FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 32, No 3 (2017), 319–327
DOI:10.22190/FUMI1703319M

SOME NEW OSTRWOSKI'S INEQUALITIES FOR n-TIMES DIFFERENTIABLE MAPPINGS WHICH ARE QUASI-CONVEX

Badreddine Meftah

Abstract. Some new Ostrowski's inequalities for functions whose n^{th} derivatives are quasi-convex are established.

Keywords: Ostrowski inequality, Holder inequality, power mean inequality, quasi-convex functions.

1. Introduction

In 1938, A.M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the derivative of a differentiable function by its integral mean as follows

Theorem 1.1. [3] *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping in the interior I° of I, and $a, b \in I^\circ$, with $a < b$.*

If $|f'| \leq M$ for all $x \in [a, b]$, then

$$(1.1) \qquad \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall x \in [a, b].$$

This is known as Ostrowski's inequality. In recent years, a number of authors have written about generalizations, extensions and variants of Inequality (1.1). We note that the literature in this context is broad and abundant.

In [1], Cerone et al. proved the following identity

Lemma 1.1. [1, Lemma 2.1] *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Then for all $x \in [a, b]$ we have the*

Received January 25, 2017; accepted April 13, 2017
2010 *Mathematics Subject Classification.* Primary 26D10; Secondary 26D15, 26A51

identity

$$\int_a^b f(t)dt = \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x) + (-1)^n\int_a^b K_n(x,t)\, f^{(n)}(t)dt$$

where the kernel $K_n : [a,b]^2 \to \mathbb{R}$ *is given by*

$$K_n(x,t) = \begin{cases} \frac{(t-a)^n}{n!} & \text{if } t \in [a,x] \\ \frac{(t-b)^n}{n!} & \text{if } t \in (x,b] \end{cases}, x \in [a,b]$$

and n *is a natural number,* $n \geq 1$.

Also, we recall that a function $f : I \to \mathbb{R}$ is said to be quasi-convex on I, if

$$f(tx+(1-t)y) \leq \max\{f(x), f(y)\}$$

holds for all $x, y \in I$ and all $t \in [0,1]$. see [2].

In this paper, we establish some new Ostrowski's inequalities for n-times differentiable mappings which are quasi-convex.

2. Main results

In what follows, we assume that $n \in \mathbb{N}, [a,b] \subset I \subset \mathbb{R}$ with $a < b$.

Theorem 2.1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be* n*-times differentiable on* $[a,b]$ *such that* $f^{(n)} \in L([a,b])$. *If* $\left|f^{(n)}\right|$ *is quasi-convex, then the following inequality*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \leq \frac{(x-a)^{n+1}}{(n+1)!}\max\left\{\left|f^{(n)}(x)\right|, \left|f^{(n)}(a)\right|\right\} + \frac{(b-x)^{n+1}}{(n+1)!}\max\left\{\left|f^{(n)}(b)\right|, \left|f^{(n)}(x)\right|\right\}$$

holds for all $x \in [a,b]$.

Proof. From Lemma 1.1, the properties of the modulus, and the quasi-convexity of $\left|f^{(n)}\right|$, we have

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\le\ & \int_a^x \frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right|du+\int_x^b \frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right|du \\
=\ & \frac{(x-a)^{n+1}}{n!}\int_0^1 t^n \left|f^{(n)}((1-t)a+tx)\right|dt \\
& +\frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n \left|f^{(n)}((1-t)x+tb)\right|dt \\
\le\ & \frac{(x-a)^{n+1}}{n!}\max\left\{\left|f^{(n)}(x)\right|,\left|f^{(n)}(a)\right|\right\}\int_0^1 t^n\,dt \\
& +\frac{(b-x)^{n+1}}{n!}\max\left\{\left|f^{(n)}(b)\right|,\left|f^{(n)}(x)\right|\right\}\int_0^1 (1-t)^n\,dt \\
=\ & \frac{(x-a)^{n+1}}{(n+1)!}\max\left\{\left|f^{(n)}(x)\right|,\left|f^{(n)}(a)\right|\right\}+\frac{(b-x)^{n+1}}{(n+1)!}\max\left\{\left|f^{(n)}(b)\right|,\left|f^{(n)}(x)\right|\right\},
\end{aligned}$$

which is the desired result. □

Corollary 2.1. *Let f be as in Theorem 2.1, so assume that*

(1) $\left|f^{(n)}\right|$ *is increasing, then*

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\le\ & \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right|+\frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(b)\right|. \qquad (2.1)
\end{aligned}$$

(2) $\left|f^{(n)}\right|$ *is decreasing, then*

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\le\ & \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right|+\frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right|. \qquad (2.2)
\end{aligned}$$

(3) $\left|f^{(n)}(a)\right| = \left|f^{(n)}(b)\right| = 0$, *we obtain*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$

$$\leq \left(\frac{(x-a)^{n+1}}{(n+1)!}+\frac{(b-x)^{n+1}}{(n+1)!}\right)\left|f^{(n)}(x)\right|. \tag{2.3}$$

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-times differentiable on $[a, b]$ such that $f^{(n)} \in L([a, b])$, and let $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$. If $\left|f^{(n)}\right|^q$ is quasi-convex, then the following inequality*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$

$$\leq \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left(\max\left\{\left|f^{(n)}(x)\right|^q, \left|f^{(n)}(a)\right|^q\right\}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}\,n!}\left(\left(\max\left\{\left|f^{(n)}(b)\right|^q, \left|f^{(n)}(x)\right|^q\right\}\right)\right)^{\frac{1}{q}}$$

holds for all $x \in [a, b]$.

Proof. From Lemma 1.1, the properties of the modulus, Hölder's inequality, and the quasi-convexity of $\left|f^{(n)}\right|^q$, we have

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$

$$\leq \int_a^x \frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right|du + \int_x^b \frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right|du$$

$$= \frac{(x-a)^{n+1}}{n!}\int_0^1 t^n\left|f^{(n)}((1-t)a+tx)\right|dt$$

$$
\begin{aligned}
&+\frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n \left|f^{(n)}((1-t)x+tb)\right| dt \\
\leq\ & \frac{(x-a)^{n+1}}{n!}\left(\int_0^1 t^{np}dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f^{(n)}((1-t)a+tx)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1 (1-t)^{np}\ dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f^{(n)}((1-t)x+tb)\right|^q dt\right)^{\frac{1}{q}} \\
=\ & \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}n!}\left(\max\left\{\left|f^{(n)}(x)\right|^q, \left|f^{(n)}(a)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}n!}\left(\left(\max\left\{\left|f^{(n)}(b)\right|^q, \left|f^{(n)}(x)\right|^q\right\}\right)\right)^{\frac{1}{q}},
\end{aligned}
$$

which completes the proof. □

Corollary 2.2. *Let f be as in Theorem 2.2, so assume that*

(1) $\left|f^{(n)}\right|$ *is increasing, then we have*

$$
\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\leq\ & \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}n!}\left|f^{(n)}(x)\right| + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}n!}\left|f^{(n)}(b)\right|.
\end{aligned}
$$

(2) $\left|f^{(n)}\right|$ *is decreasing, then we have*

$$
\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\leq\ & \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}n!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}n!}\left|f^{(n)}(x)\right|.
\end{aligned}
$$

(3) $\left|f^{(n)}(a)\right| = \left|f^{(n)}(b)\right| = 0$, *we obtain*

$$
\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\leq\ & \left(\frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}}n!} + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}n!}\right)\left|f^{(n)}(x)\right|.
\end{aligned}
$$

Theorem 2.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-times differentiable on $[a,b]$ such that $f^{(n)} \in L([a,b])$, and let $q > 1$. If $\left|f^{(n)}\right|^q$ is quasi-convex, then the following inequality*

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \\
\leq\; & \frac{(x-a)^{n+1}}{(n+1)!}\left(\max\left\{\left|f^{(n)}(x)\right|^q, \left|f^{(n)}(a)\right|^q\right\}\right)^{\frac{1}{q}} \\
+\; & \frac{(b-x)^{n+1}}{(n+1)!}\left(\max\left\{\left|f^{(n)}(b)\right|^q, \left|f^{(n)}(x)\right|^q\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

holds for all $x \in [a,b]$.

Proof. From Lemma 1.1, the properties of the modulus, the power mean inequality, and the quasi-convexity of $\left|f^{(n)}\right|^q$, we have

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \\
\leq\; & \int_a^x \frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right| du + \int_x^b \frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right| du \\
=\; & \frac{(x-a)^{n+1}}{n!}\int_0^1 t^n \left|f^{(n)}((1-t)a + tx)\right| dt \\
& + \frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n \left|f^{(n)}((1-t)x + tb)\right| dt \\
\leq\; & \frac{(x-a)^{n+1}}{n!}\left(\int_0^1 t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^n \left|f^{(n)}((1-t)a + tx)\right| dt\right)^{\frac{1}{q}} \\
& + \frac{(b-x)^{n+1}}{n!}\left(\int_0^1 (1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^n \left|f^{(n)}((1-t)x + tb)\right|^q dt\right)^{\frac{1}{q}} \\
& = \frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\left(\int_0^1 t^n \left|f^{(n)}((1-t)a + tx)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!}\left(\int_0^1 (1-t)^n \left|f^{(n)}((1-t)x+tb)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\left(\max\left\{\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(a)\right|^q\right\}\int_0^1 t^n dt\right)^{\frac{1}{q}}$$

$$+\frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!}\left(\max\left\{\left|f^{(n)}(b)\right|^q,\left|f^{(n)}(x)\right|^q\right\}\int_0^1 (1-t)^n dt\right)^{\frac{1}{q}}$$

$$=\frac{(x-a)^{n+1}}{(n+1)!}\left(\max\left\{\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(a)\right|^q\right\}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{(n+1)!}\left(\max\left\{\left|f^{(n)}(b)\right|^q,\left|f^{(n)}(x)\right|^q\right\}\right)^{\frac{1}{q}}.$$

The proof is completed. □

Corollary 2.3. *Let f be as in Theorem 2.3, so assume that*

(1) $\left|f^{(n)}\right|$ *is increasing, then* (2.1) *is valid.*

(2) $\left|f^{(n)}\right|$ *is decreasing, then* (2.2) *is valid.*

(3) $\left|f^{(n)}(a)\right|=\left|f^{(n)}(b)\right|=0$, *then* (2.3) *is valid.*

Theorem 2.4. *Suppose that all the assumptions of Theorem 2.3 are satisfied, then the following inequality*

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$

$$\leq \frac{(x-a)^{n+1}}{(qn+1)^{\frac{1}{q}}\,n!}\left(\max\left\{\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(a)\right|^q\right\}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{(qn+1)^{\frac{1}{q}}\,n!}\left(\max\left\{\left|f^{(n)}(b)\right|^q,\left|f^{(n)}(x)\right|^q\right\}\right)^{\frac{1}{q}}$$

holds for all $x\in[a,b]$.

Proof. From Lemma 1.1, the properties of the modulus, the power mean inequality, and the quasi-convexity of $\left|f^{(n)}\right|^q$, we have

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$

$$\leq \int_a^x \frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right|du+\int_x^b \frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right|du$$

$$
\begin{aligned}
&= \frac{(x-a)^{n+1}}{n!}\int_0^1 t^n \left|f^{(n)}((1-t)a+tx)\right| dt \\
&\quad + \frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n \left|f^{(n)}((1-t)x+tb)\right| dt \\
&\leq \frac{(x-a)^{n+1}}{n!}\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{qn}\left|f^{(n)}((1-t)a+tx)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{n+1}}{n!}\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^{qn}\left|f^{(n)}((1-t)x+tb)\right|^q dt\right)^{\frac{1}{q}} \\
&= \frac{(x-a)^{n+1}}{n!}\left(\int_0^1 t^{qn}\left|f^{(n)}((1-t)a+tx)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{n+1}}{n!}\left(\int_0^1 (1-t)^{qn}\left|f^{(n)}((1-t)x+tb)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{(x-a)^{n+1}}{n!}\left(\max\left\{\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(a)\right|^q\right\}\int_0^1 t^{qn}dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{n+1}}{n!}\left(\max\left\{\left|f^{(n)}(b)\right|^q,\left|f^{(n)}(x)\right|^q\right\}\int_0^1 (1-t)^{qn}dt\right)^{\frac{1}{q}} \\
&= \frac{(x-a)^{n+1}}{(qn+1)^{\frac{1}{q}}\,n!}\left(\max\left\{\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(a)\right|^q\right\}\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{n+1}}{(qn+1)^{\frac{1}{q}}\,n!}\left(\max\left\{\left|f^{(n)}(b)\right|^q,\left|f^{(n)}(x)\right|^q\right\}\right)^{\frac{1}{q}},
\end{aligned}
$$

which is the desired result. □

Corollary 2.4. *Let f be as in Theorem 2.4, assume that*

(1) $\left|f^{(n)}\right|$ *is increasing, then we have*

$$
\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \\
&\leq \frac{(x-a)^{n+1}}{(qn+1)^{\frac{1}{q}}\,n!}\left|f^{(n)}(x)\right| + \frac{(b-x)^{n+1}}{(qn+1)^{\frac{1}{q}}\,n!}\left|f^{(n)}(b)\right|.
\end{aligned}
$$

(2) $\left|f^{(n)}\right|$ *is decreasing, then we have*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$
$$\leq \frac{(x-a)^{n+1}}{(qn+1)^{\frac{1}{q}}\,n!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(qn+1)^{\frac{1}{q}}\,n!}\left|f^{(n)}(x)\right|.$$

(3) $\left|f^{(n)}(a)\right| = \left|f^{(n)}(b)\right| = 0$, *we obtain*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$
$$\leq \left(\frac{(x-a)^{n+1}}{(qn+1)^{\frac{1}{q}}\,n!} + \frac{(b-x)^{n+1}}{(qn+1)^{\frac{1}{q}}\,n!}\right)\left|f^{(n)}(x)\right|.$$

REFERENCES

1. P. Cerone, S.S. Dragomir, J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications. Demonstratio Math. 32 (1999), no. 4, 697–712.
2. D. A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions. An. Univ. Craiova Ser. Mat. Inform. 34 (2007), 83–88.
3. D. S. Mitrinović, J. E. Pečarić, A. M. Fink, Classical and new inequalities in analysis. Mathematics and its Applications (East European Series), 61. Kluwer Academic Publishers Group, Dordrecht, 1993.

Badreddine Meftah
Laboratoire des télécommunications
Facult des Sciences et de la Technologie
University of 8 May 1945 Guelma
24000 Guelma, Algeria
badrimeftah@yahoo.fr