i\} = -(\mathrm{tr}\tilde h^2)\xi. \tag{3.8}$$

By (3.1), and using (3.4) we get

$$\begin{aligned}\tilde Q\xi + 2n\alpha^2\xi &= \sum_{i=1}^{2n+1}\varepsilon_i(\tilde\nabla_{X_i}\tilde\varphi\tilde h)X_i = -\sum_{i=1}^{2n+1}\varepsilon_i(\tilde\nabla_{X_i}\tilde h\tilde\varphi)X_i\\ &= -\sum_{i=1}^{2n+1}\varepsilon_i\{(\tilde\nabla_{X_i}\tilde h)\tilde\varphi X_i + \tilde h(\tilde\nabla_{X_i}\tilde\varphi)X_i\}\\ &= -\sum_{i=1}^{2n+1}\varepsilon_i(\tilde\nabla_{X_i}\tilde h)\tilde\varphi X_i.\end{aligned} \tag{3.9}$$

Substituting (3.3) into (3.8) we obtain

$$\sum_{i=1}^{2n+1} \varepsilon_i \tilde{\varphi}(\tilde{\nabla}_{X_i}\tilde{h})X_i = \tilde{Q}\xi + (2n\alpha^2 - \mathrm{tr}\tilde{h}^2)\xi,$$

finally, we get the required result acting by $\tilde{\varphi}$ and using $\sum_{i=1}^{2n+1} \varepsilon_i \tilde{g}((\tilde{\nabla}_{X_i}\tilde{h})X_i, \xi) = 0$, which, by direct calculation, follows from the fact that $\tilde{g}(\tilde{\varphi}\tilde{h}^2 X_i, X_i) = 0$ and $\mathrm{tr}(\tilde{h}\tilde{\varphi}) = 0$. □

Next we study almost α-para-Kenmotsu manifolds under assumption that the curvature satisfies $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-nullity condition

$$\tilde{R}(X,Y)\xi = \eta(Y)BX - \eta(X)BY, \tag{3.10}$$

where B is Jacobi operator of ξ, that is to say $BX = \tilde{R}(X,\xi)\xi = \tilde{\kappa}\tilde{\varphi}^2 X + \tilde{\mu}\tilde{h}X + \tilde{\nu}\tilde{\varphi}\tilde{h}X$, for $\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} \in R_\eta(M^{2n+1})$. Particularly $B\xi = 0$. If $\tilde{\mu} = 0$ or $\tilde{h} = 0$ and $\tilde{\nu} = 0$ or $\tilde{\varphi}\tilde{h} = 0$, the $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-nullity distribution is reduced to the well-known $\tilde{\kappa}$-nullity distribution $N(\tilde{\kappa})$. The $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-nullity condition (3.10) is obtained by requiring that ξ belong to some $N(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$. If almost α-para-Kenmotsu manifold satisfies (3.10), then the manifold is said to be an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-space. We observe that, in an almost α-para-Kenmotsu manifold, if $\xi \in N(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$, (3.10) and (2.5) implies $\tilde{\varphi}\tilde{h}$ is a Codazzi tensor, that is to say, $(\tilde{\nabla}_X \tilde{\varphi}\tilde{h})Y - (\tilde{\nabla}_Y \tilde{\varphi}\tilde{h})X = 0$, for any $X, Y \in \mathcal{D}$.

Proposition 3.1. *[9] Let $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-para-Kenmotsu (κ, μ, ν)-space, then the following identities hold:*

$$\tilde{h}^2 = (\tilde{\kappa} + \alpha^2)\tilde{\varphi}^2, \tag{3.11}$$

$$\tilde{\nabla}_\xi \tilde{h} = -(2\alpha + \tilde{\nu})\tilde{h} + \tilde{\mu}\tilde{h}\tilde{\varphi}, \tag{3.12}$$

$$\xi(\tilde{\kappa}) = -2(2\alpha + \tilde{\nu})(\tilde{\kappa} + \alpha^2), \tag{3.13}$$

$$\tilde{Q}\xi = 2n\tilde{\kappa}\xi. \tag{3.14}$$

Lemma 3.3. *Let $(M^3, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu}$=const.)-spaces, then one has:*

$$\tilde{Q}X = (-\tilde{\kappa} + \frac{\tau}{2})X + (3\tilde{\kappa} - \frac{\tau}{2})\eta(X)\xi + \tilde{\mu}\tilde{h}X + \tilde{\nu}\tilde{\varphi}\tilde{h}X, \tag{3.15}$$

$$\tilde{h}\mathrm{grad}\tilde{\mu} + \tilde{\varphi}\tilde{h}\mathrm{grad}\tilde{\nu} = \mathrm{grad}\tilde{\kappa} - \xi(\tilde{\kappa})\xi, \tag{3.16}$$

where $\tilde{Q}$ is the Ricci operator of M. τ denotes scalar curvature of M and $\tilde{l} = \tilde{R}(\cdot, \xi)\xi$.

Proof. Let $Y = Z = \xi$ in (2.9) and using (3.10), we can easily obtain (3.15).

By using the well known formula

$$\frac{1}{2}grad\tau = \sum_{i=1}^{3}\varepsilon_i(\nabla_{X_i}\tilde{Q})X_i \tag{3.17}$$

for any orthonormal frames $X_i, i = 1, 2, 3$, using (2.2) and (3.15), since $\mathrm{tr}\tilde{h} = \mathrm{tr}\tilde{h}\tilde{\varphi} = 0$, we have

$$\begin{aligned}
\frac{1}{2}\mathrm{grad}\tau &= \sum_{i=1}^{3}\varepsilon_i(\nabla_{X_i}Q)X_i = \sum_{i=1}^{3}\varepsilon_i(\nabla_{X_i}QX_i - Q\nabla_{X_i}X_i)\\
&= \sum_{i=1}^{3}\varepsilon_i\{\nabla_{X_i}[(-\tilde{\kappa}+\frac{\tau}{2})X_i + (3\tilde{\kappa}-\frac{\tau}{2})\eta(X_i)\xi + \tilde{\mu}\tilde{h}X_i + \tilde{\nu}\tilde{\varphi}\tilde{h}X_i]\\
&\quad -[(-\tilde{\kappa}+\frac{\tau}{2})\nabla_{X_i}X_i + (3\tilde{\kappa}-\frac{\tau}{2})\eta(\nabla_{X_i}X_i)\xi + \tilde{\mu}\tilde{h}\nabla_{X_i}X_i + \tilde{\nu}\tilde{\varphi}\tilde{h}\nabla_{X_i}X_i]\}\\
&= \sum_{i=1}^{3}\varepsilon_i\{X_i(-\tilde{\kappa}+\frac{\tau}{2})X_i + X_i(3\tilde{\kappa}-\frac{\tau}{2})\eta(X_i)\xi + X_i(\tilde{\mu})\tilde{h}X_i + X_i(\tilde{\nu})\tilde{\varphi}\tilde{h}X_i\}\\
&\quad +\sum_{i=1}^{3}\varepsilon_i\{\tilde{\mu}(\tilde{\nabla}_{X_i}\tilde{h})X_i + \tilde{\nu}(\tilde{\nabla}_{X_i}\tilde{\varphi}\tilde{h})X_i\}\\
&= -\mathrm{grad}\tilde{\kappa} + \frac{1}{2}\mathrm{grad}\tau + \tilde{h}\mathrm{grad}\tilde{\mu} + \tilde{\varphi}\tilde{h}\mathrm{grad}\tilde{\nu} + [3\xi(\tilde{\kappa}) - \frac{1}{2}\xi(\tau)]\xi\\
&\quad +\sum_{i=1}^{3}\varepsilon_i\{\tilde{\mu}(\tilde{\nabla}_{X_i}\tilde{h})X_i + \tilde{\nu}(\tilde{\nabla}_{X_i}\tilde{\varphi}\tilde{h})X_i\}. \qquad (3.18)
\end{aligned}$$

Thus, using (3.1), (3.2) and (3.14) we get

$$\sum_{i=1}^{3}\varepsilon_i(\tilde{\nabla}_{X_i}\tilde{h})X_i = 0,$$

and

$$\sum_{i=1}^{3}\varepsilon_i(\tilde{\nabla}_{X_i}\tilde{\varphi}\tilde{h})X_i = 2(\tilde{\kappa}+\alpha^2)\xi.$$

Using these two equalities in (3.15), one has

$$\xi(\tilde{\kappa})\xi - \mathrm{grad}\tilde{\kappa} + \tilde{h}\mathrm{grad}\tilde{\mu} + \tilde{\varphi}\tilde{h}\mathrm{grad}\tilde{\nu} + \xi(2\tilde{\kappa} - \frac{1}{2}\tau)\xi + 2(\alpha^2+\tilde{\kappa})\tilde{\nu}\xi = 0.$$

Since the vector field $\xi(\tilde{\kappa})\xi - \mathrm{grad}\tilde{\kappa} + \tilde{h}\mathrm{grad}\tilde{\mu} + \tilde{\varphi}\tilde{h}\mathrm{grad}\tilde{\nu}$ is orthogonal to ξ, (3.16) follows. □

Proposition 3.2. *Let $(M^3, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-para-Kenmotsu manifold. If M is η-Einstein, then $\xi \in N(\tilde{\kappa})$ for some function $\tilde{\kappa}$.*

Proof. By (2.7), choosing the $\tilde{\varphi}$-basis $\{\xi, e, \tilde{\varphi}e\}$, we get $\tilde{Q}\xi = (a+b)\xi$ and $\tau = \tilde{g}(\xi,\xi) + \tilde{g}(\tilde{Q}e, e) + \tilde{g}(\tilde{Q}\tilde{\varphi}e, \tilde{\varphi}e) = 3a + b$. Let $Z = \xi$ in (**??**) and using (2.7), we can easily obtain $\tilde{R}(X,Y)\xi = \frac{a+b}{2}(\eta(Y)X - \eta(X)Y)$, thus $\xi \in N(\frac{a+b}{2})$. □

Corollary 3.1. *Let $(M^3, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-para-Kenmotsu manifold. If M is $\xi \in N(\tilde{\kappa})$, then M is η-Einstein.*

Proof. By Lemma 3.3, we get $\tilde{Q}X = (-\tilde{\kappa} + \frac{\tau}{2})X + (3\tilde{\kappa} - \frac{\tau}{2})\eta(X)\xi$, it is simply to get that M is η-Einstein. □

If $\tilde{h} = 0$, by (2.5), we get $\tilde{R}(X,Y)\xi = -\alpha^2(\eta(Y)X - \eta(X)Y)$, thus $\xi \in N(-\alpha^2)$, by Corollary 3.1, it follows that M is η-Einstein. Therefore, from now on, we will restrict our investigations mainly on the more meaningful case $\tilde{h} \neq 0$. I. K. Erken, P. Dacko and C. Murathan analyzed the different possibilities for the tensor field $\tilde{h}$ in [9]. If $\tilde{h}$ has

$$\begin{pmatrix} \tilde{\lambda} & 0 & 0 \\ 0 & -\tilde{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.19}$$

with respect to a local orthonormal $\tilde{\varphi}$-basis $\{X, \tilde{\varphi}X, \xi\}$, the authors called the operator $\tilde{h}$ is of $\mathfrak{h}_1$ type.

If $\tilde{h}$ has

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to a pseudo orthonormal basis $\{e_1, e_2, e_3\}$, the authors called the operator $\tilde{h}$ is of $\mathfrak{h}_2$ type.

If $\tilde{h}$ has

$$\begin{pmatrix} 0 & \tilde{\lambda} & 0 \\ -\tilde{\lambda} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.20}$$

with respect to a local orthonormal $\tilde{\varphi}$-basis $\{X, \tilde{\varphi}X, \xi\}$, in this case. the authors called the operator $\tilde{h}$ is of $\mathfrak{h}_3$ type.

It follows that $\tilde{h}^2X = \tilde{\lambda}^2X$ if $\tilde{h}$ is of $\mathfrak{h}_1$ type and $\tilde{h}^2X = -\tilde{\lambda}^2X$ if $\tilde{h}$ is of $\mathfrak{h}_3$ type, but $\tilde{h}^2X = 0$ if $\tilde{h}$ is of $\mathfrak{h}_2$ type though $\tilde{h} \neq 0$, and there are examples of 3-dimensional almost α-para-Kenmotsu manifold of this case [9]. In this paper, we manly discuss the case $\tilde{h}^2 \neq 0$, that is, $\tilde{\kappa} + \alpha^2 \neq 0$.

Lemma 3.4. *Let $(M^3, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space with $\tilde{h}$ is of $\mathfrak{h}_1$ type. Then, for any point $p \in M$, there exist a neighborhood U of p and a $\tilde{\varphi}$-basis $\{X, \tilde{\varphi}X, \xi\}$ defined on U, such that*

$$\tilde{h}X = \tilde{\lambda}X, \quad \tilde{h}\tilde{\varphi}X = -\tilde{\lambda}\tilde{\varphi}X, \quad \tilde{h}\xi = 0, \quad \tilde{\lambda} = \sqrt{\tilde{\kappa} + \alpha^2} \tag{3.21}$$

at any point $q \in U$. Moreover, setting $A = X(\tilde{\lambda})$ and $B = \tilde{\varphi}X(\tilde{\lambda})$ on U the following formulas are true:

$$\tilde{\nabla}_X \xi = \alpha X + \tilde{\lambda}\tilde{\varphi}X, \quad \tilde{\nabla}_{\tilde{\varphi}X}\xi = \alpha\tilde{\varphi}X - \tilde{\lambda}X, \tag{3.22}$$

$$\tilde{\nabla}_\xi X = -\frac{\tilde{\mu}}{2}\tilde{\varphi}X, \quad \tilde{\nabla}_\xi \tilde{\varphi}X = -\frac{\tilde{\mu}}{2}X, \tag{3.23}$$

$$\tilde{\nabla}_X X = \alpha\xi - \frac{B}{2\tilde{\lambda}}\tilde{\varphi}X, \quad \tilde{\nabla}_{\tilde{\varphi}X}\tilde{\varphi}X = -\alpha\xi - \frac{A}{2\tilde{\lambda}}X, \tag{3.24}$$

$$\tilde{\nabla}_{\tilde{\varphi}X} X = -\tilde{\lambda}\xi - \frac{A}{2\tilde{\lambda}}\tilde{\varphi}X, \quad \tilde{\nabla}_X \tilde{\varphi}X = -\tilde{\lambda}\xi - \frac{B}{2\tilde{\lambda}}X \tag{3.25}$$

$$[\xi, X] = -\alpha X - (\tilde{\lambda} + \frac{\tilde{\mu}}{2})\tilde{\varphi}X, \quad [\xi, \tilde{\varphi}X] = (\tilde{\lambda} - \frac{\tilde{\mu}}{2})X - \alpha\tilde{\varphi}X, \tag{3.26}$$

$$[X, \tilde{\varphi}X] = -\frac{B}{2\tilde{\lambda}}X + \frac{A}{2\tilde{\lambda}}\tilde{\varphi}X. \tag{3.27}$$

$$\tilde{h}\mathrm{grad}\tilde{\mu} = \mathrm{grad}\tilde{\kappa} - \xi(\tilde{\kappa})\xi, \tag{3.28}$$

Proof. By [9] we know that if $\tilde{h}$ is of $\mathfrak{h}_1$ type with respect to a $\tilde{\varphi}$-basis $\{X, \tilde{\varphi}X, \xi\}$ such that $\tilde{h}X = \tilde{\lambda}X$, $\tilde{h}\tilde{\varphi}X = -\tilde{\lambda}\tilde{\varphi}X$, and by (3.11), we get $\tilde{\lambda} = \sqrt{\tilde{\kappa} + \alpha^2}$. Similar as the proof of [16], we get Lemma 3.4. □

Similarly as Lemma 3.4, we get the following Lemma.

Lemma 3.5. *Let $(M^3, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space with $\tilde{h}$ is of $\mathfrak{h}_3$ type. Then, for any point $p \in M$, there exist a neighborhood U of p and a $\tilde{\varphi}$-basis $\{X, \tilde{\varphi}X, \xi\}$ defined on U, such that*

$$\tilde{h}X = \tilde{\lambda}\tilde{\varphi}X, \quad \tilde{h}\tilde{\varphi}X = -\tilde{\lambda}X, \quad \tilde{h}\xi = 0, \quad \tilde{\lambda} = \sqrt{-(\tilde{\kappa} + \alpha^2)} \tag{3.29}$$

at any point $q \in U$. Moreover, setting $A = X(\tilde{\lambda})$ and $B = \tilde{\varphi}X(\tilde{\lambda})$ on U the following formulas are true:

$$\tilde{\nabla}_X \xi = (\alpha + \tilde{\lambda})X, \quad \tilde{\nabla}_{\tilde{\varphi}X}\xi = (\alpha - \tilde{\lambda})\tilde{\varphi}X, \tag{3.30}$$

$$\tilde{\nabla}_\xi X = -\frac{\tilde{\mu}}{2}\tilde{\varphi}X, \quad \tilde{\nabla}_\xi \tilde{\varphi}X = -\frac{\tilde{\mu}}{2}X, \tag{3.31}$$

$$\tilde{\nabla}_X X = (\alpha + \tilde{\lambda})\xi - \frac{B}{2\tilde{\lambda}}\tilde{\varphi}X, \quad \tilde{\nabla}_{\tilde{\varphi}X}\tilde{\varphi}X = (\tilde{\lambda} - \alpha)\xi - \frac{A}{2\tilde{\lambda}}X, \tag{3.32}$$

$$\tilde{\nabla}_{\tilde{\varphi}X} X = -\frac{A}{2\tilde{\lambda}}\tilde{\varphi}X, \; \tilde{\nabla}_X \tilde{\varphi}X = -\frac{B}{2\tilde{\lambda}}X \tag{3.33}$$

$$[\xi, X] = -(\alpha + \tilde{\lambda})X - \frac{\tilde{\mu}}{2}\tilde{\varphi}X, \; [\xi, \tilde{\varphi}X] = -\frac{\tilde{\mu}}{2}X + (\tilde{\lambda} - \alpha)\tilde{\varphi}X, \tag{3.34}$$

$$[X, \tilde{\varphi}X] = -\frac{B}{2\tilde{\lambda}}X + \frac{A}{2\tilde{\lambda}}\tilde{\varphi}X. \tag{3.35}$$

$$\tilde{h}\mathrm{grad}\tilde{\mu} = \mathrm{grad}\tilde{\kappa} - \xi(\tilde{\kappa})\xi. \tag{3.36}$$

4. Almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space with $d\tilde{\kappa} \wedge \eta = 0$

Locally, an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space with $\tilde{h}$ is of $\mathfrak{h}_1$ type and $d\tilde{\kappa} \wedge \eta = 0$ can be described as follows.

Theorem 4.1. *Let $(M^3, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space with $\tilde{h}$ is of $\mathfrak{h}_1$ type and $d\tilde{\kappa} \wedge \eta = 0$. Then, in a neighbourhood U of every point $p \in M$, there exist coordinates x,y,z and an orthonormal frame $\{X, \tilde{\varphi}X, \xi\}$ of eigenvectors of $\tilde{h}$ with $\tilde{h}X = \tilde{\lambda}X$, such that on U $\tilde{\kappa}, \tilde{\mu}$ only depends on z and*

$$X = \frac{\partial}{\partial x}, \; \tilde{\varphi}X = \frac{\partial}{\partial y}, \; \xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

and the tensor fields $\tilde{\varphi}, \tilde{g}, \tilde{h}$ are given by the relations:

$$\tilde{g} = \begin{pmatrix} -1 & 0 & a \\ 0 & 1 & -b \\ a & -b & 1 - a^2 + b^2 \end{pmatrix}, \; \tilde{\varphi} = \begin{pmatrix} 0 & 1 & -b \\ 1 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix}, \; \tilde{h} = \begin{pmatrix} \tilde{\lambda} & 0 & -a\tilde{\lambda} \\ 0 & -\tilde{\lambda} & b\tilde{\lambda} \\ 0 & 0 & 0 \end{pmatrix}.$$

where $a = \alpha x + (\frac{\tilde{\mu}}{2} - \tilde{\lambda})y + f(z), \; b = (\frac{\tilde{\mu}}{2} + \tilde{\lambda})x - \alpha y - g(z), f(z), g(z)$ are arbitrary smooth functions of z, α is a constant value.

Proof. The condition $d\tilde{\kappa} \wedge \eta = 0$ and (3.28) means that $d\tilde{\mu} \wedge \eta = 0$, since $\tilde{h} \neq 0$ and $\ker\tilde{h} = \mathrm{Span}\{\xi\}$. Moreover, we have $E(\tilde{\lambda}) = 0$ for all $E \in \mathcal{D}$. By lemma 3.4, we get that for any point $p \in M$, there exist a neighborhood U of p and a $\tilde{\varphi}$-basis $\{X, \tilde{\varphi}X, \xi\}$ defined on U, such that $\tilde{h}X = \tilde{\lambda}X, \; \tilde{h}\tilde{\varphi}X = -\tilde{\lambda}\tilde{\varphi}X, \; \tilde{\lambda} = \sqrt{\tilde{\kappa} + \alpha^2}$.

Hence $A = X(\tilde{\lambda}) = B = \tilde{\varphi}X(\tilde{\lambda}) = 0$, that is to say, by Lemma 3.4, we get that $[X, \tilde{\varphi}X] = 0$. So, fixed the point $p \in M$, there exist coordinates (x, y, t) on an open neighbourhood V of p such that

$$X = \frac{\partial}{\partial x},\ \tilde{\varphi}X = \frac{\partial}{\partial y},\ \xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial t},$$

where a, b and c are smooth functions on V with $c \neq 0$ everywhere. Since we get $[X, \xi] \in \mathcal{D}$ and $[X, \xi] \in \mathcal{D}$ for any $X \in \mathcal{D}$ by $d\eta = 0$, we obtain that $\frac{\partial c}{\partial x} = 0$ and $\frac{\partial c}{\partial y} = 0$. Therefore, if we consider on V the linearly independent vector fields $X, \tilde{\varphi}X$ and $Z = c\frac{\partial}{\partial t}$, we have

$$[X, \tilde{\varphi}X] = 0,\ [X, Z] = 0,\ [\tilde{\varphi}X, Z] = 0.$$

This implies that there exists a coordinate system $\{U, (x, y, z)\}$ around p in V such that $X = \frac{\partial}{\partial x}$, $\tilde{\varphi}X = \frac{\partial}{\partial y}$ and $Z = \frac{\partial}{\partial z}$.Thus, on the open set U we have $\xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. From (3.13) and (3.21), we get that $\xi(\tilde{\lambda}) = -(2\alpha + \tilde{\nu})\tilde{\lambda}$, and since $A = X(\tilde{\lambda}) = B = \tilde{\varphi}X(\tilde{\lambda}) = 0$, it follows that $\tilde{\lambda} = ce^{-(2\alpha+\tilde{\nu})z}$, and $\tilde{\kappa} = \tilde{\lambda}^2 - \alpha^2 = c^2e^{-2(2\alpha+\tilde{\nu})z} - \alpha^2$ for some real constant $c > 0$. Since $d\tilde{\mu} \wedge \eta = 0$, we get that $\tilde{\mu} = \tilde{\mu}(z)$. Next, we need to compute the functions a, b. To this end,

$$[\xi, X] = -\frac{\partial a}{\partial x}\frac{\partial}{\partial x} - \frac{\partial b}{\partial x}\frac{\partial}{\partial y},\quad [\xi, \tilde{\varphi}X] = -\frac{\partial a}{\partial y}\frac{\partial}{\partial x} - \frac{\partial b}{\partial y}\frac{\partial}{\partial y}$$

And by Lemma 3.4, we obtain

$$[\xi, X] = -\alpha\frac{\partial}{\partial x} - (\tilde{\lambda} + \frac{\tilde{\mu}}{2})\frac{\partial}{\partial y},\quad [\xi, \tilde{\varphi}X] = (\tilde{\lambda} - \frac{\tilde{\mu}}{2})\frac{\partial}{\partial x} - \alpha\frac{\partial}{\partial y},$$

The comparison of these relations with the previous leads to

$$\frac{\partial a}{\partial x} = \alpha,\quad \frac{\partial a}{\partial y} = \frac{\tilde{\mu}}{2} - \tilde{\lambda},\quad \frac{\partial b}{\partial x} = \tilde{\lambda} + \frac{\tilde{\mu}}{2},\quad \frac{\partial b}{\partial y} = \alpha. \tag{4.1}$$

By integration of these system, considering $\tilde{\lambda}, \tilde{\mu}$ functions depending only on z, we get $a = \alpha x + (\frac{\tilde{\mu}}{2} - \tilde{\lambda})y + f(z)$, $b = (\frac{\tilde{\mu}}{2} + \tilde{\lambda})x - \alpha y - g(z), f(z), g(z)$ are arbitrary smooth functions of z.

We will continue calculate the tensor fields $\eta, \tilde{\varphi}, \tilde{g}$ and $\tilde{h}$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. The expression of the 1-form $\eta = dz$ immediately follows from $\eta(\xi) = 1, \eta(X) = \eta(\tilde{\varphi}X) = 0$. For the components of $\tilde{g}_{ij}$ of the pseudo-Riemannian metric, we have

$$\tilde{g}_{11} = \tilde{g}(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \tilde{g}(X, X) = -1,\quad \tilde{g}_{22} = \tilde{g}(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = \tilde{g}(\tilde{\varphi}X, \tilde{\varphi}X) = 1,$$

$$\tilde{g}_{33} = \tilde{g}(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = \tilde{g}(\xi - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}, \xi - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}) = 1 - a^2 + b^2.$$

$$\tilde{g}_{12} = \tilde{g}_{21} = \tilde{g}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \tilde{g}(X, \tilde{\varphi}X) = 0, \quad \tilde{g}_{13} = \tilde{g}_{31} = \tilde{g}(\frac{\partial}{\partial x}, \xi - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}) = a,$$

$$\tilde{g}_{23} = \tilde{g}_{32} = \tilde{g}(\frac{\partial}{\partial y}, \xi - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}) = -b,$$

thus the matrix form of $\tilde{g}$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ is given by

$$\tilde{g} = \begin{pmatrix} -1 & 0 & a \\ 0 & 1 & -b \\ a & -b & 1 - a^2 + b^2 \end{pmatrix}.$$

The components of the tensor field $\tilde{\varphi}$ are followed by:

$$\tilde{\varphi}(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}, \quad \tilde{\varphi}(\frac{\partial}{\partial y}) = \tilde{\varphi}^2(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x}, \quad \tilde{\varphi}(\frac{\partial}{\partial z}) = \tilde{\varphi}(\xi - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}) = -a\frac{\partial}{\partial y} - b\frac{\partial}{\partial y},$$

thus the matrix form of $\tilde{\varphi}$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ is given by

$$\begin{pmatrix} 0 & 1 & -b \\ 1 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix}.$$

The components of the tensor field $\tilde{h}$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are given as follows:

$$\tilde{h}(\frac{\partial}{\partial x}) = \tilde{h}(X) = \tilde{\lambda}X = \tilde{\lambda}\frac{\partial}{\partial x}, \ \tilde{h}(\frac{\partial}{\partial y}) = \tilde{h}\tilde{\varphi}X = -\tilde{\lambda}\tilde{\varphi}X = -\tilde{\lambda}\frac{\partial}{\partial y},$$

$$\tilde{h}(\frac{\partial}{\partial z}) = \tilde{h}(\xi - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}) = -a\tilde{\lambda}\frac{\partial}{\partial x} + b\tilde{\lambda}\frac{\partial}{\partial y}.$$

Thus the matrix form of $\tilde{h}$ is given by

$$\begin{pmatrix} \tilde{\lambda} & 0 & a\tilde{\lambda} \\ 0 & -\tilde{\lambda} & b\tilde{\lambda} \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Now we consider the case of $\tilde{h}$ is of $\mathfrak{h}_3$ type.

Theorem 4.2. *Let $(M^3, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space with $\tilde{h}$ is of $\mathfrak{h}_3$ type and $d\tilde{\kappa} \wedge \eta = 0$. Then, in a neighbourhood U of every point $p \in M$, there exist coordinates x,y,z and an orthonormal frame $\{X, \tilde{\varphi}X, \xi\}$ with $\tilde{h}X = \tilde{\lambda}\tilde{\varphi}X$, $\tilde{h}\tilde{\varphi}X = -\tilde{\lambda}X$, such that on U $\tilde{\kappa}, \tilde{\mu}$ only depends on z and*

$$X = \frac{\partial}{\partial x}, \ \tilde{\varphi}X = \frac{\partial}{\partial y}, \ \xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

and the tensor fields $\tilde{\varphi}, \tilde{g}, \tilde{h}$ are given by the relations:

$$\tilde{g} = \begin{pmatrix} -1 & 0 & a \\ 0 & 1 & -b \\ a & -b & 1-a^2+b^2 \end{pmatrix}, \ \tilde{\varphi} = \begin{pmatrix} 0 & 1 & -b \\ 1 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix}, \ \tilde{h} = \begin{pmatrix} 0 & -\tilde{\lambda} & b\tilde{\lambda} \\ \tilde{\lambda} & 0 & -a\tilde{\lambda} \\ 0 & 0 & 0 \end{pmatrix}.$$

where $a = (\alpha+\tilde{\lambda})x + \frac{\tilde{\mu}}{2}y + f(z),\ b = \frac{\tilde{\mu}}{2}x + (\alpha-\tilde{\lambda})y + g(z), f(z), g(z)$ *are arbitrary smooth functions of* z.

Proof. The condition $d\tilde{\kappa} \wedge \eta = 0$ and (3.28) means that $d\tilde{\mu} \wedge \eta = 0$, since $\tilde{h} \neq 0$ and $\ker\tilde{h} = \mathrm{Span}\{\xi\}$. Moreover, we have $E(\tilde{\lambda}) = 0$ for all $E \in \mathcal{D}$. By lemma 3.5, we get that for any point $p \in M$, there exist a neighborhood U of p and a $\tilde{\varphi}$-basis $\{X, \tilde{\varphi}X, \xi\}$ defined on U, such that $\tilde{h}X = \tilde{\lambda}\tilde{\varphi}X,\ \tilde{h}\tilde{\varphi}X = -\tilde{\lambda}X,\ \tilde{\lambda} = \sqrt{-(\tilde{\kappa}+\alpha^2)}$. Hence $A = X(\tilde{\lambda}) = B = \tilde{\varphi}X(\tilde{\lambda}) = 0$, that is to say, by Lemma 3.5, we get that $[X, \tilde{\varphi}X] = 0$. So, fixed the point $p \in M$, there exist coordinates (x, y, t) on an open neighbourhood V of p such that

$$X = \frac{\partial}{\partial x},\ \tilde{\varphi}X = \frac{\partial}{\partial y},\ \xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial t},$$

where a, b and c are smooth functions on V with $c \neq 0$ everywhere. Since we get $[X, \xi] \in \mathcal{D}$ and $[X, \xi] \in \mathcal{D}$ for any $X \in \mathcal{D}$ by $d\eta = 0$, we obtain that $\frac{\partial c}{\partial x} = 0$ and $\frac{\partial c}{\partial y} = 0$. Therefore, if we consider on V the linearly independent vector field $X, \tilde{\varphi}X$ and $Z = c\frac{\partial}{\partial t}$, we have

$$[X, \tilde{\varphi}X] = 0,\ [X, Z] = 0,\ [\tilde{\varphi}X, Z] = 0.$$

This implies that there exists a coordinate system $\{U, (x, y, z)\}$ around p in V such that $X = \frac{\partial}{\partial x},\ \tilde{\varphi}X = \frac{\partial}{\partial y}$ and $Z = \frac{\partial}{\partial z}$.Thus, on the open set U we have $\xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. From (3.13) and (3.21), we get that $\xi(\tilde{\lambda}) = -(2\alpha+\tilde{\nu})\tilde{\lambda}$, and since $A = X(\tilde{\lambda}) = B = \tilde{\varphi}X(\tilde{\lambda}) = 0$, it follows that $\tilde{\lambda} = ce^{-(2\alpha+\tilde{\nu})z}$, and $\tilde{\kappa} = -\tilde{\lambda}^2 - \alpha^2 = -c^2e^{-2(2\alpha+\tilde{\nu})z} - \alpha^2$ for some real constant $c > 0$. Since $d\tilde{\mu} \wedge \eta = 0$, we get that $\tilde{\mu} = \tilde{\mu}(z)$. Next, we need to compute the functions a, b. To this end,

$$[X, \xi] = \frac{\partial a}{\partial x}\frac{\partial}{\partial x} + \frac{\partial b}{\partial x}\frac{\partial}{\partial y},\quad [\tilde{\varphi}X, \xi] = \frac{\partial a}{\partial y}\frac{\partial}{\partial x} + \frac{\partial b}{\partial y}\frac{\partial}{\partial y}$$

And by Lemma 3.5, we obtain

$$[X, \xi] = (\alpha+\tilde{\lambda})\frac{\partial}{\partial x} + \frac{\tilde{\mu}}{2}\frac{\partial}{\partial y},\quad [\tilde{\varphi}X, \xi] = \frac{\tilde{\mu}}{2}\frac{\partial}{\partial x} + (\alpha-\tilde{\lambda})\frac{\partial}{\partial y}.$$

The comparison of these relations with the previous leads to

$$\frac{\partial a}{\partial x} = \alpha+\tilde{\lambda},\quad \frac{\partial a}{\partial y} = \frac{\tilde{\mu}}{2},\quad \frac{\partial b}{\partial x} = \frac{\tilde{\mu}}{2},\quad \frac{\partial b}{\partial y} = \alpha-\tilde{\lambda}. \tag{4.2}$$

By integration of these system, considering $\tilde{\lambda}, \tilde{\mu}$ functions depending only on z, we get $a = (\alpha + \tilde{\lambda})x + \frac{\tilde{\mu}}{2}y + f(z),\ b = \frac{\tilde{\mu}}{2}x - (\alpha - \tilde{\lambda})y - g(z), f(z), g(z)$ are arbitrary smooth functions of z.

We will continue calculate the tensor fields $\eta, \tilde{\varphi}, \tilde{g}$ and $\tilde{h}$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. The expression of the 1-form $\eta = dz$ immediately follows from $\eta(\xi) = 1, \eta(X) = \eta(\tilde{\varphi}X) = 0$. For the components of $\tilde{g}_{ij}$ of the pseudo-Riemannian metric and the components of the tensor field $\tilde{\varphi}$, the proof is the same with that of Theorem 4.1, we omit here. The components of the tensor field $\tilde{h}$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are given as follows:

$$\tilde{h}(\frac{\partial}{\partial x}) = \tilde{h}(X) = \tilde{\lambda}\tilde{\varphi}X = \tilde{\lambda}\frac{\partial}{\partial y},\ \tilde{h}(\frac{\partial}{\partial y}) = \tilde{h}\tilde{\varphi}X = -\tilde{\lambda}X = -\tilde{\lambda}\frac{\partial}{\partial x},$$

$$\tilde{h}(\frac{\partial}{\partial z}) = \tilde{h}(\xi - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}) = b\tilde{\lambda}\frac{\partial}{\partial x} - a\tilde{\lambda}\frac{\partial}{\partial y}.$$

Thus the matrix form of $\tilde{h}$ is given by

$$\begin{pmatrix} 0 & \tilde{\lambda} & b\tilde{\lambda} \\ -\tilde{\lambda} & 0 & -a\tilde{\lambda} \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Theorem 4.1 and Theorem 4.2 allow us to obtain a complete local classification of 3-dimensional almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-spaces with $\tilde{h}$ is of $\mathfrak{h}_1$ type or $\mathfrak{h}_3$ type and $d\tilde{\kappa} \wedge \eta = 0$. In fact, we can construct in $\mathbb{R}^3$ almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space for each of them as follows.

Let M be the open submanifold of $\mathbb{R}^3$ defined by $M := \{(x, y, z) \in \mathbb{R}^3\}$ and

$$\tilde{\lambda} = ce^{-(2\alpha+\tilde{\nu})z}, \tilde{\mu},\ f, g : M \to \mathbb{R}$$

be four smooth functions of z, where $\alpha, c, \tilde{\nu}$ are constant functions. Let us denote again by x, y, z the coordinates induced on M by the standard ones on $\mathbb{R}^3$. We consider on M

$$\xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z},\quad \eta = dz,$$

the pseudo-Riemannian metric $\tilde{g}$, the tensor fields $\tilde{\varphi}$ and $\tilde{h}$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are given by the relations:

$$\tilde{g} = \begin{pmatrix} -1 & 0 & a \\ 0 & 1 & -b \\ a & -b & 1 - a^2 + b^2 \end{pmatrix},\ \tilde{\varphi} = \begin{pmatrix} 0 & 1 & -b \\ 1 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix},\ \tilde{h} = \begin{pmatrix} \tilde{\lambda} & 0 & -a\tilde{\lambda} \\ 0 & -\tilde{\lambda} & b\tilde{\lambda} \\ 0 & 0 & 0 \end{pmatrix}.$$

where $a = \alpha x + (\frac{\tilde{\mu}}{2} - \tilde{\lambda})y + f(z),\ b = (\frac{\tilde{\mu}}{2} + \tilde{\lambda})x - \alpha y + g(z), \alpha$ is a constant value. It is easy to check that $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ ia an almost paracontact metric manifold. Since $d\eta = 0$ and $\Phi = -\frac{1}{2}dx \wedge dy + \frac{b}{2}dx \wedge dz - \frac{a}{2}dy \wedge dz$, thus we get $d\Phi = -\alpha dx \wedge dy \wedge dz = 2\alpha\eta \wedge \Phi$, that is to say, $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ ia an almost α-para-Kenmotsu manifold and that $\{X = \frac{\partial}{\partial x}, \tilde{\varphi}X = \frac{\partial}{\partial y}, \xi\}$ makes up a global $\tilde{\varphi}$-basis on M. Moreover, by direct computation, we get

$$[X, \tilde{\varphi}X] = 0, \quad [X, \xi] = \alpha X + (\tilde{\lambda} + \frac{\tilde{\mu}}{2})\tilde{\varphi}X, \quad [\tilde{\varphi}X, \xi] = (\frac{\tilde{\mu}}{2} - \tilde{\lambda})X + \alpha\tilde{\varphi}X.$$

and

$$\tilde{h}(X) = \tilde{h}(\frac{\partial}{\partial x}) = \tilde{\lambda}\frac{\partial}{\partial x} = \tilde{\lambda}X,\ \tilde{h}\tilde{\varphi}X = \tilde{h}(\frac{\partial}{\partial y}) = -\tilde{\lambda}\frac{\partial}{\partial y} = -\tilde{\lambda}\tilde{\varphi}X,\ \tilde{h}\xi = 0.$$

In this case $\tilde{h}$ is of $\mathfrak{h}_1$ type with respect to the $\tilde{\varphi}$-basis $\{X, \tilde{\varphi}X, \xi\}$. By the well-known formula

$$\begin{aligned} & 2\tilde{g}(\tilde{\nabla}_Z W, T) \\ = \ & Z\tilde{g}(W,T) + W\tilde{g}(T,Z) - T\tilde{g}(Z,W) - \tilde{g}(Z,[W,T]) + \tilde{g}(W,[T,Z]) + \tilde{g}(T,[Z,W]) \end{aligned}$$

and by (2.3), we obtain the following identities

$$\tilde{\nabla}_X\xi = \alpha X + \tilde{\lambda}\tilde{\varphi}X, \quad \tilde{\nabla}_{\tilde{\varphi}X}\xi = \alpha\tilde{\varphi}X - \tilde{\lambda}X, \quad \tilde{\nabla}_\xi X = -\frac{\tilde{\mu}}{2}\tilde{\varphi}X, \quad \tilde{\nabla}_\xi\tilde{\varphi}X = -\frac{\tilde{\mu}}{2}X,$$

$$\tilde{\nabla}_X X = \alpha\xi, \quad \tilde{\nabla}_{\tilde{\varphi}X}\tilde{\varphi}X = -\alpha\xi, \quad \tilde{\nabla}_{\tilde{\varphi}X}X = -\tilde{\lambda}\xi,\ \tilde{\nabla}_X\tilde{\varphi}X = -\tilde{\lambda}\xi.$$

By direct calculation we obtain

$$\tilde{R}(X,\xi)\xi = (\tilde{\lambda}^2 - \alpha^2)X + \tilde{\mu}\tilde{h}X + \tilde{\nu}\tilde{\varphi}\tilde{h}X,$$

$$\tilde{R}(\tilde{\varphi}X,\xi)\xi = (\tilde{\lambda}^2 - \alpha^2)\tilde{\varphi}X + \tilde{\mu}\tilde{h}\tilde{\varphi}X + \tilde{\nu}\tilde{\varphi}\tilde{h}\tilde{\varphi}X,$$

$$\tilde{R}(X,\tilde{\varphi}X)\xi = 0.$$

Therefore, for any Z, W on M, it holds

$$\tilde{R}(Z,W)\xi = (\tilde{\kappa}I + \tilde{\mu}\tilde{h} + \tilde{\nu}\tilde{\varphi}\tilde{h})(\eta(W)Z - \eta(Z)W),$$

and since $\tilde{\kappa} = \tilde{\lambda}^2 - \alpha^2 = c^2e^{-2(2\alpha+\tilde{\nu})z} - \alpha^2$, it satisfies $d\tilde{\kappa} \wedge \eta = 0$. In this way, we construct an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space with $\tilde{h}$ is of $\mathfrak{h}_1$ type and $d\tilde{\kappa} \wedge \eta = 0$.

If we consider on M

$$\xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad \eta = dz,$$

the pseudo-Riemannian metric $\tilde{g}$, the tensor fields $\tilde{\varphi}$ and $\tilde{h}$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are given by the relations:

$$\tilde{g} = \begin{pmatrix} -1 & 0 & a \\ 0 & 1 & -b \\ a & -b & 1-a^2+b^2 \end{pmatrix}, \ \tilde{\varphi} = \begin{pmatrix} 0 & 1 & -b \\ 1 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix}, \ \tilde{h} = \begin{pmatrix} \tilde{\lambda} & 0 & -a\tilde{\lambda} \\ 0 & -\tilde{\lambda} & b\tilde{\lambda} \\ 0 & 0 & 0 \end{pmatrix}.$$

where $a = (\alpha + \tilde{\lambda})x + \frac{\tilde{\mu}}{2}y + f(z),\ b = \frac{\tilde{\mu}}{2}x + (\alpha - \tilde{\lambda})y + g(z), \alpha$ is a constant value. It is also easy to check that $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is an almost α-para-Kenmotsu manifold and that $\{X = \frac{\partial}{\partial x}, \tilde{\varphi}X = \frac{\partial}{\partial y}, \xi\}$ makes up a global $\tilde{\varphi}$-basis on M. Moreover, by direct calculation, we get

$$[X, \tilde{\varphi}X] = 0, \quad [X, \xi] = (\alpha + \tilde{\lambda})X + \frac{\tilde{\mu}}{2}\tilde{\varphi}X, \quad [\tilde{\varphi}X, \xi] = \frac{\tilde{\mu}}{2}X + (\alpha - \tilde{\lambda})\tilde{\varphi}X.$$

and

$$\tilde{h}(X) = \tilde{h}(\frac{\partial}{\partial x}) = \tilde{\lambda}\frac{\partial}{\partial y} = \tilde{\lambda}\tilde{\varphi}X,\ \tilde{h}\tilde{\varphi}X = \tilde{h}(\frac{\partial}{\partial y}) = -\tilde{\lambda}\frac{\partial}{\partial x} = -\tilde{\lambda}X,\ \tilde{h}\xi = 0.$$

In this case $\tilde{h}$ is of $\mathfrak{h}_3$ type with respect to the $\tilde{\varphi}$-basis $\{X, \tilde{\varphi}X, \xi\}$.

By the well-known Koszul's formula and by (2.3), we obtain the following identities

$$\tilde{\nabla}_X\xi = (\alpha + \tilde{\lambda})X, \quad \tilde{\nabla}_{\tilde{\varphi}X}\xi = (\alpha - \tilde{\lambda})\tilde{\varphi}X, \quad \tilde{\nabla}_\xi X = -\frac{\tilde{\mu}}{2}\tilde{\varphi}X, \quad \tilde{\nabla}_\xi\tilde{\varphi}X = -\frac{\tilde{\mu}}{2}X,$$

$$\tilde{\nabla}_X X = (\alpha + \tilde{\lambda})\xi, \quad \tilde{\nabla}_{\tilde{\varphi}X}\tilde{\varphi}X = (\tilde{\lambda} - \alpha)\xi, \quad \tilde{\nabla}_{\tilde{\varphi}X}X = 0,\ \tilde{\nabla}_X\tilde{\varphi}X = 0.$$

After long but direct calculation we obtain

$$\tilde{R}(X, \xi)\xi = -(\tilde{\lambda}^2 + \alpha^2)X + \tilde{\mu}\tilde{h}X + \tilde{\nu}\tilde{\varphi}\tilde{h}X,$$

$$\tilde{R}(\tilde{\varphi}X, \xi)\xi = -(\tilde{\lambda}^2 + \alpha^2)\tilde{\varphi}X + \tilde{\mu}\tilde{h}\tilde{\varphi}X + \tilde{\nu}\tilde{\varphi}\tilde{h}\tilde{\varphi}X,$$

$$\tilde{R}(X, \tilde{\varphi}X)\xi = 0.$$

therefore, for any Z, W on M, it holds

$$\tilde{R}(Z, W)\xi = (\tilde{\kappa}I + \tilde{\mu}\tilde{h} + \tilde{\nu}\tilde{\varphi}\tilde{h})(\eta(W)Z - \eta(Z)W),$$

And since $\tilde{\kappa} = -(\tilde{\lambda}^2 + \alpha^2) = -c^2e^{-2(2\alpha+\tilde{\nu})z} - \alpha^2$, it satisfies $d\tilde{\kappa} \wedge \eta = 0$. In this way, we construct an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space with $\tilde{h}$ is of $\mathfrak{h}_3$ type and $d\tilde{\kappa} \wedge \eta = 0$.

5. Further Characterizations

Proposition 5.1. *Let $(M^3, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space with $\tilde{h}^2 \neq 0$ and $d\tilde{\kappa} \wedge \eta = 0$. Then the leaves of the canonical foliation of M are flat para-Kähler manifolds.*

Proof. Let M' be a leaf of $\mathcal{D}$ and $(J, <,>)$ be the induced almost para-Hermitain structure. M' is a para-Kähler manifold since it is almost para-Kähler manifold of dimension 2. In order to prove the flatness of $(M', <,>)$, we consider the Weingarten operator A of M', if $\tilde{h}$ is of $\mathfrak{h}_1$ type, then $AX = -\alpha X - \tilde{\varphi}\tilde{h}X = -(\alpha X + \tilde{\lambda}\tilde{\varphi}X)$ for a unit timelike vector field X such that $\tilde{h}X = \tilde{\lambda}X$ and using the Gauss equation, the sectional curvature K' of $<,>$ is given by $K'(X, \tilde{\varphi}X) = K(X, \tilde{\varphi}X) - (\alpha^2 + \tilde{\lambda}^2)$. By Lemma 3.4, we obtain $\tilde{R}(X, \tilde{\varphi}X)\tilde{\varphi}X = -(\alpha^2 + \tilde{\lambda}^2)X$, thus $K(X, \tilde{\varphi}X) = -(\alpha^2 + \tilde{\lambda}^2)\tilde{g}(X, X) = \alpha^2 + \tilde{\lambda}^2$. Therefore, we get $K'(X, \tilde{\varphi}X) = 0$. If $\tilde{h}$ is of $\mathfrak{h}_3$ type, then $AX = -\alpha X - \tilde{\varphi}\tilde{h}X = -(\alpha + \tilde{\lambda})X$ for the unit timelike vector field X such that $\tilde{h}X = \tilde{\lambda}\tilde{\varphi}X$, and using the Gauss equation, the sectional curvature K' of $<,>$ is given by $K'(X, \tilde{\varphi}X) = K(X, \tilde{\varphi}X) + \tilde{\lambda}^2 - \alpha^2$. By Lemma 3.5, we obtain $K(X, \tilde{\varphi}X) = \tilde{R}(X, \tilde{\varphi}X, \tilde{\varphi}X, X) = \alpha^2 - \tilde{\lambda}^2$. Therefore, we get $K'(X, \tilde{\varphi}X) = 0$. □

Remark 5.1. This conclusion is in accord with Corollary 3 of [9].

Proposition 5.2. *Let $(M^3, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space. If $\tilde{h}$ is of $\mathfrak{h}_1$ type, then*

$$\mathcal{L}_\xi \tilde{h} = -(2\alpha + \tilde{\nu})\tilde{h} + \tilde{\mu}\tilde{h}\tilde{\varphi} - 2\tilde{\lambda}^2\tilde{\varphi}. \tag{5.1}$$

If $\tilde{h}$ is of $\mathfrak{h}_3$ type, then

$$\mathcal{L}_\xi \tilde{h} = -(2\alpha + \tilde{\nu})\tilde{h} + \tilde{\mu}\tilde{h}\tilde{\varphi} + 2\tilde{\lambda}^2\tilde{\varphi}. \tag{5.2}$$

Proof. By (2.3) and (3.12), it is easy to get that

$$\mathcal{L}_\xi \tilde{h} = \tilde{\nabla}_\xi \tilde{h} + \tilde{h}(\tilde{\nabla}\xi) - (\tilde{\nabla}\xi)\tilde{h} = -(2\alpha + \tilde{\nu})\tilde{h} + \tilde{\mu}\tilde{h}\tilde{\varphi} - 2\tilde{h}^2\tilde{\varphi}.$$

Hence, If $\tilde{h}$ is of $\mathfrak{h}_1$ type, $\tilde{h}^2X = \tilde{\lambda}^2X$, If $\tilde{h}$ is of $\mathfrak{h}_3$ type, $\tilde{h}^2X = -\tilde{\lambda}^2X$, the relations (5.1) and (5.2) are easily obtained. □

Now we give the following further characterization.

Theorem 5.1. *Let $(M^3, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an almost paracontact metric manifold $\tilde{h}^2 \neq 0$, and $\tilde{\kappa}, \tilde{\mu}$ are smooth functions on M such that $d\tilde{\kappa} \wedge \eta = 0$. Then, M^3 is an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space if and only if for any point $p \in M$, there exists an open neighbourhood U of p with coordinates x_1, x_2, t such that $\tilde{\kappa}$ and $\tilde{\mu}$*

depend only on t *and the tensor fields of the structure are expressed in the following way:*

$$(5.3)\ \tilde{\varphi} = \sum_{i,j=1}^{2} \tilde{\varphi}_j^i dx_j \otimes \frac{\partial}{\partial x_i},\ \xi = \frac{\partial}{\partial t},\ \eta = dt,\ \tilde{g} = dt \otimes dt + \sum_{i,j=1}^{2} \tilde{g}_{ij} dx_i \otimes dx_j,$$

where $\tilde{\varphi}_j^i, \tilde{g}_{ij}$ *are functions only of* t*; The fundamental 2-form* Φ *is given by*

$$\Phi = e^{2t} dx_1 \wedge dx_2, \tag{5.4}$$

and the non-zero components $\tilde{h}_j^i, \tilde{B}_j^i$ *in* U *of* $\tilde{h}$ *and* $B := \tilde{\varphi}\tilde{h}$*, respectively, are functions of* t *satisfying the condition* $\sum_k B_k^i B_j^k = e^{-2(2\alpha+\tilde{\nu})t}\delta_j^i$ *and the following system of differential equations:*

$$\frac{d\tilde{\varphi}_j^i}{dt} = 2\tilde{h}_j^i,\ \frac{d\tilde{h}_j^i}{dt} = \mp 2\tilde{\lambda}^2 \tilde{\varphi}_j^i - (2\alpha + \tilde{\nu})\tilde{h}_j^i - \tilde{\mu}\tilde{B}_j^i, \tag{5.5}$$
$$\frac{d\tilde{B}_j^i}{dt} = -(2\alpha + \tilde{\nu})\tilde{B}_j^i - \tilde{\mu}\tilde{h}_j^i,$$

where $\tilde{\lambda} = e^{-(2\alpha+\tilde{\nu})t}$, and it takes " $-$ " if $\tilde{h}$ is of $\mathfrak{h}_1$ type, it takes " $+$ " if $\tilde{h}$ is of $\mathfrak{h}_3$ type.

Proof. Suppose that M carries a structure locally represented as in (5.3)-(5.5). Obviously $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ are followed by (5.3)-(5.4), therefore, M is an almost α-para-Kenmotsu manifold. Now we need to prove that M satisfies the $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-nullity condition. Notice that $X_1 = \frac{\partial}{\partial x_1}$ and $X_2 = \frac{\partial}{\partial x_2}$ are Killing vector fields and thus we get $\tilde{g}(\tilde{\nabla}_{X_i} X_j, X_k) = 0$ for any $i, j, k \in \{1, 2\}$. Since the distribution orthogonal to $\xi = \frac{\partial}{\partial t}$ is spanned by X_1 and X_2, it follows that $\tilde{\nabla}_{X_i} X_j \in [\xi]$ for all $i, j \in \{1, 2\}$. Consequently, for the Levi-Civita connection $\tilde{\nabla}$ determined by $\tilde{g}$, we obtain

$$(5.6)\tilde{\nabla}_{X_i} X_j = \tilde{\nabla}_{X_j} X_i = -\tilde{g}(X_i, \alpha X_j + BX_j)\xi,\quad \tilde{\nabla}_\xi X_i = \tilde{\nabla}_{X_i}\xi = \alpha X_i + BX_i.$$

Using (5.5) and (5.6) and by direct computations, we get

$$\tilde{R}(X_i, X_j)\xi = 0,$$

and

$$\begin{aligned}
\tilde{R}(X_i, \xi)\xi &= -\tilde{\nabla}_\xi \tilde{\nabla}_{X_i}\xi = -\alpha(\alpha X_i + BX_i) - [\frac{dB_i^k}{dt} X_k + B_i^k \tilde{\nabla}_\xi X_i] \\
&= -\alpha^2 X_i - 2\alpha BX_i + (2\alpha + \tilde{\nu})BX_i + \tilde{\mu}\tilde{h}X_i - B^2 X_i \\
&= (\tilde{h}^2 - \alpha^2 I)X_i + \tilde{\mu}\tilde{h}X_i + \tilde{\nu}\tilde{\varphi}\tilde{h}X_i.
\end{aligned}$$

If $\tilde{h}$ is of $\mathfrak{h}_1$ type, $\tilde{R}(X_i, \xi)\xi = (\tilde{\lambda}^2 - \alpha^2)X_i + \tilde{\mu}\tilde{h}X_i + \tilde{\nu}\tilde{\varphi}\tilde{h}X_i$. Thus, M^3 is an almost α-para-Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu} = const.)$-space, where $\tilde{\kappa} = \tilde{\lambda}^2 - \alpha^2$. If $\tilde{h}$ is of $\mathfrak{h}_3$ type,

$\tilde{R}(X_i,\xi)\xi = -(\tilde{\lambda}^2+\alpha^2)X_i+\tilde{\mu}\tilde{h}X_i+\tilde{\nu}\tilde{\varphi}\tilde{h}X_i$. Thus, M^3 is an almost α-para-Kenmotsu $(\tilde{\kappa},\tilde{\mu},\tilde{\nu} = const.)$-space, where $\tilde{\kappa} = -(\tilde{\lambda}^2+\alpha^2)$.

Suppose M^3 is an almost α-para-Kenmotsu $(\tilde{\kappa},\tilde{\mu},\tilde{\nu} = const.)$-space, we have (5.3)-(5.5) as similar as the proof of Theorem 6.1 in [16], we omit here. □

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