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APPLICATIONS OF INFINITE MATRICES IN NON-NEWTONIAN CALCULUS FOR PARANORMED SPACES AND THEIR TOEPLITZ DUALS

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Abstract. The main purpose of this paper is to construct some difference sequence spaces over the geometric complex numbers for an infinite matrix and Museilak-Orlicz function. We also make an effort to study some inclusion relations, topological and geometric properties of these spaces. An endeavor has been made to prove that these are Banach spaces. Furthermore, we compute the α-, β-, γ-dual of these spaces.

Keywords: geometric difference, Orlicz function, paranorm space, geometric complex numbers, non-Newtonian calculus, Köthe- Toeplitz duals

1. Introduction and Preliminaries

In the period from 1967 to 1972, Grossman and Katz [17] introduced the non-Newtonian calculus consisting of the branches of geometric, bigeometric, quadratic, biquadratic calculus and so forth. Also, Grossman in [18] extended this notion to other fields. All these calculi can be described simultaneously within the framework of general theory. We prefer to use the name non-Newtonian to indicate any of the calculi other than the classical calculus. Every property in classical calculus has an analogue in non-Newtonian calculus which is a methodology that allows one to have a different look at problems which can be investigated via calculus. In some cases, for example, for wage-rate (in dollars, euro, etc.) related problems, the use of bigeometric calculus which is a kind of non-Newtonian calculus is advocated instead of the traditional Newtonian one. Bashirov et al. [3] have recently focused on non-Newtonian calculus and gave the results with applications corresponding to the well-known properties of derivatives and integrals in classical calculus. Some authors have also worked on classical sequence spaces and related topics by using non-Newtonian calculus ([6], [29]).

Geometric calculus is an alternative to the usual calculus by Newton and Leibniz. It provides differentiation and integration tools based on multiplication instead of

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addition. Every property in Newtonian calculus has an analog in multiplicative calculus.
Kórus [11] studied some recent results concerning Λ^2-strong convergence of numerical sequences. He gave a new appropriate definition for the Λ^2-strong convergence. Moreover, Kórus [12] generalized the results on the L^1-convergence of Fourier series. In [13], he also studied the uniform convergence of mearurable functions by extended results of Móricz and gave examples for appropriate functions. Recently, Raj and Sharma [26] used the idea of Kórus [11] and study some applications of strongly convergent sequences to Fourier series by means of modulus function.
Let w, l_∞, c and c_0 be the classical sequence spaces of all, bounded, convergent and null sequences respectively, normed by $\|x\|_\infty = \sup_k |x_k|$ and $\mathbb{C}(G)$ be the set of geometric complex numbers [30].

The notion of difference sequence spaces was introduced by Kızmaz [19], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [16] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Later the concept have been studied by Bektaş et al. [3] and Et et al. [15]. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [32] who studied the spaces $l_\infty(\Delta_n)$, $c(\Delta_n)$ and $c_0(\Delta_n)$. Recently, Esi et al. [8] and Tripathy et al. [31] have introduced a new type of generalized difference operators and unified those as follows.
Let n, m be non-negative integers, then for Z a given sequence space, we have

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\Delta_n^m x = (\Delta^m n_k x_k) = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $(\Delta_n^0 x_k) = (n_k x_k) = (x_k)$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+nv}.$$

Taking $n = 1$, we get the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ studied by Et and Çolak [16]. Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz [19].

Türkmen and Başar [30] defined the geometric complex numbers $\mathbb{C}(G)$ as follows:

$$\mathbb{C}(G) = \{e^z : z \in \mathbb{C}\} = \mathbb{C} \setminus \{0\}.$$

Then $(\mathbb{C}(G), \oplus, \odot)$ is a field with geometric zero 1 and geometric identity exponential e. They have also proved $w(G) = \{(x_k) : x_k \in \mathbb{C}(G) \text{ for all } k \in \mathbb{N}\}$ is a vector space over $\mathbb{C}(G)$ with the algebric operations $\oplus$ addition and $\odot$ multiplication

$$\oplus : w(G) \times w(G) \to w(G)$$

$$(x, y) \to x \oplus y = (x_k) \oplus (y_k) = (x_k y_k)$$

$$\odot : \mathbb{C}(G) \times w(G) \rightarrow w(G)$$
$$(\alpha, y) \rightarrow \alpha \odot y = \alpha \odot (y_k) = (\alpha^{\ln y_k}),$$

where $x = (x_k), y = (y_k) \in w(G)$ and $\alpha \in \mathbb{C}(G)$. Further, these results have been generalized and studied by K. Boruah and et.al [5].

Lemma 1.1. *[30] (Triangle inequality) Let $x, y \in \mathbb{C}(G)$. Then*

$$|x \oplus y|_G \leq |x|_G \oplus |y|_G. \tag{1.1}$$

Lemma 1.2. *[30] (Minkowski's inequality) Let $p \geq 1$ and $a_k, b_k \in \mathbb{C}(G)$ with $a_k = e^{c_k}, b_k = e^{d_k}$ for $k \in \{1, 2, ..., n\}$. Then*

$$\sqrt[p]{{}_G\sum_{k=1}^{n} |a_k \oplus b_k|_G^{p_G}}^{\,G} \leq \sqrt[p]{{}_G\sum_{k=1}^{n} |a_k|_G^{p_G}}^{\,G} \oplus \sqrt[p]{{}_G\sum_{k=1}^{n} |b_k|_G^{p_G}}^{\,G}. \tag{1.2}$$

Let $A = (a_{nk})$ be an infinite matrix of real numbers and $x = (x_k) \in \omega$ be an infinite sequence. Then we obtain the sequence $(Ax)_n$, denoted by A-transform of x, as

$$Ax = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1k} & ... \\ a_{21} & a_{22} & ... & a_{2k} & ... \\ \vdots & \vdots & ... & \vdots & ... \\ a_{n1} & a_{n2} & ... & a_{nk} & ... \\ \vdots & \vdots & ... & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 & + a_{12}x_2 & + a_{13}x_3 & + ... \\ a_{21}x_1 & + a_{22}x_2 & + a_{23}x_3 & + ... \\ & \vdots & & \\ a_{n1}x_1 & + a_{n2}x_2 & + a_{n3}x_3 & + ... \\ & \vdots & & \end{pmatrix} \begin{pmatrix} (Ax)_1 \\ (Ax)_2 \\ \vdots \\ (Ax)_n \\ \vdots \end{pmatrix}$$

In this case, we transform the sequence x into the sequence $Ax = \{(Ax)_n\}$ with

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N}) \tag{1.3}$$

provided the series on the right hand side of (1.3) converges for each n.
Let X and Y be any two sequence spaces. If Ax exists and is in Y for every sequence $x = (x_k) \in X$, then we say that A defines a matrix transformation from X into Y, that is, $A : X \rightarrow Y$. By $(X : Y)$, we denote the class of all matrices A from X into Y.

Definition 1.1. An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. If convexity of Orlicz function is replaced by $M(x+y) \leq M(x)+M(y)$, then this function is called modulus function. Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

is known as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also it was shown in [10] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M, is a right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$. For more details (see [7], [22], [23], [25], [27], [28]) and references therein.

Definition 1.2. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be Musielak-Orlicz function (see [20, 24]). A Musielak-Orlicz function $\mathcal{M} = (M_k)$ is said to satisfy Δ_2-*condition* if there exist constants a, $K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in l_+^1$ (the positive cone of l^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^+$, whenever $M_k(u) \leq a$.

Definition 1.3. Let X be a linear space. A function $p : X \to \mathbb{R}$ is called paranorm, if
$(PN1)$ $p(x) \geq 0$ for all $x \in X$,
$(PN2)$ $p(-x) = p(x)$ for all $x \in X$,
$(PN3)$ $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
$(PN4)$ if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.
A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [33] Theorem 10.4.2, pp. 183).

Let $w(G)$ denote the set of all sequences over the geometric complex field $\mathbb{C}(G)$. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. In present paper we define the following classes of sequences:

$l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] =$

$$\Big\{x = (x_k) \in w(G) : \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\},$$

$c[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] =$

$$\Big\{x = (x_k) \in w(G) : {}^G\lim_{k\to\infty} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k \ominus l|_G}{\rho}\Big)\Big]^{p_k} = 1, \text{ for some } l \text{ and } \rho > 0\Big\},$$

$c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] =$

$$\Big\{x = (x_k) \in w(G) : {}^G\lim_{k\to\infty} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k} = 1, \text{ for some } \rho > 0\Big\},$$

where $m, n \in \mathbb{N}$ and

$$\begin{aligned}
{}_G\Delta_n^0 x &= ({}_G\Delta_n^0 x_k) = (x_k)\\
{}_G\Delta_n x &= ({}_G\Delta_n x_k) = (x_k \ominus x_{k+1})\\
{}_G\Delta_n^2 x &= ({}_G\Delta_n^2 x_k) = ({}_G\Delta_n x_k \ominus {}_G\Delta_n x_{k+1})\\
&= (x_k \ominus x_{k+1} \ominus x_{k+1} \oplus x_{k+2})\\
&= (x_k \ominus e^2 \odot x_{k+1} \oplus x_{k+2})\\
{}_G\Delta_n^3 x &= ({}_G\Delta_n^3 x_k) = ({}_G\Delta_n^2 x_k \ominus {}_G\Delta_n^2 x_{k+1})\\
&= (x_k \ominus e^3 \odot x_{k+1} \oplus e^3 \odot x_{k+2} \ominus x_{k+3})\\
\dots\dots\dots & \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
{}_G\Delta_n^m x &= ({}_G\Delta_n^m x_k) = ({}_G\Delta_n^{m-1} x_k \ominus {}_G\Delta_n^{m-1} x_{k+1})\\
&= \Big({}_G\sum_{v=0}^{m} (\ominus e)^{v_G} \odot e^{\binom{m}{v}} \odot x_{k+nv}\Big), \text{ with } (\ominus e)^{0_G} = e.
\end{aligned}$$

If $\mathcal{M} = M_k(x) = x$ for all $k \in \mathbb{N}$, then above sequence spaces reduces to $l_\infty[G, {}_G\Delta_n^m, A, p, u]$, $c[G, {}_G\Delta_n^m, A, p, u]$ and $c_0[G, {}_G\Delta_n^m, A, p, u]$.

By taking $p = (p_k) = 1$, for all k then we get the sequence spaces $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, u]$, $c[G, {}_G\Delta_n^m, A, \mathcal{M}, u]$ and $c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, u]$.

The following inequality will be used throughout the paper. If $0 \le p_k \le \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k \oplus b_k|_G^{p_k} \le K\{|a_k|_G^{p_k} \oplus |b_k|_G^{p_k}\} \tag{1.4}$$

for all k and $a_k, b_k \in \mathbb{C}(G)$. Also $|a|_G^{p_k} \le \max(1, |a|_G^H)$ for all $a \in \mathbb{C}(G)$.

In this paper, we first give a description of some new difference sequence spaces for an infinite matrix and Musielak-Orlicz function over the geometric complex field which forms a Banach space with the norm defined on it. We investigate some topological properties of these sequence spaces and establish some inclusion relations concerning these spaces. Furthermore, we devote the final section of the paper to compute their algebraic duals such as the $\alpha-, \beta-, \gamma$-duals.

2. Main Results

In this section we study some topological properties and some inclusion relations between the sequence spaces which we have defined above.

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$, $c[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ and $c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ are linear spaces over the field $\mathbb{C}(G)$ of geometric complex numbers.*

Proof. We shall prove the assertion for $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ only and the others can be proved similarly. Let $x = (x_k)$ and $y = (y_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ and $\alpha, \beta \in \mathbb{C}(G)$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_1}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho_1 > 0$$

and

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m y_k|_G}{\rho_2}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho_2 > 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is a non-decreasing and convex so by using inequality (1.4), we have

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m(\alpha x_k \oplus \beta y_k)|_G}{\rho_3}\Big)\Big]^{p_k}\\
&= \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m \alpha x_k|_G}{\rho_3}\Big) \oplus M_k\Big(\frac{|u_{kG}\Delta_n^m \beta y_k|_G}{\rho_3}\Big)\Big]^{p_k}\\
&\leq K\sup_{k\in\mathbb{N}} \frac{1}{2^{p_k}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_1}\Big)\Big]^{p_k} \oplus K\sup_{k\in\mathbb{N}} \frac{1}{2^{p_k}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m y_k|_G}{\rho_2}\Big)\Big]^{p_k}\\
&\leq K\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_1}\Big)\Big]^{p_k} \oplus K\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m y_k|_G}{\rho_2}\Big)\Big]^{p_k}\\
&< \infty.
\end{aligned}$$

Therefore, $(\alpha x \oplus \beta y) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. This proves that $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ is a linear space. Similarly, we can prove that $c[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ and $c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ are linear spaces. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ is a paranormed space with paranormed defined by*

$$g(x)=\inf\left\{(\rho)^{(p_k \oslash M)_G} : \Big(\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k}\Big)^{(1\oslash M)_G} \leq 1, \text{ for some } \rho>0\right\},$$

where $0 < p_k \leq \sup p_k = H < \infty$ and $M = \max(1, H)$.

Proof. (i) Clearly $g(x) \geq 0$ for $x = (x_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. Since $M_k(0) = 0$, we get $g(0) = 0$.
(ii) $g(-x) = g(x)$.
(iii) Let $x = (x_k)$ and $y = (y_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$, then there exist positive numbers ρ_1 and ρ_2 such that

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_1}\Big)\Big]^{p_k} \leq 1$$

and

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_2}\Big)\Big]^{p_k} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality (1.2), we have

$$\begin{aligned}
\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m (x_k \oplus y_k)|_G}{\rho}\Big)\Big]^{p_k} &= \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m (x_k \oplus y_k)|_G}{\rho_1+\rho_2}\Big)\Big]^{p_k}\\
&\leq \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_1+\rho_2}\Big)\Big]^{p_k}\\
&\oplus \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m y_k|_G}{\rho_1+\rho_2}\Big)\Big]^{p_k}\\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_1}\Big)\Big]^{p_k}\\
&\oplus \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m y_k|_G}{\rho_2}\Big)\Big]^{p_k}\\
&\leq 1
\end{aligned}$$

and thus,

$$\begin{aligned}
g(x\oplus y) &= \inf\Big\{(\rho)^{(p_k \oslash M)_G} : \Big(\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m (x_k \oplus y_k)|_G}{\rho}\Big)\Big]^{p_k}\Big)^{(1\oslash M)_G} \leq 1\Big\}\\
&\leq \inf\Big\{(\rho_1)^{(p_k \oslash M)_G} : \Big(\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_1}\Big)\Big]^{p_k}\Big)^{(1\oslash M)_G} \leq 1\Big\}\\
&\oplus \inf\Big\{(\rho_2)^{(p_k \oslash M)_G} : \Big(\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m y_k|_G}{\rho_2}\Big)\Big]^{p_k}\Big)^{(1\oslash M)_G} \leq 1\Big\}.
\end{aligned}$$

Therefore, $g(x \oplus y) \leq g(x) \oplus g(y)$. Finally, we prove that the scalar multiplication is continuous. Let λ be any geometric complex number. By definition,

$$\begin{aligned} g(\lambda \odot x) &= \inf\Big\{(\rho)^{(p_k \oslash M)_G} : \Big(\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m(\lambda \odot x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big)^{(1\oslash M)_G} \leq 1\Big\} \\ &= \inf\Big\{(|\lambda|t)^{(p_k \oslash M)_G} : \Big(\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k}\Big)^{(1\oslash M)_G} \leq 1\Big\} \end{aligned}$$

where $t = \frac{\rho}{|\lambda|_G} > 0$. Since $|\lambda|_G^{p_k} \leq \max(1, |\lambda|_G^{\sup p_k})$, we have

$$g(\lambda\odot x) \leq \max(1, |\lambda|_G^{\sup p_k}) \inf\Big\{t^{(p_k \oslash M)_G} : \Big(\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k}\Big)^{(1\oslash M)_G} \leq 1\Big\}.$$

So, the fact that the scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

Theorem 2.3. *If $0 < p_k < q_k < \infty$ for each k, then we have $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] \subset l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, q, u]$.*

Proof. Let $x = (x_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. Then there exists $\rho > 0$ such that

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k} < \infty,$$

This implies that $a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k} < 1$ for sufficiently large values of k. Since M_k is non-decreasing, we get

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{q_k} \leq \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k} < \infty.$$

Thus, $x = (x_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, q, u]$. This completes the proof. □

Theorem 2.4. *Suppose $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the following inclusions hold:*
(i) If $0 < \inf p_k \leq p_k \leq 1$ then $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] \subset l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, u]$,
(ii) If $1 < p_k \leq \sup p_k < \infty$ then $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, u] \subset l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$.

Proof. (i) Let $x = (x_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. Since $0 < \inf p_k \leq p_k \leq 1$, we obtain the following

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big] \leq \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G)}{\rho}\Big)\Big]^{p_k} < \infty,$$

and hence, $x = (x_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, u]$.
(ii) Let $p_k \geq 1$ for each k and $\sup p_k < \infty$. Let $x = (x_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, u]$. Then for each $0 < \epsilon < 1$ there exists a positive integer N such that

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big] \leq \epsilon < 1 \text{ for all } k \in N.$$

This implies that

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k} \le \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big] < \infty.$$

Therefore, $x = (x_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. This completes the proof. □

Theorem 2.5. *If* $0 < h = \inf p_k \le p_k \le \sup p_k = H < \infty$. *Let* $\mathcal{M} = (M_k)$ *and* $\mathcal{M}' = (M_k')$ *be two Musielak-Orlicz functions satisfying* Δ_2*-condition, then we have*

$$l_\infty(G, {}_G\Delta_n^m, A, \mathcal{M}', p, u) \subset l_\infty(G, {}_G\Delta_n^m, A, \mathcal{M}o\mathcal{M}', p, u).$$

Proof. Let $x = (x_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. Then we have,

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k'\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0$$

Let $\epsilon > 0$ and choose $\delta > 0$ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \le t \le \delta$. Then

Let $y_k = a_{nk}\Big[M_k'\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]$ for all $k \in \mathbb{N}$ and consider

$${}_G\sum_k [M_k(y_k)]^{p_k} = {}_G\sum_1 [M_k(y_k)]^{p_k} + {}_G\sum_2 [M_k(y_k)]^{p_k},$$

where the first summation is over $y_k \le \delta$ and second summation is over $y_k > \delta$. Since $\mathcal{M} = (M_k)$ continuous, so we have

$$(2.1) \qquad {}_G\sum_1 [M_k(y_k)]^{p_k} < \epsilon^H$$

and for $y_k > \delta$, we use the fact that $y_k < \frac{y_k}{\delta} \le 1 + \frac{y_k}{\delta}$.
By the definition, we have for $y_k > \delta$

$$M_k(y_k) < 2M_k(1)\frac{y_k}{\delta}$$

Hence

$$(2.2) \qquad {}_G\sum_2 [M_k(y_k)]^{p_k} \le \max(1, (2M_k(1)\delta^{-1}))_G \sum_k [y_k]^{p_k}.$$

From equation (2.1) and (2.2), we have

$$l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}', p, u,] \subset l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}o\mathcal{M}', p, u].$$

□

Theorem 2.6. *Let* $\mathcal{M} = (M_k)$ *be a Musielak-Orlicz function and* $\beta = \lim_{t\to\infty} \frac{M_k(t)}{t} > 0$. *Then* $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] = l_\infty[G, {}_G\Delta_n^m, A, p, u]$.

Proof. In order to prove that $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] = l_\infty[G, {}_G\Delta_n^m, A, p, u]$. It is sufficient to show that $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] \subseteq l_\infty[G, {}_G\Delta_n^m, A, p, u]$. Now, let $\beta > 0$. By definition of β, we have $M_k(t) \geq \beta t$ for all $t \geq 0$. Since $\beta > 0$, we have $t \leq \frac{1}{\beta} M_k(t)$ for all $t \geq 0$.
Let $x = (x_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. Then, we have

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k} \leq \frac{1}{\beta}\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k} < \infty.$$

which implies that $x = (x_k) \in l_\infty[G, {}_G\Delta_n^m, A, p, u]$. This completes the proof. □

Theorem 2.7. *For a Musielak-Orlicz function $\mathcal{M} = (M_k)$, $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then*
(i) $c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] \subset l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$,
(ii) $c[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] \subset l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$.

Proof. The proof is easy so we omit it. □

Theorem 2.8. *Let $\mathcal{M}' = (M_k')$ and $\mathcal{M}'' = (M_k'')$ are two Musielak-Orlicz functions,*

$$l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}', p, u] \cap l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}'', p, u] \subset l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}' + \mathcal{M}'', p, u].$$

Proof. Let $x = (x_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}', p, u] \cap l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}'', p, u]$. Then

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k'\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_1}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho_1 > 0$$

and

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k''\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_2}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho_2 > 0.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The result follows from the inequality,

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} a_{nk}\Big[(M_k' + M_k'')\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k} \\
&= \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k'\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_1}\Big)\Big]^{p_k} + \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k''\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_2}\Big)\Big]^{p_k} \\
&\leq K\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k'\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_1}\Big)\Big]^{p_k} + K\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k''\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_2}\Big)\Big]^{p_k} \\
&< \infty.
\end{aligned}$$

Thus, $\sup_{k\in\mathbb{N}} a_{nk}\Big[(M_k' + M_k'')\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k} < \infty$. Therefore,
$x = (x_k) \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}' + \mathcal{M}'', p, u]$. This completes the proof. □

Theorem 2.9. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz functions. Then*
(i) $X[G, {}_G\Delta_n^m, \mathcal{M}, p, u] \subset X[G, {}_G\Delta_n^{m+1}, A, \mathcal{M}, p, u]$ *and the inclusion is strict, for* $X = l_\infty, c$ *and* c_0.
(ii) $c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] \subset c[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] \subset l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$.

Proof. (i) We give the proof for $X = l_\infty$ only. Let $x \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. Since

$$\begin{aligned} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^{m+1}x_k|_G}{\rho}\Big)\Big]^{p_k} &\leq a_{nk}\Big[M_k\Big(\frac{|u_{kG}(\Delta_n^m x_k \ominus \Delta_n^m x_{k+1})|_G}{\rho}\Big)\Big]^{p_k} \\ &\leq a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho_1}\Big)\Big]^{p_k} \\ &\oplus a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_{k+1}|_G}{\rho}\Big)\Big]^{p_k} \end{aligned}$$

we obtain $x \in l_\infty[G, {}_G\Delta_n^{m+1}, A, \mathcal{M}, p, u]$. For $A = (C, 1)$, $M_k(x) = x$, $p_k = 1$, $u_k = 1$ for all $k \in \mathbb{N}$, this inclusion is strict since the sequence $x = (e^{k^m})$ belongs to $l_\infty[G, {}_G\Delta_n^{m+1}, A, \mathcal{M}, p, u]$ but does not belong to $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$, where $n = (e^k)$.
(ii) The proof is trivial. □

Theorem 2.10. *The spaces* $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$, $c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ *and* $c[G, {}_G\Delta_n^m,$
$A, \mathcal{M}, p, u]$ *are normed linear spaces with norm*

$$\|x\|_G^{{}_G\Delta_n^m, A, \mathcal{M}, p, u} = {}_G\sum_{i=1}^m |x_i|_G \oplus \Big\|a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k}\Big\|_{G_\infty}.$$

Proof. It can be easily proved so we omit it. □

Theorem 2.11. *The spaces* $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$, $c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ *and* $c[G, {}_G\Delta_n^m,$
$A, \mathcal{M}, p, u]$ *are Banach spaces with norm*

$$\|x\|_G^{{}_G\Delta_n^m, A, \mathcal{M}, p, u} = {}_G\sum_{i=1}^m |x_i|_G \oplus \Big\|a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k}\Big\|_{G_\infty}.$$

Proof. Since the proof is similar for the space $c[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ and
$c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$, we prove the theorem only for $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. Let (x_j) be a Cauchy sequences in $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$, where
$x_j = (x_i^{(j)}) = (x_1^{(j)}, x_2^{(j)}, x_3^{(j)}, ...)$ for $j \in \mathbb{N}$ and $x_k^{(j)}$ is the k^{th} coordinate of x_j. Then
(2.3)

$$\|x_j \ominus x_l\|_G^{{}_G\Delta_n^m, A, \mathcal{M}, p, u} = {}_G\sum_{i=1}^m |x_i^{(j)} \ominus x_i^{(l)}|_G \oplus \Big\|a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m (x_j \ominus x_l)|_G}{\rho}\Big)\Big]^{p_k}\Big\|_{G_\infty}$$

$$= {}_G\sum_{i=1}^{m} |x_i^{(j)} \ominus x_i^{(l)}|_G \oplus \sup_{k\in\mathbb{N}} \Big\{ a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m(x_j \ominus x_l)|_G}{\rho}\Big)\Big]^{p_k}\Big\}$$

$$\to 1 \text{ as } l, j \to \infty.$$

This implies that $|x_k^{(j)} \ominus x_k^{(l)}|_G \to 1$ as $l, j \to \infty$ for each $k \in \mathbb{N}$. Therefore, $(x_k^{(j)}) = (x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ...)$ is a Cauchy sequence in $\mathbb{C}(G)$. Then by completeness of $\mathbb{C}(G)$, $(x_k^{(j)})$ is convergent. Let us suppose ${}^G\lim_{n\to\infty} x_k^{(j)} = x_k$, for each $k \in \mathbb{N}$. Since (x_j) is a Cauchy sequence, for each $\epsilon > 1$, there exists $N = N(\epsilon)$ such that $\|x_j \ominus x_l\|_G^{G\Delta_n^m, A, \mathcal{M}, p, u} < \epsilon$ for all $j, l \geq N$. Hence, from equation (2.3) we have

$${}_G\sum_{i=1}^{m} |x_i^{(j)} \ominus x_i^{(l)}|_G < \epsilon$$

and

$$a_{nk}\left[M_k\left(\frac{|u_{k_G}\sum_{v=0}^{m}(\ominus e)^{v_G} \odot e^{\binom{m}{v}} \odot (x_{k+nv}^{(j)} \ominus x_{k+nv}^{(l)})|_G}{\rho}\right)\right]^{p_k} < \epsilon$$

for all $k \in \mathbb{N}$ and $j, l \geq N$ we have

$${}^G\lim_{l\to\infty} {}_G\sum_{i=1}^{m} |x_i^{(j)} \ominus x_i^{(l)}|_G = {}_G\sum_{i=1}^{m} |x_i^{(j)} \ominus x_i|_G < \epsilon \text{ and}$$

${}^G\lim_{l\to\infty} a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m(x_k^{(j)} \ominus x_k^{(l)})|_G}{\rho}\Big)\Big]^{p_k} = a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m(x_k^{(j)} \ominus x_k)|_G}{\rho}\Big)\Big]^{p_k} <$ ϵ for all $n \geq N$. This implies $\|x_j \ominus x\|_G^{G\Delta_n^m, \mathcal{M}, A, p, u} < \epsilon^2$ for all $n \geq N$, that is $x_j \xrightarrow{G} x$ as $j \to \infty$, where $x = (x_k)$. Now we must show that $x \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. We have

$$\begin{aligned}
&a_{nk}\Big[M_k\Big(\tfrac{|u_{kG}\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k} \\
&= a_{nk}\left[M_k\left(\frac{|u_{k_G}\sum_{v=0}^{m}(\ominus e)^{v_G} \odot e^{\binom{m}{v}} \odot x_{k+nv}|_G}{\rho}\right)\right]^{p_k} \\
&= a_{nk}\left[M_k\left(\frac{|u_{k_G}\sum_{v=0}^{m}(\ominus e)^{v_G} \odot e^{\binom{m}{v}} \odot (x_{k+nv} \ominus x_{k+nv}^N \oplus x_{k+nv}^N)|_G}{\rho}\right)\right]^{p_k} \\
&\leq a_{nk}\left[M_k\left(\frac{|u_{k_G}\sum_{v=0}^{m}(\ominus e)^{v_G} \odot e^{\binom{m}{v}} \odot (x_{k+nv}^N \ominus x_{k+nv})|_G}{\rho}\right)\right]^{p_k} \\
&\oplus a_{nk}\left[M_k\left(\frac{|u_{k_G}\sum_{v=0}^{m}(\ominus e)^{v_G} \odot e^{\binom{m}{v}} \odot x_{k+nv}^N|_G}{\rho}\right)\right]^{p_k} \\
&\leq \|x^N \ominus x\|_G^{G\Delta_n^m, A, \mathcal{M}, p, u} \oplus a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^m x_k^N|_G}{\rho}\Big)\Big]^{p_k} = \mathrm{O}(e).
\end{aligned}$$

Therefore, $x \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. Hence $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ is a Banach space. $\square$

Furthermore, since $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$, $c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ and $c[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ are Banach spaces with continuous coordinates, i.e., $\|x_j \ominus x\|_G^{{}_G\Delta_n^m, A, \mathcal{M}, p, u} \to 1$ implies $|x_k^{(j)} \ominus x_k|_G \to 1$ for each $k \in \mathbb{N}$ as $n \to \infty$, these are also BK-spaces.

Let us define the operator

$$D : X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] \to X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] \text{ by}$$

$Dx = (1, 1..., x_{m+1}, x_{m+2}, ...),$, where $x = (x_1, x_2, ..., x_m, ...)$. It is trivial that D is a bounded linear operator on $X[G, {}_G\Delta_n^m, \mathcal{M}, A, p, u], X = l_\infty, c$ and c_0. Furthermore, the set

$$\begin{aligned} &D\Big[X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big] = DX[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] \\ &= \{x = (x_k) : x \in X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u], x_1 = x_2 = ... = x_m = 1\} \end{aligned}$$

is a subspace of $X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ and normed by

$$\begin{aligned} \|x\|_G^{{}_G\Delta_n^m, A, \mathcal{M}, p, u} &= |x_1|_G \oplus |x_2|_G \oplus ... \oplus |x_m|_G \oplus \Big\|a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k}\Big\|_{G_\infty} \\ &= 1 \oplus 1 \oplus ... \oplus 1 \oplus \Big\|a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k}\Big\|_{G_\infty} \\ &= \Big\|a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k}\Big\|_{G_\infty} \\ &= \Big\|a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k}\Big\|_{G_\infty} \in DX[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]. \end{aligned}$$

$DX[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ and $X[G, A, \mathcal{M}, p, u]$ are equivalent as topological spaces [21] since

$$ {}_G\Delta_n^m : DX[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u] \to X[G, A, \mathcal{M}, p, u] \text{ defined by}$$

$$ {}_G\Delta_n^m x = y = ({}_G\Delta_n^m x_k) = ({}_G\Delta_n^{m-1} x_k \ominus {}_G\Delta_n^{m-1} x_{k+1}),$$

is a linear homomorphism.

3. The $\alpha-, \beta-$ and $\gamma-$ duals of the spaces $l_\infty[G, {}_G\Delta_n^m, \mathcal{M}, p, u]$, $c[G, {}_G\Delta_n^m, \mathcal{M}, p, u]$ and $c_0[G, {}_G\Delta_n^m, \mathcal{M}, p, u]$

The aim here lies in this section is to determine the Köthe-Toeplitz duals of the classical sequence spaces.

Definition 3.1. ([9], [14], [21]) Let X be a sequence space and one can define

$$\begin{aligned} X^\alpha &= \Big\{b=(b_k): \sum_{k=1}^{\infty} |b_k x_k| < \infty, \text{ for each } x \in X\Big\}, \\ X^\beta &= \Big\{b=(b_k): \sum_{k=1}^{\infty} b_k x_k \text{ is convergent, for each } x \in X\Big\}, \\ X^\gamma &= \Big\{b=(b_k): \sup_n \Big|\sum_{k=1}^{n} b_k x_k\Big| < \infty, \text{ for each } x \in X\Big\}. \end{aligned}$$

Then X^α, X^β and X^γ are called $\alpha-$dual (or Köthe-Toeplitz dual), $\beta-$dual and $\gamma-$dual spaces of X, respectively. Then $X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$, then $Y^\eta \subset X^\eta$ for $\eta = \alpha, \beta$ or γ. It is clear that $X = X^{\alpha\alpha}$ then X is called an α-space. In particular, an α-space is a Köthe space or perfect sequence space.

Lemma 3.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz functions, $u = (u_k)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the following conditions are equivalent*

(i) $\sup_{k\in\mathbb{N}} \Big\{a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_k \ominus {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$ *for some* $\rho > 0$,

(ii)(a) $\sup_{k\in\mathbb{N}} \Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-1}} \odot {}_G\Delta_n^{m-1}x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$ *for some* $\rho > 0$,

(b) $\sup_{k\in\mathbb{N}} \Big\{a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_k \ominus e^{k(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty.$

Proof. Let (i) be true, that is, $\sup_{k\in\mathbb{N}} \Big\{a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_k \ominus {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$.

Now, $a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_1 \ominus_G \Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}$

$$\begin{aligned} &= a_{nk}\Big[M_k\Big(\frac{|u_k \sum_{l=1}^{k}({}_G\Delta_n^{m-1}x_l \ominus {}_G\Delta_n^{m-1}x_{l+1})|_G}{\rho}\Big)\Big]^{p_k} \\ &= a_{nk}\Big[M_k\Big(\frac{|u_k \sum_{l=1}^{k} {}_G\Delta_n^{m}x_l|_G}{\rho}\Big)\Big]^{p_k} \\ &\le \sum_{l=1}^{k} a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^{m}x_l|_G}{\rho}\Big)\Big]^{p_k} = \mathrm{O}(e^k) \end{aligned}$$

and

$$a_{nk}\Big[M_k\Big(\frac{|u_{k\,G}\Delta_n^{m-1}x_k|_G}{\rho}\Big)\Big]^{p_k}$$
$$= a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_1 \ominus {}_G\Delta_n^{m-1}x_1 \oplus {}_G\Delta_n^{m-1}x_{k+1} \oplus {}_G\Delta_n^{m-1}x_k \ominus {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}$$
$$\leq\ a_{nk}\Big[M_k\Big(\frac{|u_{k\,G}\Delta_n^{m-1}x_1|_G}{\rho}\Big)\Big]^{p_k} \oplus a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_1 \ominus {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}$$
$$\oplus\quad a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_k \ominus {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k} =\ \mathrm{O}(e^k).$$

Therefore, $\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-1}} \odot {}_G\Delta_n^{m-1}x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$. This completes the proof of $(ii)(a)$.
Again, we have

$$\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_k \ominus e^{k(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}\Big\}$$
$$= a_{nk}\Big[M_k\Big(\frac{|u_k(\{e^{(k+1)} \odot e^{(k+1)^{-1}}\} \odot {}_G\Delta_n^{m-1}x_k \ominus e^{k(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}$$
$$= a_{nk}\Big[M_k\Big(\frac{|u_k(\{(e^k \oplus e) \odot e^{(k+1)^{-1}}\} \odot {}_G\Delta_n^{m-1}x_k \ominus e^{k(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}$$
$$= a_{nk}\Big[M_k\Big(\frac{|u_k(\{e^{k(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_k \oplus e^{(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_k\} \ominus e^{k(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}$$
$$= a_{nk}\Big[M_k\Big(\frac{|u_k(\{e^{k(k+1)^{-1}} \odot ({}_G\Delta_n^{m-1}x_k \ominus {}_G\Delta_n^{m-1}x_{k+1})\} \oplus \{e^{(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_k\})|_G}{\rho}\Big)\Big]^{p_k}$$
$$\leq a_{nk}\Big[M_k\Big(\frac{|u_k\{e^{k(k+1)^{-1}} \odot ({}_G\Delta_n^{m-1}x_k \ominus {}_G\Delta_n^{m-1}x_{k+1})\}|_G}{\rho}\Big)\Big]^{p_k}$$
$$\oplus\, a_{nk}\Big[M_k\Big(\frac{|u_k(e^{(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_k)|_G}{\rho}\Big)\Big]^{p_k}$$
$$=\ \mathrm{O}(e).$$

Therefore, $\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_k \ominus e^{k(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$.
This completes the proof of $(ii)(b)$.
Conversely, let (ii) be true. Then

$$a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_k \ominus e^{k(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}$$
$$=\quad a_{nk}\Big[M_k\Big(\frac{|u_k(e^{(k+1)(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_k \ominus e^{k(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}$$
$$\geq\quad a_{nk}\Big[M_k\Big(\frac{|u_k\{e^{k(k+1)^{-1}} \odot ({}_G\Delta_n^{m-1}x_k \ominus {}_G\Delta_n^{m-1}x_{k+1})\}|_G}{\rho}\Big)\Big]^{p_k}$$
$$\ominus\quad a_{nk}\Big[M_k\Big(\frac{|u_k(e^{(k+1)^{-1}} \odot {}_G\Delta_n^{m-1}x_k)|_G}{\rho}\Big)\Big]^{p_k},$$

we can write

$$a_{nk}\Big[M_k\Big(\frac{|u_k\{e^{k(k+1)^{-1}}\odot({}_G\Delta_n^{m-1}x_k\ominus_G\Delta_n^{m-1}x_{k+1})\}|_G}{\rho}\Big)\Big]^{p_k}$$
$$\leq a_{nk}\Big[M_k\Big(\frac{|u_k(e^{(k+1)^{-1}}\odot{}_G\Delta_n^{m-1}x_k)|_G}{\rho}\Big)\Big]^{p_k}$$
$$\oplus\ a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_k\ominus e^{k(k+1)^{-1}}\odot{}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}.$$

Thus, $\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_k\ominus{}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}<\infty$. as both $(ii)(a)$ and $(ii)(b)$ holds. □

Lemma 3.2. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function. Then*
$\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-i}}\odot{}_G\Delta_n^{m-i}x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\}<\infty$ *implies*
$\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-(i+1)}}\odot{}_G\Delta_n^{m-(i+1)}x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\}<\infty$, *for all $i\in\mathbb{N}$ and $\rho>0$.*

Proof. For $i=1$ in Lemma (3.2), the proof is obvious. Let the result is true for $i=r$, we have $\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-r}}\odot{}_G\Delta_n^{m-r}x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\}<\infty$. Then

$$a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-(r+1)}x_k\ominus_G\Delta_n^{m-(r+1)}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}$$
$$= a_{nk}\Big[M_k\Big(\frac{|u_k\sum_{v=1}^{k}{}_G\Delta_n^{m-r}x_v|_G}{\rho}\Big)\Big]^{p_k}$$
$$\leq \sum_{v=1}^{k}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^{m-r}x_v|_G}{\rho}\Big)\Big]^{p_k}$$
$$= \mathrm{O}\Big((e^{k^r})^k\Big)=\mathrm{O}\Big(e^{k^{(r+1)}}\Big),\ \text{as}\ \sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-r}}\odot{}_G\Delta_n^{m-r}x_v)|_G}{\rho}\Big)\Big]^{p_k}\Big\}<\infty\ \text{and}$$

$$a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^{m-(r+1)}x_k|_G}{\rho}\Big)\Big]^{p_k}$$
$$=a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-(r+1)}x_k\oplus_G\Delta_n^{m-(r+1)}x_1\ominus_G\Delta_n^{m-(r+1)}x_1\oplus_G\Delta_n^{m-(r+1)}x_{k+1}\ominus_G\Delta_n^{m-(r+1)}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}$$
$$\leq a_{nk}\Big[M_k\Big(\frac{|u_{kG}\Delta_n^{m-(r+1)}x_1|_G}{\rho}\Big)\Big]^{p_k}\oplus a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-(r+1)}x_1\ominus{}_G\Delta_n^{m-(r+1)}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}$$
$$\oplus\, a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-(r+1)}x_k\ominus{}_G\Delta_n^{m-(r+1)}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}=\mathrm{O}\Big(e^{k^{(r+1)}}\Big).$$

From this, we have $\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-(r+1)}}\odot{}_G\Delta_n^{m-(r+1)}x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\}<\infty$.
Thus,

$\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-(i+1)}} \odot {}_G\Delta_n^{m-(i+1)}x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$, for all $i \in \mathbb{N}$ and $\rho > 0$. □

Lemma 3.3. *If* $\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-1}} \odot {}_G\Delta_n^{m-1}x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$ *then* $\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-m}} \odot n_k x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$*, for all* $i \in \mathbb{N}$ *and* $\rho > 0$.

Proof. For $i = 1$ in Lemma (3.3), we obtain

$$\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-1}} \odot {}_G\Delta_n^{m-1}x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$$
$$\Rightarrow \sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-2}} \odot {}_G\Delta_n^{m-2}x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty.$$

Again for $i = 2$ in Lemma (3.3), we obtain

$$\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-2}} \odot {}_G\Delta_n^{m-2}x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$$
$$\Rightarrow \sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-3}} \odot {}_G\Delta_n^{m-3}x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty.$$

Continuing this procedure for $i = m - 1$, we arrive

$$\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-(m-1)}} \odot {}_G\Delta_n x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$$
$$\Rightarrow \sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-m}} \odot n_k x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty. \quad \square$$

Lemma 3.4. *If* $x \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$*, then*

$$\sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-m}} \odot n_k x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty.$$

Proof. Let $x \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$

$$\Rightarrow \sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k {}_G\Delta_n^m x_k|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$$
$$\Rightarrow \sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k({}_G\Delta_n^{m-1}x_k \ominus {}_G\Delta_n^{m-1}x_{k+1})|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty$$
$$\Rightarrow \sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-1}} \odot {}_G\Delta_n^m x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty \text{ by Lemma (3.1)}$$
$$\Rightarrow \sup_{k\in\mathbb{N}}\Big\{a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-m}} \odot n_k x_k)|_G}{\rho}\Big)\Big]^{p_k}\Big\} < \infty \text{ by Lemma (3.3).}$$

□

Theorem 3.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then*

(i) $\Big[c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^\alpha = \Big[c[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^\alpha$
$= \Big[l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^\alpha = D_1,$

(ii) $D_1^\alpha = D_2$*, where*

$$D_1 = \Big\{b = (b_k) : {}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^m} \odot n_k^{-1} b_k)|_G}{\rho}\Big)\Big]^{p_k} < \infty\Big\},$$
$$D_2 = \Big\{b = (b_k) : \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-m}} \odot n_k b_k)|_G}{\rho}\Big)\Big]^{p_k} < \infty\Big\}.$$

Proof. (i) First we suppose that $b \in D_1$, then

$${}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^m} \odot n_k^{-1} x_k)|_G}{\rho}\Big)\Big]^{p_k} < \infty.$$

Now, for any $x \in l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$, we have

$$\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-m}} \odot n_k x_k)|_G}{\rho}\Big)\Big]^{p_k} < \infty.$$

Then we have
$${}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(b_k \odot x_k)|_G}{\rho}\Big)\Big]^{p_k}$$
$$= {}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(\{e^{k^{-m}} \odot n_k x_k\} \odot \{e^{k^m} \odot n_k^{-1} b_k\})|_G}{\rho}\Big)\Big]^{p_k}$$
$$\leq {}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^m} \odot n_k^{-1} b_k)|_G}{\rho}\Big)\Big]^{p_k} < \infty.$$

Hence, $b \in \Big[l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^\alpha$.
Conversely, let $b \in \Big[X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^\alpha$ for $X = c$ and l_∞. Then,
${}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(b_k \odot x_k)|_G}{\rho}\Big)\Big]^{p_k} < \infty$ for each $x \in X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. So we take
$x_k = e^{k^m} \odot n_k^{-1},\ k \geq 1$. Then,

$${}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^m} \odot n_k^{-1} b_k)|_G}{\rho}\Big)\Big]^{p_k} = {}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(b_k \odot x_k)|_G}{\rho}\Big)\Big]^{p_k} < \infty.$$

This implies that $b \in D_1$.
Again suppose that $b \in \Big[c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^{\alpha}$ and $b \notin D_1$. Then there exists a strictly increasing sequence (v_i) of positive integers v_i with $v_1 < v_2 < ...$ such that

$$ {}_G\sum_{k=v_i+1}^{v_{i+1}} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^m} \odot n_k^{-1} b_k)|_G}{\rho}\Big)\Big]^{p_k} > i. $$

Now we define a sequence $x = (x_k)$ by

$$ x_k = \begin{cases} 1, & 1 \le k \le v_i, \\ \frac{e^{k^m} \odot n_k^{-1}}{i}, & v_i + 1 < k \le v_{i+1}, \quad i = 1, 2, ... \end{cases} $$

Then it is easy to verify that $x \in c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$. But

$$ \begin{aligned} & {}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(b_k \odot x_k)|_G}{\rho}\Big)\Big]^{p_k} \\ = \quad & {}_G\sum_{k=v_1+1}^{v_2} a_{nk}\Big[M_k\Big(\frac{|u_k(b_k \odot x_k)|_G}{\rho}\Big)\Big]^{p_k} + ... \\ + \quad & {}_G\sum_{k=v_i+1}^{v_{i+1}} a_{nk}\Big[M_k\Big(\frac{|u_k(b_k \odot x_k)|_G}{\rho}\Big)\Big]^{p_k} + ... \\ = \quad & {}_G\sum_{k=v_1+1}^{v_2} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^m} \odot n^{-1} b_k)|_G}{\rho}\Big)\Big]^{p_k} + ... \\ + \quad & \frac{1}{i}\,{}_G\sum_{k=v_i+1}^{v_{i+1}} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^m} \odot n^{-1} b_k)|_G}{\rho}\Big)\Big]^{p_k} + ... \\ \ge \quad & {}_G\sum_{i=1}^{\infty} 1 = \infty, \end{aligned} $$

where $A = (C, 1)$, $M_k(x) = x$, $p_k = 1$, $u_k = 1$ for all $k \in \mathbb{N}$. Hence, $b \notin \Big[c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^{\alpha}$ which contradicts our assumption and $b \in D_1$. This completes the proof.
(ii) The proof is similar to that of part (i). $\square$

Theorem 3.2. *Let X stand for l_∞ or c then* $\Big[X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^{\alpha\alpha} = D_2$.

where $D_2 = \Big\{b = (b_k) : \sup_{k \in \mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-m}} \odot n_k b_k)|_G}{\rho}\Big)\Big]^{p_k} < \infty\Big\}$,

Proof. Let $b \in D_2$ and $x \in \Big[X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^{\alpha}$, then we have

$$
\begin{aligned}
&{}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(b_k \odot x_k)|_G}{\rho}\Big)\Big]^{p_k} \\
&\qquad = {}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(\{e^{k^m} \odot n_k^{-1}x_k\} \odot \{e^{k^{-m}} \odot n_k b_k\})|_G}{\rho}\Big)\Big]^{p_k} \\
&\qquad \leq {}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^m} \odot n_k^{-1}x_k)|_G}{\rho}\Big)\Big]^{p_k} \\
&\qquad \odot \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-m}} \odot n_k b_k)|_G}{\rho}\Big)\Big]^{p_k} < \infty.
\end{aligned}
$$

Hence, $b \in \Big[X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^{\alpha\alpha}$.
Conversely, let $b \in \Big[X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^{\alpha\alpha}$ and $b \notin D_2$. Then we must have

$$
\sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-m}} \odot n_k b_k)|_G}{\rho}\Big)\Big]^{p_k} = \infty.
$$

Therefore, there exists a strictly increasing sequence $(e^k(i))$ of geometric integers [30], where $k(i)$ is a strictly increasing sequence of positive integers such that

$$
a_{nk}\Big[M_k\Big(\frac{|u_k(e^{[k(i)^{-m}]} \odot n_{k(i)} b_{k(i)})|_G}{\rho}\Big)\Big]^{p_k} > e^{i^m}.
$$

Let us define the sequence $x = (x_k)$ by

$$
x_k = \begin{cases} (|n_{k(i)} b_{k(i)}|_G)^{-1_G}, & k = k(i), \\ 1, & k \neq k(i), \end{cases}
$$

where $(|n_{k(i)} b_{k(i)}|_G)^{-1_G}$ is a geometric inverse of $|n_{k(i)} b_{k(i)}|_G$ so that $|n_{k(i)} b_{k(i)}|_G \odot (|n_{k(i)} b_{k(i)}|_G)^{-1_G} = e$. Then we have

$$
\begin{aligned}
&{}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^m} \odot n_k^{-1}x_k)|_G}{\rho}\Big)\Big]^{p_k} \\
&\qquad = {}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{[k(i)^m]} \odot n_{k(i)} b_{k(i)})|_G}{\rho}\Big)\Big]^{p_k} \\
&\qquad \leq e^{i^{-m}} < \infty.
\end{aligned}
$$

Hence, $x \in \Big[X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^{\alpha}$ and ${}_G\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{|u_k(b_k \odot x_k)|_G}{\rho}\Big)\Big]^{p_k} = {}_G\sum e = \infty$, where $A = (C, 1)$, $M_k(x) = x$, $p_k = 1$, $u_k = 1$ for all $k \in \mathbb{N}$. This is a contradiction as $b \in \Big[X[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]\Big]^{\alpha\alpha}$. Therefore, $b \in D_2$. □

Corollary 3.1. *Let X stand for l_∞ or c, we have*

$$\Big[X[G, {}_G\Delta_n^2, A, \mathcal{M}, p, u]\Big]^{\alpha\alpha} =$$

$$\Big\{b = (b_k) : \sup_{k\in\mathbb{N}} a_{nk}\Big[M_k\Big(\frac{|u_k(e^{k^{-2}} \odot n_k b_k)|_G}{\rho}\Big)\Big]^{p_k} < \infty\Big\}.$$

Proof. By putting $m = 2$ in Theorem (3.2),we obtain the result. □

Corollary 3.2. *The sequence spaces $l_\infty[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$, $c[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ and $c_0[G, {}_G\Delta_n^m, A, \mathcal{M}, p, u]$ are not perfect.*

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