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EXISTENCE OF $N(k)$-QUASI EINSTEIN MANIFOLDS

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Abstract. The aim of the present paper is to study the properties of pseudo Ricci symmetric quasi Einstein and $N(k)-$quasi Einstein manifolds. We construct some examples of $N(k)-$quasi Einstein manifolds which support the existence of such manifolds.

Keywords: quasi Einstein, weakly Ricci-symmetric, pseudo Ricci symmetric quasi Einstein manifolds, $k-$nullity distribution, $N(k)-$quasi Einstein manifold, different curvature tensors

1. Introduction

An $n-$dimensional semi-Riemannian or Riemannian manifold (M_n, g), $(n > 2)$, is said to be an Einstein manifold if its Ricci tensor S satisfies the condition $S = \frac{r}{n}g$, where r denotes the scalar curvature of (M_n, g). In other words, an Einstein manifold is a Riemannian or pseudo Riemannian manifold whose Ricci tensor is proportional to the metric. The notion of quasi Einstein manifolds arose during the study of exact solutions to Einstein field equations, as well as during consideration of quasi-umbilical hyper surfaces. A non-flat $n-$dimensional Riemannian manifold is said to be quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \qquad X, Y \in TM \tag{1.1}$$

for smooth functions a and $b \neq 0$, where η is a non-zero $1-$form such that

$$g(X, \xi) = \eta(X), \qquad g(\xi, \xi) = \eta(\xi) = 1 \tag{1.2}$$

for all vector fields X and the associated vector fields ξ [4]. The $1-$form η is called the associated $1-$form and the unit vector field ξ is called the generator of the manifold. If the generator of a quasi Einstein manifold is a parallel vector field then the manifold is locally a product manifold of one-dimensional distribution U

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and $(n-1)$ dimensional distribution $U^{\perp}$, where $U^{\perp}$ is involutive and integrable [25]. In an $n-$dimensional quasi Einstein manifold the Ricci tensor has precisely two distinct eigenvalues a and $a+b$, where the multiplicity of a is $n-1$ and $a+b$ is simple [4]. A proper $\eta-$Einstein contact metric manifold is a natural example of a quasi Einstein manifold ([5], [6]). Different geometrical properties of quasi Einstein manifolds have been studied by Chaki [26], Guha [28], De and Ghosh ([8], [18], [27]), Shaikh, Yoon and Hui [32], Shaikh, Kim and Hui [33], Deszcz, Hotlos and Senturk [19], Mantica and Suh [20], and others.

Let R denote the Riemannian curvature tensor of a Riemannian manifold M_n. The $k-$nullity distribution $N(k)$ of a Riemannian manifold is defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k\,[g(Y,Z)X - g(X,Z)Y]\}, \tag{1.3}$$

where k is a smooth function [7]. If the generator ξ belongs to $k-$nullity distribution $N(k)$, then the quasi Einstein manifold is called an $N(k)-$quasi Einstein manifold [9]. A conformally flat quasi Einstein manifolds are certain $N(k)-$quasi Einstein manifolds [9]. The derivation conditions $R(\xi,X).R = 0$, $R(\xi,X).S = 0$ have also been studied in [9], where R and S denote the curvature and Ricci tensors of the manifold, respectively. In 2007, Özgür and Tripathi [10] studied the deviation conditions $\hat{Z}(\xi,X).\hat{Z} = 0$, $\hat{Z}(\xi,X).R = 0$ and $\hat{Z}(\xi,X).R = 0$ on $N(k)-$quasi Einstein manifolds, where $\hat{Z}$ denotes the concircular curvature tensor. Özgür and Sular [11] continued the study of $N(k)-$quasi Einstein manifolds with conditions $R(\xi,X).C = 0$ and $R(\xi,X).\tilde{C} = 0$, where C and $\tilde{C}$ denote the Weyl conformal and quasi conformal curvature tensors, respectively. Again, in 2008, Özgür [12] studied the deviation conditions $R(\xi,X).P = 0$, $P(\xi,X).S = 0$ and $P(\xi,X).P = 0$ for an $N(k)-$quasi Einstein manifold, where P denotes the projective curvature tensor and some physical examples of $N(k)-$quasi Einstein manifolds are given. In 2010, Singh, Pandey and Gautam [13], Taleshian and Hosseinzadeh [17], Dwivedi [24] have studied the $N(k)-$quasi Einstein manifolds with the deviation conditions $R(\xi,X).\tilde{P} = 0$, $R(\xi,X).W_2 = 0$, $W_2(\xi,X).S = 0$ and $\tilde{P}(\xi,X).\tilde{P} = 0$, where $\tilde{P}$ and W_2 denote the pseudo projective curvature and W_2-projective curvature tensors, respectively. Several geometrical properties of $N(k)-$quasi Einstein manifolds have studied by Yildiz, De and Cetinkaya [15], Taleshian and Hosseinzadeh ([16], [23]), De, De and Gazi [14], Yang and Xu [22] and others. Motivated by the above studies, the author continues the study of N$(k)-$quasi Einstein manifolds.

The paper is organized as follows. Section 2 is about the prerequisites of $N(k)-$quasi Einstein manifolds. In Section 3, we give some examples of $N(k)-$quasi Einstein manifolds which support the existence of such manifolds. We also prove that the $m-$projectively flat quasi Einstein manifold is an $N(\frac{2a+b}{2(n-1)})-$quasi Einstein manifold but the converse is not true. In Sections 4 and 5, we show that there does not exist $N(k)-$quasi Einstein manifolds under certain conditions. Section 6 gives the answer to the question:
Que: "What condition is to be imposed on $N(k)-$quasi Einstein manifold that makes it $m-$projectively flat?"
Section 7 is concerned with the study of the pseudo Ricci symmetric quasi Einstein manifold.

2. Preliminaries

In consequence of (1.1) and (1.2), we get

$$S(X,\xi) = (a+b)\eta(X), \tag{2.1}$$

and

$$r = na + b, \tag{2.2}$$

where r denotes the scalar curvature of the Riemannian manifold (M_n, g).

Lemma 2.1. *[10] In an $n-$dimensional $N(k)-$quasi Einstein manifold it follows that*

$$k = \frac{a+b}{n-1}.$$

In an $n-$dimensional $N(k)-$quasi Einstein manifold (M_n, g), the following relations hold ([9], [10])

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{2.3}$$

$$R(X,\xi)Y = k[\eta(Y)X - g(X,Y)\xi] = -R(\xi,X)Y, \tag{2.4}$$

$$R(\xi,X)\xi = k[\eta(X)\xi - X], \tag{2.5}$$

$$Q\xi = k(n-1)\xi, \tag{2.6}$$

$$\eta(R(X,Y)Z) = k[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)] \tag{2.7}$$

for arbitrary vector fields X, Y and Z.

The projective curvature tensor P [3], concircular curvature tensor $\hat{C}$ ([1], [2]) and $m-$projective curvature tensor W^* [34] on the Riemannian manifold (M_n, g) are defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\left[S(Y,Z)X - S(X,Z)Y\right], \tag{2.8}$$

$$\hat{C}(X,Y)Z = R(X,Y)Z - \frac{r}{(n-1)(n-2)}\left[g(Y,Z)X - g(X,Z)Y\right], \tag{2.9}$$

and

$$\begin{aligned} W^*(X,Y)Z &= R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X \\ &- S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \end{aligned} \tag{2.10}$$

respectively, for arbitrary vector fields X, Y, Z; where $S(X,Y) = g(QX,Y)$. Here Q denotes the Ricci operator of the Riemannian manifold. The properties of this curvature tensor have been noticed in ([35], [36], [37], [38], [39], [40], etc.).

The curvature conditions $P.R$, $P.\hat{C}$ and $P.W^*$ are defined by

$$\begin{aligned}(P(W,X).R)(Y,Z)U &= P(W,X)R(Y,Z)U - R((P(W,X)Y,Z)U \\ &\quad -R(Y,P(W,X)Z)U - R(Y,Z)P(W,X)U,\end{aligned} \tag{2.11}$$

$$\begin{aligned}(P(W,X).\hat{C})(Y,Z)U &= P(W,X)\hat{C}(Y,Z)U - \hat{C}(P(W,X)Y,Z)U \\ &\quad -\hat{C}(Y,P(W,X)Z)U - \hat{C}(Y,Z)P(W,X)U,\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}(P(W,X).W^*)(Y,Z)U &= P(W,X)W^*(Y,Z)U - W^*((P(W,X)Y,Z)U \\ &\quad -W^*(Y,P(W,X)Z)U - W^*(Y,Z)P(W,X)U,\end{aligned} \tag{2.13}$$

respectively, for all vector fields W, X, Y, Z, U, where $P(W,X)$ acts on R, $\hat{C}$ and W^* as a deviation.

3. Examples of $N(k)-$quasi Einstein manifolds

In this section, we prove the existence of $N(k)-$quasi Einstein manifolds.

Theorem 3.1. *Every $m-$projectively flat quasi Einstein manifold of dimension n is an $N(\frac{2a+b}{2(n-1)})-$ quasi Einstein manifold.*

Proof. Let us suppose that the quasi Einstein manifold (M_n, g) is $m-$projectively flat, *i.e.*, $W^*(X,Y)Z = 0$, therefore, (2.10) gives

$$R(X,Y)Z = \frac{1}{2(n-1)}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\}. \tag{3.1}$$

In consequence of (1.1) and (1.2), (3.1) becomes

$$\begin{aligned}R(X,Y)Z &= \frac{a}{(n-1)}\{g(Y,Z)X - g(X,Z)Y\} \\ &+\frac{b}{2(n-1)}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + \eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi\}.\end{aligned} \tag{3.2}$$

Replacing Z by ξ in (3.2) and then using (1.2), we have

$$R(X,Y)\xi = \frac{(2a+b)}{2(n-1)}\{\eta(Y)X - \eta(X)Y\}, \tag{3.3}$$

which shows that the generator ξ belongs to the $\frac{(2a+b)}{2(n-1)}-$nullity distribution of $N(\frac{(2a+b)}{2(n-1)})$. Hence the statement of the theorem. The converse part is not true. □

Remark 3.1. From Theorem (3.1), one natural question arises here as: *"Under what condition is $N(\frac{2a+b}{2(n-1)})-$quasi Einstein manifold $m-$projectively flat""*.

Theorem (6.1) gives the answer to this question.

Theorem 3.2. *An $m-$projectively flat $(WRS)_n$ $(n > 2)$ is an $N(\frac{r}{2})-$quasi Einstein manifolds.*

Proof. A non-flat Riemannian manifold (M_n, g), $n > 2$, is called a weakly-Ricci symmetric if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the relation

$$(D_X S)(Y, Z) = A(X)g(Y, Z) + B(Y)S(X, Z) + D(Z)S(X, Y), \tag{3.4}$$

for arbitrary vector fields X, Y and Z [21]. Here, A, B and D are $1-$forms associated with vectors ρ_1, ρ_2 and ρ_3, respectively, *i.e.*,

$$A(X) = g(X, \rho_1), \quad B(X) = g(X, \rho_2) \quad and \quad D(X) = g(X, \rho_3).$$

In consequence of (3.4) and symmetric properties of S, it follows that

$$\{B(Y) - D(Y)\} S(X, Z) = \{B(Z) - D(Z)\} S(X, Y). \tag{3.5}$$

Let $\sigma(X) = B(X) - D(X)$ for any vector field X, then (3.5) becomes

$$\sigma(Y)S(X, Z) = \sigma(Z)S(X, Y). \tag{3.6}$$

Let $\{e_i\}$, $i = 1, 2, \ldots, n$, be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = Z = e_i$ in (3.6) and then taking summation over i, $1 \le i \le n$, we get

$$r\sigma(Y) = \sigma(QY), \tag{3.7}$$

where $\sigma(X) = g(X, \delta)$ for any vector field X and r is the scalar curvature. From (3.6), we have

$$\sigma(\delta)S(X, Z) = \sigma(Z)S(X, \delta) = \sigma(Z)\sigma(QX). \tag{3.8}$$

From (3.7) and (3.8), we get

$$S(X, Z) = rH(X)H(Z), \tag{3.9}$$

where $H(X) = \frac{\sigma(X)}{\sqrt{\sigma(\delta)}}$ and $g(X, \rho) = H(X)$, ρ is a unit vector field. Let us consider an $m-$projectively flat $(WRS)_n$ manifold, then (2.10) gives

$$\begin{aligned} {}'R(X, Y, Z, U) &= \frac{1}{2(n-1)}[S(Y, Z)g(X, U) - S(X, Z)g(Y, U) \\ &\quad + g(Y, Z)S(X, U) - g(X, Z)S(Y, U)]. \end{aligned} \tag{3.10}$$

In view of (3.9), (3.10) becomes

$$\begin{aligned} 'R(X,Y,Z,U) &= \frac{r}{2(n-1)}[H(Y)H(Z)g(X,U)-H(X)H(Z)g(Y,U) \\ &\quad +g(Y,Z)H(X)H(U)-g(X,Z)H(Y)H(U)]. \end{aligned} \tag{3.11}$$

Substituting $X=U=e_i$ in (3.11) and then taking summation over i, $1\leq i\leq n$, we get

$$S(Y,Z)=ag(Y,Z)+bH(Y)H(Z), \tag{3.12}$$

where

$$a=\frac{r}{2(n-1)} \qquad \textit{and} \qquad b=\frac{r(n-2)}{2(n-1)}. \tag{3.13}$$

Equation (3.12) shows that the manifold is quasi-Einstein [4]. In view of (3.13) and lemma (2.1), we get

$$k=\frac{a+b}{(n-1)}=\frac{r}{2}.$$

Hence the theorem. □

In 2004, P. Alegre, D. E. Blair and A. Carriazo introduced the idea of the generalized Sasakian space form and they constructed many examples by using different geometric techniques such as Riemannian submersions, warped products or conformal and related transformations [41]. A Riemannian manifold of dimension n equipped with a tensor field ϕ of type $(1,1)$, a structure vector field ξ and a covariant vector field η associated with the Riemannian metric g satisfies the relations

$$\phi^2(X)=-X+\eta(X)\xi, \qquad \eta(\xi)=1, \qquad g(X,\xi)=\eta(X), \qquad \textit{and} \qquad \phi\xi=0 \tag{3.14}$$

and

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{3.15}$$

for arbitrary vector fields X and Y, is called an almost contact metric manifold (M,ϕ,ξ,η,g) [5]. An almost contact metric manifold (M,ϕ,ξ,η,g) is said to be a generalized Sasakian space form if the Riemannian curvature tensor R satisfies the tensorial relation

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X-g(X,Z)Y\} \\ &\quad +f_2\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\} \\ &\quad +f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{3.16}$$

for arbitrary vector fields X, Y and Z, where f_1, f_2 and f_3 are smooth functions on M. Here we consider that $f_1\neq f_3$. Replacing Z with the structure vector field ξ in (3.16) and then using Equation (3.14), we get

$$R(X,Y)\xi=(f_1-f_3)\{\eta(Y)X-\eta(X)Y\}, \tag{3.17}$$

which demonstrates that $(f_1 - f_3) \in (f_1 - f_3)$ nullity distribution. Contracting (3.16) along the vector field X and then using (3.14) and (3.15), we find that

$$S(Y, Z) = \{(n-1)f_1 + 3f_2 - f_3\}g(Y, Z) - \{3f_2 + (n-2)f_3\}\eta(Y)\eta(Z), \tag{3.18}$$

which proves that the generalized Sasakian space form is a certain class of quasi Einstein manifold with scalar functions $a = (n-1)f_1 + 3f_2 - f_3$ and $b = -\{3f_2 + (n-2)f_3\}$. In consequence of lemma (2.1) and equation (3.18), we can calculate the value of k as

$$k = f_1 - f_3. \tag{3.19}$$

This shows that the manifold is an $N(f_1 - f_3)-$quasi Einstein manifold. Hence from the above discussions, we can state the following example:

Example 3.1. An $n-$dimensional generalized Sasakian space form is an $N(f_1 - f_3)-$quasi Einstein manifold.

Example 3.2. Let $(x^1, x^2, \dots, x^n) \in \mathbb{R}^n$, where $\mathbb{R}^n$ denotes $n-$dimensional real number space. We consider a Lorentzian metric g on $\mathbb{R}^4 = (x^1, x^2, x^3, x^4; q\pi < x^1 < q\pi + \frac{\pi}{2}, q \in \mathbb{Z})$, ($\mathbb{Z}$ is the set of integer), by

$$ds^2 = g_{ij}dx^i dx^j = \{sin(x^1) - tan(x^1)\}\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] - (dx^4)^2, \tag{3.20}$$

where $(i, j = 1, 2, 3, 4)$. With the help of (3.20), we can see that the non-vanishing components of the Lorentzian metric are

$$g_{11} = g_{22} = g_{33} = sin(x^1) - tan(x^1), \quad g_{44} = -1 \tag{3.21}$$

and its associated components are

$$g^{11} = g^{22} = g^{33} = \frac{cot(x^1)}{cos(x^1) - 1}, \quad g^{44} = -1. \tag{3.22}$$

In consequence of (3.21) and (3.22), it can be calculated that the non-vanishing components of Christoffel symbols, curvature tensor, Ricci tensor and scalar curvature are given by

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \frac{cos^2(x^1) + cos(x^1) + 1}{sin2(x^1)}, \quad \Gamma^1_{22} = \Gamma^1_{33} = -\frac{cos^2(x^1) + cos(x^1) + 1}{sin2(x^1)},$$

$$R_{1331} = \frac{(1 - cos(x^1))(cos^2(x^1) + cos(x^1) + 1)^2}{2sin2(x^1)cos^2(x^1)}, \quad S_{33} = \frac{(cos^2(x^1) + cos(x^1) + 1)^2}{sin^2(2x^1)},$$

$$r = \frac{(cos^2(x^1) + cos(x^1) + 1)^2}{2sin2(x^1)sin^2(x^1)(cos(x^1) - 1)}(\neq 0) \tag{3.23}$$

and the components obtained by the symmetric properties. From (3.23), it is clear that the manifold $(\mathbb{R}^4, g)$ is a Lorentzian manifold. Now, we are going to prove that the manifold $(\mathbb{R}^4, g)$ is a certain class of $N(k)-$quasi Einstein manifold. For this purpose we take the associated smooth functions a and b as follows:

$$a = 0, \quad b = \frac{(cos^2(x^1) + cos(x^1) + 1)^2}{2sin2(x^1)sin^2(x^1)(1 - cos(x^1))}(\neq 0). \tag{3.24}$$

Now we define the 1−forms A_i as follows:

$$A_i = \begin{cases} \sqrt{tan(x^1) - sin(x^1)} & , \quad i = 3 \\ 0 & , \quad otherwise \end{cases} \tag{3.25}$$

Now, we have to prove the following:

$$S_{ij} = ag_{ij} + bA_iA_j \tag{3.26}$$

for $i, j = 1, 2, 3, 4$. For instance, we have to show that

$$S_{33} = ag_{33} + bA_3A_3. \tag{3.27}$$

The left-hand side of (3.27) $= S_{33} = \frac{(cos^2(x^1)+cos(x^1)+1)^2}{sin^2(2x^1)}$ (from (3.23)). In view of (3.21), (3.24) and (3.25), the right-hand side of (3.27) $= ag_{33} + bA_3A_3 = \frac{(cos^2(x^1)+cos(x^1)+1)^2}{sin^2(2x^1)}$. In a similar way, we can verify for other components of S_{ij}. From (1.1), (1.2), (1.3) and (3.24), it can be easily proven that

$$k = \frac{a-b}{n-1} = \frac{(cos^2(x^1) + cos(x^1) + 1)^2}{6sin2(x^1)sin^2(x^1)(cos(x^1) - 1)}$$

and $r = 4a - b$ hold on $(\mathbb{R}^4, g)$ and, therefore, it is an $N\left(\frac{(cos^2(x^1)+cos(x^1)+1)^2}{6sin2(x^1)sin^2(x^1)(cos(x^1)-1)}\right)$ −quasi Einstein manifold.

Example 3.3. If $\mathbb{R}^3 = (x^1, x^2, x^3); x^1 \neq q\pi, q \in \mathbb{Z})$, ($\mathbb{Z}$ is a set of integers), a a three dimensional real number space, then we define a Lorentzian metric on $\mathbb{R}^3$ as

$$ds^2 = g_{ij}dx^idx^j = -(dx^1)^2 + e^{(x^1)}sin(x^1)[(dx^2)^2 + (dx^3)^2], \tag{3.28}$$

where $(i, j = 1, 2, 3)$. It can be seen from (3.28) that the non-vanishing components of the Lorentzian metric are

$$g_{11} = -1, \quad g_{22} = e^{(x^1)}sin(x^1) = g_{33} \tag{3.29}$$

and its associated components of the Lorentzian metric are

$$g^{11} = -1, \quad g^{22} = g^{33} = e^{-(x^1)}csc(x^1). \tag{3.30}$$

From the equations (3.29) and (3.30), it can be easily calculated that the only non-vanishing components of the Christoffel symbols, curvature tensors and Ricci tensors are given by the following relations

$$\begin{aligned} &\Gamma^1_{22} = \Gamma^1_{33} = \frac{e^{(x^1)}(sin(x^1) + cos(x^1))}{2}, \quad \Gamma^2_{12} = \Gamma^3_{13} = \frac{sin(x^1) + cos(x^1)}{2sin(x^1)}, \\ &R_{2323} = \frac{e^{2(x^1)}(1 + sin2(x^1))}{4}, \quad R_{1212} = R_{1313} = -e^{(x^1)}\left\{\frac{3 + sin2(x^1)}{4sin(x^1)}\right\}, \\ &S_{11} = \frac{sin2(x^1) - 1}{2sin^2(x^1)}, \quad S_{22} = S_{33} = -e^{(x^1)}cos(x^1) \end{aligned} \tag{3.31}$$

and the components which can be obtained from these by the symmetric properties. Here S_{ij} represents the components of the Ricci tensor. From equation (3.31), it is clear that the 3−dimensional space $\mathbb{R}^3$ with the Lorentzian metric g defined in (3.28) is a Lorentzian manifold of dimension 3. From (3.29), (3.30) and (3.31), it can be easily seen that the scalar curvature of the manifold $(\mathbb{R}^3, g)$ is non-zero, *i.e.*,

$$r = \frac{csc^2(x^1) - 6cot(x^1)}{2} \neq 0. \tag{3.32}$$

Now, we choose the scalars a and b as follows:

$$a = -cot(x^1), \qquad b = -\frac{csc^2(x^1)}{2} \neq 0. \tag{3.33}$$

We define the 1−form A as follows:

$$A_i = \begin{cases} 1 & , \quad i = 1 \\ 0 & , \quad otherwise \end{cases} \tag{3.34}$$

Now, we have to prove that

$$S_{ij} = ag_{ij} + bA_iA_j. \tag{3.35}$$

In consequence of (3.29), (3.31), (3.33), (3.34) and (3.35), it can be easily verified that $S_{11} = ag_{11} + bA_1A_1$ and all the other components of S satisfy the relation (3.35). Hence $(\mathbb{R}^3, g)$ is a quasi-Einstein manifold. Now, $k = \frac{a-b}{n-1} = \frac{1}{4}\left\{csc^2(x^1) - 2cot(x^1)\right\}$ and $r = 3a - b$ hold on $(\mathbb{R}^3, g)$, therefore $(\mathbb{R}^3, g)$ be an $N\left(\frac{1}{4}(csc^2(x^1) - 2cot(x^1))\right)$ −quasi Einstein manifold.

Let us consider a conformally flat perfect fluid space time (M^4, g) satisfying Einstein's equation without cosmological constant. Further, let ξ be the unit time like velocity vector of the fluid. The Einstein equation without cosmological constant can be written as

$$S(X, Y) - \frac{1}{2}rg(X, Y) = \kappa T(X, Y), \tag{3.36}$$

where κ is the gravitational constant and T is the energy momentum tensor of type $(0, 2)$. In the present case (3.36) can be written as

$$S(X, Y) - \frac{1}{2}rg(X, Y) = \kappa[(\sigma + p)\eta(X)\eta(Y) + pg(X, Y)],$$

where σ is the energy density and p is the isotropic pressure of the fluid. Then we have from the above equation

$$S(X, Y) = \left(\kappa p + \frac{1}{2}r\right) g(X, Y) + \kappa(\sigma + p)\eta(X)\eta(Y), \tag{3.37}$$

which gives

$$r = \kappa(\sigma - 3p).$$

Putting the value of r in (3.37), we get

$$S(X,Y) = \Big(\frac{\kappa}{2}(\sigma - p)\Big)\, g(X,Y) + \kappa(\sigma + p)\eta(X)\eta(Y),$$

which is of the form (1.2), where $a = \frac{\kappa}{2}(\sigma - p)$ and $b = \kappa(\sigma + p)$. Since the space time is conformally flat, therefore it is a $N(k)-$quasi Einstein manifold. Hence we state the following example:

Example 3.4. [12] A conformally flat perfect fluid space time (M^4, g) satisfying Einstein's equation without cosmological constant is an $N(\frac{\kappa(3\sigma+p)}{6})-$quasi Einstein manifold.

Again we consider a conformally flat perfect fluid space time (M^4, g) satisfying Einstein's equation with the cosmological constant λ. If ξ is the unit time like velocity vector of the fluid, then Einstein's equation takes the form

$$S(X,Y) + (\lambda - \frac{1}{2}r)g(X,Y) = \kappa[(\sigma + p)\eta(X)\eta(Y) + pg(X,Y)],$$

which is equivalent to

$$S(X,Y) = \Big(\kappa p + \frac{1}{2}r - \lambda\Big)\, g(X,Y) + \kappa(\sigma + p)\eta(X)\eta(Y). \tag{3.38}$$

From (3.38), it can be seen that

$$r = 4\lambda + \kappa(\sigma - 3p).$$

On substituting the value of r in (3.38), we find

$$S(X,Y) = \Big(\lambda + \frac{\kappa}{2}(\sigma - p)\Big)\, g(X,Y) + \kappa(\sigma + p)\eta(X)\eta(Y)$$

, where $a = \lambda + \frac{\kappa}{2}(\sigma - p)$ and $b = \kappa(\sigma + p)$. Now,

$$k = \frac{a+b}{3} = \frac{\lambda}{3} + \frac{\kappa(\sigma + p)}{6}.$$

Hence we state the following example:

Example 3.5. [12] A conformally flat perfect fluid space time (M^4, g) satisfying Einstein's equation with the cosmological constant λ is an $N(\frac{\lambda}{3} + \frac{\kappa(3\sigma+p)}{6})-$quasi Einstein manifold.

In [14], we viewed the following examples:

Example 3.6. [14] A special para-Sasakian manifold with a vanishing $D-$concircular curvature tensor is an $N(k)-$quasi Einstein manifold.

Example 3.7. [14] A perfect fluid pseudo Ricci symmetric space time is an $N(\frac{2r}{9})-$quasi Einstein manifold.

4. $N(k)-$quasi Einstein manifolds satisfying $P(\xi, X).R = 0$

From (1.1), (1.2), (2.1), (2.3), (2.4), (2.8) and lemma (2.1), it follows that

$$P(\xi, Y)Z = \frac{b}{n-1}\left[g(Y, Z) - \eta(Y)\eta(Z)\right]\xi. \tag{4.1}$$

Let us assume that $P(\xi, X).R = 0$, then from (2.11) we have

$$\begin{aligned} &P(\xi, X)R(Y, Z)U - R(P(\xi, X)Y, Z)U \\ &-R(Y, P(\xi, X)Z)U - R(Y, Z)P(\xi, X)U = 0. \end{aligned} \tag{4.2}$$

In view of (1.2), (1.3), (2.3), (2.4), (2.7) and (4.1), Equation (4.2) becomes

$$\begin{aligned} &\frac{b}{n-1}[{}'R(Y, Z, U, X) + k(g(X, Z)g(Y, U) - g(X, Y)g(Z, U) \\ &+\eta(U)\eta(Z)g(X, Y) - \eta(U)\eta(Y)g(X, Z))] = 0, \end{aligned} \tag{4.3}$$

where ${}'R(Y, Z, U, X) = g(R(Y, Z)U, X)$. Since $b \neq 0$, therefore, (4.3) gives

$$\begin{aligned} &{}'R(Y, Z, U, X) + k(g(X, Z)g(Y, U) - g(X, Y)g(Z, U) \\ &+\eta(U)\eta(Z)g(X, Y) - \eta(U)\eta(Y)g(X, Z)) = 0. \end{aligned} \tag{4.4}$$

Let $\{\{e_i\}, i = 1, 2, ..., n\}$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (4.4) and taking summation over i, $1 \leq i \leq n$, we get

$$S(Z, U) = k(n-1)[g(Z, U) - \eta(Z)\eta(U)]. \tag{4.5}$$

Since, on a quasi-Einstein manifold the smooth functions a and b are unique, as if $S = a_1 g + b_1 \eta \otimes \eta$, then $(a - a_1)g + (b - b_1)\eta \otimes \eta$=0 and thus g is of rank ≤ 1, a contradiction. Hence we can state the following theorem:

Theorem 4.1. *There exists no $N(k)-$quasi Einstein manifold (M_n, g) satisfying the condition $P(\xi, X).R = 0$.*

5. $N(k)-$quasi Einstein manifolds satisfying $P(\xi, X).\hat{C} = 0$

From (1.3), (2.2), (2.7) and (2.9) it follows that

$$\eta(\hat{C}(Y, Z)U) = \sigma[g(Z, U)\eta(Y) - g(Y, U)\eta(Z)], \tag{5.1}$$

where $\sigma = \frac{b(n-3)-2a}{(n-1)(n-2)} \neq 0$.

Let us suppose that the $N(k)-$quasi Einstein manifold satisfies $P(\xi, X).\hat{C} = 0$, then (2.12) gives

$$\begin{aligned} &P(\xi, X)\hat{C}(Y, Z)U - \hat{C}(P(\xi, X)Y, Z)U \\ &-\hat{C}(Y, P(\xi, X)Z)U - \hat{C}(Y, Z)P(\xi, X)U = 0. \end{aligned} \tag{5.2}$$

In view of (1.2) and (4.1), (5.2) becomes

$$\frac{b}{n-1}[\left\{{'\hat{C}}(Y,Z,U,X)-\eta(X)\eta(\hat{C}(Y,Z)U)\right\}\xi-\{g(X,Y)-\eta(X)\eta(Y)\}\hat{C}(\xi,Z)U$$
$$-\{g(X,Z)-\eta(X)\eta(Z)\}\hat{C}(Y,\xi)U-\{g(X,U)-\eta(X)\eta(U)\}\hat{C}(Y,Z)\xi]=0,$$

where ${'\hat{C}}(Y,Z,U,X)=g(\hat{C}(Y,Z)U,X)$. Since $b\neq 0$, therefore the above equation becomes

$$\left\{{'\hat{C}}(Y,Z,U,X)-\eta(X)\eta(\hat{C}(Y,Z)U)\right\}\xi-\{g(X,Y)-\eta(X)\eta(Y)\}\hat{C}(\xi,Z)U$$
$$-\{g(X,Z)-\eta(X)\eta(Z)\}\hat{C}(Y,\xi)U-\{g(X,U)-\eta(X)\eta(U)\}\hat{C}(Y,Z)\xi=0,$$

which becomes

$$\begin{aligned}&{'\hat{C}}(Y,Z,U,X)-\{g(X,Y)-\eta(X)\eta(Y)\}\eta(\hat{C}(\xi,Z)U)\\&-\{g(X,Z)-\eta(X)\eta(Z)\}\eta(\hat{C}(Y,\xi)U)-\eta(X)\eta(\hat{C}(Y,Z)U)\\&-\{g(X,U)-\eta(X)\eta(U)\}\eta(\hat{C}(Y,Z)\xi)=0.\end{aligned}\tag{5.3}$$

Using (5.1) in (5.3), we find

$$\begin{aligned}&{'\hat{C}}(Y,Z,U,X)=\sigma[g(X,Y)g(Z,U)\\&-g(X,Z)g(Y,U)+\eta(Y)\eta(U)g(X,Z)-\eta(U)\eta(Z)g(X,Y)].\end{aligned}\tag{5.4}$$

In consequence of (2.2) and (2.9), (5.4) becomes

$$\begin{aligned}{'R}(Y,Z,U,X)\;=\;&k\{g(X,Y)g(Z,U)-g(X,Z)g(Y,U)\}\\&+\sigma\{\eta(U)\eta(Y)g(X,Z)-\eta(U)\eta(Z)g(X,Y)\}.\end{aligned}\tag{5.5}$$

Let $\{e_i\}$, $i=1,2,...,n$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=Y=e_i$ in (5.5) and taking summation over i, $1\leq i\leq n$, we get

$$S(Z,U)=a'g(Z,U)+b'\eta(U)\eta(Z),\tag{5.6}$$

where $a'=(a+b)$, $b'=-(n-1)\sigma$. Since, on a quasi-Einstein manifold the smooth functions a and b are unique, as if $S=a_1g+b_1\eta\otimes\eta$, then $(a-a_1)g+(b-b_1)\eta\otimes\eta=0$ and thus g is of rank ≤ 1, a contradiction. Hence we can state the following theorem:

Theorem 5.1. *There exists no $N(k)-$quasi Einstein manifold (M_n,g) satisfying the condition $P(\xi,X).\hat{C}=0$.*

6. $N(k)-$quasi Einstein manifolds satisfying $P(\xi,X).W^*=0$

In view of equations (1.1), (1.2), (1.3), (2.7) and (2.10), we can find that

$$\begin{aligned}{'W^*}(Y,Z,U,X)\;=\;&-\frac{b}{2(n-1)}[\eta(Z)\eta(U)g(X,Y)-\eta(U)\eta(Y)g(Z,X)\\&+g(Z,U)\eta(X)\eta(Y)-g(Y,U)\eta(X)\eta(Z)]\\&+(k-\frac{a}{n-1})\{g(Z,U)g(Y,X)-g(Y,U)g(Z,X)\}.\end{aligned}\tag{6.1}$$

and

$$\eta(W^*(X,Y)Z) = \lambda\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{6.2}$$

where $\lambda = (k - \frac{a}{n-1} - \frac{b}{2(n-1)})$ and $'W^*(Y,Z,U,X) = g(W^*(Y,Z)U,X)$. Let us suppose that $N(k)-$quasi Einstein manifold satisfies $P(\xi,X).W^* = 0$, then (2.13) gives

$$P(\xi,X).W^*(Y,Z)U - W^*(P(\xi,X)Y,Z)U - W^*(X,P(\xi,Y)Z)U - W^*(Y,Z)P(\xi,X)U = 0.$$

In consequence of (1.2) and (4.1), the above equation becomes

$$\begin{aligned}&\frac{b}{n-1}[{}'W^*(Y,Z,U,X)\xi - \eta(X)\eta(W^*(Y,Z)U)\xi - g(X,Y)W^*(\xi,Z)U\\&+\eta(X)\eta(Y)W^*(\xi,Z)U - g(X,Z)W^*(Y,\xi)U + \eta(X)\eta(Z)W^*(Y,\xi)U\\&-g(X,U)W^*(Y,Z)\xi + \eta(X)\eta(U)W^*(Y,Z)\xi] = 0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac{b}{n-1}[{}'W^*(Y,Z,U,X) - \eta(X)\eta(W^*(Y,Z)U) - g(X,Y)\eta(W^*(\xi,Z)U)\\&+\eta(X)\eta(Y)\eta(W^*(\xi,Z)U) - g(X,Z)\eta(W^*(Y,\xi)U) + \eta(X)\eta(Z)\eta(W^*(Y,\xi)U)\\&-g(X,U)\eta(W^*(Y,Z)\xi) + \eta(X)\eta(U)\eta(W^*(Y,Z)\xi)] = 0,\end{aligned} \tag{6.3}$$

Since $b \neq 0$, therefore (6.3) becomes

$$\begin{aligned}&{}'W^*(Y,Z,U,X) - \eta(X)\eta(W^*(Y,Z)U) - g(X,Y)\eta(W^*(\xi,Z)U)\\&+\eta(X)\eta(Y)\eta(W^*(\xi,Z)U) - g(X,Z)\eta(W^*(Y,\xi)U) + \eta(X)\eta(Z)\eta(W^*(Y,\xi)U)\\&-g(X,U)\eta(W^*(Y,Z)\xi) + \eta(X)\eta(U)\eta(W^*(Y,Z)\xi) = 0.\end{aligned} \tag{6.4}$$

With the help of (1.2), (6.1) and (6.2), (6.4) becomes

$$\begin{aligned}&(k - \frac{a}{n-1} - \lambda)\{g(Z,U)g(X,Y) - g(Y,U)g(Z,X)\}\\&-\frac{b}{2(n-1)}[\eta(Z)\eta(U)g(X,Y) - \eta(Y)\eta(U)g(Z,X) + \eta(X)\eta(Y)g(Z,U)\\&-\eta(X)\eta(Z)g(Y,U)] - \lambda[\eta(Y)\eta(U)g(X,Z) - \eta(U)\eta(Z)g(X,Y)] = 0.\end{aligned} \tag{6.5}$$

Let $\{e_i\}$, $i = 1,2,...,n$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (6.5) and taking summation over i, $1 \leq i \leq n$, we get

$$\frac{b(n-2)}{2(n-1)}g(Z,U) + \{(n-1)k - a - \frac{b}{2}(1 + \frac{n-2}{n-1})\eta(U)\eta(Z) = 0. \tag{6.6}$$

Replacing Z by ξ in equation (6.6) and using (1.2), we get

$$k = \frac{2a+b}{2(n-1)}. \tag{6.7}$$

From the equations (6.2) and (6.7), we obtain

$$\eta(W^*(X,Y)Z) = 0. \tag{6.8}$$

In consequence of (6.4) and (6.8), we can prove that $W^* = 0$. Hence we can state the following theorem:

Theorem 6.1. *An $N(k)-$quasi Einstein manifold (M_n, g) satisfies $P(\xi, X).W^* = 0$ if and only if it is $m-$projectively flat.*

7. Pseudo Ricci symmetric quasi Einstein manifold

R. N. Sen and M. C. Chaki [30] obtained the following expressions of covariant derivative of Ricci tensor during the study of certain curvature restrictions on a certain kind of conformally flat space of class one:

$$S_{ij,l} = 2\lambda_l S_{ij} + \lambda_i S_{lj} + \lambda_j S_{il}, \tag{7.1}$$

where λ_i is a non-zero covariant vector and $','$ denotes covariant differentiation with respect to the metric g_{ij}.

A non-flat Riemannian manifold (M_n, g) is called a pseudo Ricci symmetric (briefly $(PRS)_n$) [29] if its Ricci tensor of type $(0, 2)$ is not identically zero and satisfies (7.1), that is (in index free notation),

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(Z,X) + A(Z)S(X,Y), \tag{7.2}$$

where ∇ denotes the Levi-Civita connection and A is a non-zero $1-$form such that

$$A(X) = g(X, \rho)$$

for all vector fields X; ρ is the basic vector field of (M_n, g) associated with the $1-$form A. It is a particular case of weakly Ricci symmetric manifold introduced by Tamassy and Binh [21].

Theorem 7.1. *If M_n is a pseudo Ricci symmetric quasi Einstein manifold, then*

$$X(a+b) = (a+b)\{A(X) + 2\eta(X)\eta(\rho)\}. \tag{7.3}$$

Proof. Using (7.2) in

$$(\nabla_X S)(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z),$$

we get

$$2A(X)S(Y,Z) + A(Y)S(Z,X) + A(Z)S(X,Y) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).$$

Putting $Y = Z = \xi$ in the above equation and using (1.3) and (2.1), we get (7.3). Hence the theorem. □

If $(a+b)(\neq 0)$ is a constant, then (7.3) gives $A(X) = -2\eta(X)\eta(\rho)$. Hence we can state the following corollary:

Corollary 7.1. *If M_n is a pseudo Ricci symmetric quasi Einstein manifold, then $A(X) = -2\eta(X)\eta(\rho)$ if and only if $(a+b)$ is non-zero constant.*

Remark 7.1. If an $n-$dimensional generalized Sasakian space form is pseudo Ricci symmetric and satisfies the condition $f_1 = f_3$, then from the equation (3.18) we can easily observe that $a+b=0$ and, therefore, the theorem (7.1) is verified.

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