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SOME HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR TWICE DIFFERENTIABLE MAPPINGS VIA FRACTIONAL INTEGRALS

Mehmet Zeki Sarikaya, * **Huseyin Budak**

Abstract. In this paper, a general integral identity for fractional integrals is derived. Using this identity, we establish some new generelized inequalities of the Hermite-Hadamard's type for functions whose absolute values of derivatives are convex.
Keywords: Hermte-Hadamard Integral Inequalities; fractional integrals; convex function.

1. Introduction

The following definition for convex functions is well known in the mathematical literature:

The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Many inequalities have been established for convex functions but the most famous inequality is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications(see, e.g.,[15, p.137], [9]). These inequalities state that if $f:I\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety

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*Corresponding author

of refinements and generalizations have been found (see, for example, [1, 2, 9, 11, 12, 15],[18]-[20],[26],[27]) and the references cited therein.

The following lemma was proved for twice differentiable mappings in [9]:

Lemma 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^o, $a, b \in I$ with $a < b$ and f'' of integrable on $[a, b]$, the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)\, f''(ta+(1-t)b)\,dt. \tag{1.2}$$

In [11], by using Lemma 1.1, Hussain et al. proved some inequalities related to Hermite-Hadamard's inequality for s-convex functions:

Theorem 1.1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. If $\left|f''\right|$ is $s-$convex on $[a,b]$ for some fixed $s \in [0,1]$ and $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2\times 6^{\frac{1}{p}}}\left[\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}, \tag{1.3}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 1.1. If we take $s = 1$ in (1.3), then we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{1.4}$$

In [18], Sarikaya and Aktan gave the following inequalities:

Theorem 1.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° with $f'' \in L_1[a,b]$. If $\left|f''\right|$ is a convex on $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \leq \frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right]. \tag{1.5}$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [10, 13, 14, 16].

Definition 1.1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Meanwhile, Sarikaya et al.[22] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]=\frac{b-a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(ta+(1-t)b)\,dt. \tag{1.6}$$

It is remarkable that Sarikaya et al.[22] first gave the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\le\frac{f(a)+f(b)}{2} \tag{1.7}$$

with $\alpha > 0$.

On the other hand, in [25], Wang et al. extended Lemma 1.2 to the case of including a twice differentiable function involving Riemann-Liouville fractional integrals as follows:

Lemma 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b) with $0 \le a < b$. If $f'' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\\ =\ &\frac{(b-a)^2}{2(\alpha+1)}\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]f''[ta+(1-t)b]\,dt.\end{aligned} \tag{1.8}$$

For some recent results connected with fractional integral inequalities see ([3]-[8],[17],[21],[23],[24],[25],[28])

The aim of this paper is to establish generalized Hermite-Hadamard type integral inequalities for Riemann-Liouville fractional integral and some other integral inequalities using the generalized identity are obtained for fractional integrals.

2. Main Results

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable mapping on (a,b) with $0\le a<b$. If $f''\in L[a,b]$, then the following equality for fractional integrals holds:*

$$
\begin{aligned}
&\frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2}-\frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\\
&\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]
\end{aligned}
$$

$$
(2.1)\quad = \int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt
$$

where $\lambda\in[0,1]\backslash\{\frac{1}{2}\}$ and $\alpha>0$.

Proof. Denote

$$
\begin{aligned}
I &= \int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt.\\
&= \int_0^1 f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\\
&\quad-\int_0^1(1-t)^{\alpha+1}f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\\
&\quad-\int_0^1 t^{\alpha+1}f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt
\end{aligned}
$$

$$
(2.2)\quad = I_1-I_2-I_3.
$$

Calculating I_1, I_2 and I_3, we have

$$
\begin{aligned}
I_1 &= \int_0^1 f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\\
(2.3)\qquad &= \frac{1}{(1-2\lambda)(b-a)}\left[f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)\right],
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \int_0^1 (1-t)^{\alpha+1} f''\left[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)\right] dt \\
&= \frac{(1-t)^{\alpha+1}}{(1-2\lambda)(b-a)} f'\left[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)\right]\Big|_0^1 \\
&\quad + \frac{\alpha+1}{(1-2\lambda)(b-a)} \int_0^1 (1-t)^{\alpha+1} f'\left[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)\right] dt \\
&= -\frac{f'(\lambda b + (1-\lambda)a)}{(1-2\lambda)(b-a)} + \frac{\alpha+1}{(1-2\lambda)(b-a)} \left[-\frac{f(\lambda b + (1-\lambda)a)}{(1-2\lambda)(b-a)} + \frac{\alpha}{(1-2\lambda)(b-a)} \right. \\
&\quad \left. \times \int_0^1 (1-t)^{\alpha-1} f\left[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)\right] dt \right] \\
&= -\frac{f'(\lambda b + (1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{(\alpha+1) f(\lambda b + (1-\lambda)a)}{(1-2\lambda)^2 (b-a)^2} + \frac{\alpha(\alpha+1)}{(1-2\lambda)^2 (b-a)^2} \\
&\quad \times \int_0^1 (1-t)^{\alpha-1} f\left[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)\right] dt \qquad (2.4)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
I_3 &= \int_0^1 t^{\alpha+1} f''\left[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)\right] dt \\
&= \frac{f'(\lambda a + (1-\lambda)b)}{(1-2\lambda)(b-a)} - \frac{(\alpha+1) f(\lambda a + (1-\lambda)b)}{(1-2\lambda)^2(b-a)^2} + \frac{\alpha(\alpha+1)}{(1-2\lambda)^2(b-a)^2} \\
&\quad \times \int_0^1 t^{\alpha-1} f\left[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)\right] dt. \qquad (2.5)
\end{aligned}
$$

Using (2.3), (2.4) and (2.5) in (2.2), it follows that

$$
\begin{aligned}
I &= I_1 - I_2 - I_3 \\
&= \frac{(\alpha+1)\left[f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2} - \frac{\alpha(\alpha+1)}{(1-2\lambda)^2(b-a)^2}
\end{aligned}
$$

$$\times\left[\int_0^1 (1-t)^{\alpha-1} f\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\right.$$

$$\left.+\int_0^1 t^{\alpha-1} f\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\right]$$

$$=\frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2}-\frac{\alpha(\alpha+1)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}$$

$$\times\left[\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b}\left[(\lambda a+(1-\lambda)b)-x\right]f(x)dx+\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b}\left[x-(\lambda b+(1-\lambda)a)\right]f(x)dx\right]$$

$$=\frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2}-\frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}$$

$$\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}} f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}} f(\lambda b+(1-\lambda)a)\right].$$

□

Corollary 2.1. *Under the same assumptions of Lemma 2.1 with $\lambda=0$, we have*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{b^-} f(a)+J^{\alpha}_{a^+} f(b\right]$$

$$=\frac{(b-a)^2}{2(\alpha+1)}\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]f''\left[tb+(1-t)a\right]dt.$$

Remark 2.1. If we take $\lambda=1$ in Lemma 2.1, we obtain

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{(-1)^{\alpha+2}\,2(b-a)^{\alpha}}\left[J^{\alpha}_{a^-} f(b)+J^{\alpha}_{b^+} f(a)\right]$$

$$=\frac{(b-a)^2}{2(\alpha+1)}\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]f''\left[ta+(1-t)b\right]dt. \tag{2.6}$$

By using $J^{\alpha}_{b^+} f(a)+J^{\alpha}_{a^-} f(b)=(-1)^{\alpha}\left[J^{\alpha}_{a^+} f(b)+J^{\alpha}_{b^-} f(a)\right]$ in (2.6), the identity (2.6) reduce to the identity (1.8) which was proved by Wang et al. in [25].

Remark 2.2. If we take $\alpha=1$ in (2.6), then the equality in (2.6) becomes to the equality (1.2).

Theorem 2.1. *$f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping on (a,b) with $0\le a<b$. If $\left|f''\right|^q$ is convex on $[a,b]$ for some fixed $q\ge 1$ then following inequality for fractional holds:*

$$\left|\frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2}-\frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\right.$$

$$\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|$$

$$\le\frac{\alpha}{\alpha+2}\left[\frac{\left|f''(\lambda a+(1-\lambda)b)\right|^q+\left|f''(\lambda b+(1-\lambda)a)\right|^q}{2}\right]^{\frac{1}{q}}\tag{2.7}$$

where $\lambda\in[0,1]\backslash\{\frac{1}{2}\}$ and $\alpha>0$.

Proof. Firstly, we suppose that $q=1$. From Lemma 2.1, we have

$$\left|\frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2}-\frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\right.$$

$$\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|$$

$$\le\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt.$$

$$\le\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left[t\left|f''(\lambda a+(1-\lambda)b)\right|+(1-t)\left|f''(\lambda b+(1-\lambda)a)\right|\right]dt$$

$$=\left|f''(\lambda a+(1-\lambda)b)\right|\int_0^1\left[t-(1-t)^{\alpha+1}t-t^{\alpha+2}\right]dt$$

$$+\left|f''(\lambda b+(1-\lambda)a)\right|\int_0^1\left[(1-t)-(1-t)^{\alpha+2}-t^{\alpha+1}(1-t)\right]dt$$

$$=\frac{\alpha}{(\alpha+2)}\left[\frac{\left|f''(\lambda a+(1-\lambda)b)\right|+\left|f''(\lambda b+(1-\lambda)a)\right|}{2}\right].$$

Here we use $(1-t)^{\alpha+1}+t^{\alpha+1}\le 1$ for any $t\in[0,1]$ and the convexity of $\left|f''\right|$.

Secondly, we suppose that $q>1$. Using Lemma 2.1 and the power mean equality for q, we have

$$\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt.$$

$$\leq \left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}}. \tag{2.8}$$

Using Lemma 2.1, (2.8) and the convexity of $\left|f''\right|^q$, we have

$$\left|\frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2}-\frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\right.$$
$$\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq \left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)$$
$$\leq \left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left[t\left|f''(\lambda a+(1-\lambda)b)\right|^q+(1-t)\left|f''(\lambda b+(1-\lambda)a)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\left(\left[t+\frac{(1-t)^{\alpha+2}}{\alpha+2}-\frac{t^{\alpha+2}}{\alpha+2}\right]\Big|_0^1\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{\alpha}{\alpha+2}\left[\frac{\left|f''(\lambda a+(1-\lambda)b)\right|^q+\left|f''(\lambda b+(1-\lambda)a)\right|^q}{2}\right]\right)^{\frac{1}{q}}$$
$$=\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{\alpha}{\alpha+2}\right)^{\frac{1}{q}}\left[\frac{\left|f''(\lambda a+(1-\lambda)b)\right|^q+\left|f''(\lambda b+(1-\lambda)a)\right|^q}{2}\right]^{\frac{1}{q}}$$
$$=\frac{\alpha}{\alpha+2}\left[\frac{\left|f''(\lambda a+(1-\lambda)b)\right|^q+\left|f''(\lambda b+(1-\lambda)a)\right|^q}{2}\right]^{\frac{1}{q}}.$$

□

Corollary 2.2. *Under the same assumptions of Theorem 2.1 with $\lambda = 0$, then we take*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{b^-} f(a) + J^{\alpha}_{a^+} f(b)\right]\right| \leq \frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}\left[\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{2.9}$$

Remark 2.3. If we take $\lambda = 1$ in Theorem 2.1, then we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{(-1)^{\alpha+2}2(b-a)^{\alpha}}\left[J^{\alpha}_{b^+} f(a) + J^{\alpha}_{a^-} f(b)\right]\right| \leq \frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}\left[\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{2.10}$$

By using $J^{\alpha}_{b^+} f(a) + J^{\alpha}_{a^-} f(b) = (-1)^{\alpha}\left[J^{\alpha}_{a^+} f(b) + J^{\alpha}_{b^-} f(a)\right]$ in (2.10), the inequality (2.10) reduces to the inequality (2.9).

Remark 2.4. If we take $\alpha = 1$ in (2.9), then the inequality in (2.9) becomes to the inequality (1.4).

Remark 2.5. If we take $\alpha = 1$ and $q = 1$ in (2.9), then the inequality in (2.9) becomes to the inequality (1.5).

Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b) with $0 \leq a < b$. If $\left|f''\right|^q$ is convex on $[a, b]$ for some fixed $q > 1$, then following inequality for fractional holds:*

$$\left|\frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b) + f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2} - \frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\right.$$
$$\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+} f(\lambda a+(1-\lambda)b) + J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq \left(1 - \frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{\left|f''(\lambda a+(1-\lambda)b)\right|^q + \left|f''(\lambda b+(1-\lambda)a)\right|^q}{2}\right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in [0, 1] \setminus \{\frac{1}{2}\}$ and $\alpha > 0$.

Proof. From Lemma 2.1 and using the well-known Hölder's inequality and convexity of $\left|f''\right|^q$, we have

$$\left|\frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b) + f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2} - \frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\right.$$
$$\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+} f(\lambda a+(1-\lambda)b) + J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(\lambda b+(1-\lambda)a)\right]\right|$$

$$\leq \int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right| dt$$

$$\leq \left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1 \left[1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)}\right] dt\right)^{\frac{1}{p}}$$

$$\times\left(\left|f''(\lambda a+(1-\lambda)b)\right|^q \int_0^1 t dt+\left|f''(\lambda b+(1-\lambda)a)\right|^q \int_0^1 (1-t) dt\right)^{\frac{1}{q}}$$

$$\leq \left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{\left|f''(\lambda a+(1-\lambda)b)\right|^q+\left|f''(\lambda b+(1-\lambda)a)\right|^q}{2}\right)^{\frac{1}{q}} \tag{2.11}$$

Here we use

$$\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]^p \leq 1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)}$$

for any $t \in [0,1]$ which follows from

$$(A-B)^p \leq A^p - B^p,$$

for any $A > B \geq 0$ and $p \geq 1$. which completes the proof. □

Corollary 2.3. *Under the same assumptions of Theorem 2.2 with $\lambda = 0$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\right| \tag{2.12}$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Remark 2.6. If we take $\lambda = 1$ in Theorem 2.2, then we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{(-1)^{\alpha+2}2(b-a)^\alpha}\left[J_{b^+}^\alpha f(a)+J_{a^-}^\alpha f(b)\right]\right|$$

$$\leq \frac{(b-a)^2}{2(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}. \tag{2.13}$$

By using $J^{\alpha}_{b^+}f(a) + J^{\alpha}_{a^-}f(b) = (-1)^{\alpha}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\right]$ in (2.13), the inequality (2.13) reduces to the inequality (2.12).

Theorem 2.3. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $0 \le a < b$. If $\left|f''\right|^q$ is convex on $[a,b]$, for some fixed $q \ge 1$, then following inequality for fractional hold:*

$$\left|\frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2} - \frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\right.$$
$$\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b) + J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|$$
$$\le \left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{\left|f''(\lambda a+(1-\lambda)b)\right|^q+\left|f''(\lambda b+(1-\lambda)a)\right|^q}{2}\right)^{\frac{1}{q}}$$

where $\lambda \in [0,1]\backslash\{\frac{1}{2}\}$ and $\alpha > 0$.

Proof. From Lemma 2.1 and using well-known Hölder's inequality, we have

$$\left|\frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2} - \frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\right.$$
$$\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b) + J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|$$
$$\le \int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right| dt$$
$$\le \left(\int_0^1 1dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]^q\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}}$$
$$\le \left(\int_0^1 \left[1-(1-t)^{q(\alpha+1)}-t^{q(\alpha+1)}\right]\left[t\left|f''(\lambda a+(1-\lambda)b)\right|^q+(1-t)\left|f''(\lambda b+(1-\lambda)a)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$= \left(\left|f''(\lambda a+(1-\lambda)b)\right|^q\int_0^1\left[t-(1-t)^{q(\alpha+1)}t-t^{q(\alpha+1)+1}\right]dt\right.$$
$$\left.+\left|f''(\lambda b+(1-\lambda)a)\right|^q\int_0^1\left[(1-t)-(1-t)^{q(\alpha+1)+1}-t^{q(\alpha+1)}(1-t)\right]\right)^{\frac{1}{q}}$$
$$= \left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{\left|f''(\lambda a+(1-\lambda)b)\right|^q+\left|f''(\lambda b+(1-\lambda)a)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Here we use

$$\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]^{q}\leq 1-(1-t)^{q(\alpha+1)}-t^{q(\alpha+1)}$$

for any $t\in[0,1]$ which follows from

$$(A-B)^{q}\leq A^{q}-B^{q},$$

for any $A>B\geq 0$ and $q\geq 1$ which completes the proof. □

Corollary 2.4. *Under the same assumptions of Theorem 2.3 with $\lambda=0$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\right| \leq \frac{(b-a)^{2}}{2(\alpha+1)}\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{\left|f''(a\right|^{q}+\left|f''(b)\right|^{q}}{2}\right)^{\frac{1}{q}}. \tag{2.14}$$

Corollary 2.5. *If we take $\lambda=1$ in Theorem 2.3, then we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{(-1)^{\alpha+2}2(b-a)^{\alpha}}\left[J_{b^{+}}^{\alpha}f(a)+J_{a^{-}}^{\alpha}f(b)\right]\right| \leq \frac{(b-a)^{2}}{2(\alpha+1)}\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}}{2}\right)^{\frac{1}{q}}. \tag{2.15}$$

By using $J_{b^{+}}^{\alpha}f(a)+J_{a^{-}}^{\alpha}f(b)=(-1)^{\alpha}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]$ in (2.15), the inequality (2.15) reduces to the inequality (2.14).

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Mehmet Zeki Sarikaya

Department of Mathematics,Faculty of Science and Arts

Düzce University, Düzce, Turkey

sarikayamz@gmail.com

Huseyin Budak

Department of Mathematics,Faculty of Science and Arts

Düzce University, Düzce, Turkey

hsyn.budak@gmail.com